Qubit Thermalization by Random Pulses: Asymptotic State Factorization

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Received: 13.08.2025 & Accepted: 14.11.2025

Doi: 10.12693/APhysPolA.149.17

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Here we consider an analytically tractable model of a two-level quantum system subject to random shocks and prove that it decays asymptotically to a trivial state, that is, to a state in which the two levels have equal probability of being occupied. In a two-qubit system, if the shocks affect each qubit independently, the equilibrium density matrix becomes a simple product of the one-qubit equilibrium density matrix, regardless of the form of the initial state. This has potential applications to entangled qubits in quantum computers.

topics: quantum systems subject to random pulses, qubit thermalization, quantum systems in random environments, disentanglement and decorrelation

1. Introduction

If we type "disentanglement" into our favorite browser, we will be provided with links covering almost every type of human endeavor. If we refine the search to "disentanglement in quantum mechanics", we will find references to works from many domains of quantum mechanics application. It therefore makes sense to recall the meaning of separability, correlation, and entanglement.

When dealing with a system composed of two subsystems, in the Schrödinger description one says that the state Ψ of the whole system is an entangled state when there do not exist two states Ψ_1 and Ψ_2 such that $\Psi = \Psi_1 \otimes \Psi_2$. In the Heisenberg representation things are more complicated. From [1] or [2], we recall that

Definition 1. Consider a composite quantum system made up K subsystems. A quantum state of the composite system, described by a density matrix ρ is separable if it can be written as

$$\rho = \sum_{k=1}^{N} \mu_k \left|\psi_1(k)\right\rangle\left\langle\psi_i(k)\right| \otimes ... \otimes \left|\psi_K(k)\right\rangle\left\langle\psi_K(k)\right|. \tag{1}$$

Here, for $k = 1, \ldots, N$, $|\psi_i(k)\rangle$ denote a state in the i-th subsystem (here $i = 1, \ldots, K$), and $\{\mu_k : \; k = 1, ..., N\}$ are positive numbers such that $\sum_{k=1}^{N} \mu_k = 1$. The definition in terms of density matrices is similar, but more general, because densities may not be pure states. The state ρ is separable, classically correlated, or mixed, whenever

$$\rho = \sum_{k=1}^{N} \mu_k \, \rho_1(k) \otimes ... \otimes \rho_K(k). \tag{2}$$

Now, for $k = 1, ..., N$, $\{\rho_i(k) : \; i = 1, ..., K\}$ denote states in the corresponding Heisenberg representation, and $\{\mu_k : \; k = 1, ..., N\}$ are positive numbers such that $\sum_{k=1}^{N} \mu_k = 1$.
A state ρ is called entangled if it is not separable.

Also, making use of the Bloch representation (or using Pauli spin matrices as basis vectors for a class of density matrices), we shall see that the asymptotic behavior of any density matrix can be obtained from the asymptotic behavior of the time evolution operator acting on Pauli matrices. So proving entanglement may not be easy. Nevertheless, there is a partial test — the *purity* test — that discriminates between pure and non-pure states. Given a state ρ, it is a pure state if $\rho^2 = \rho$. Therefore, if $\mathrm{Tr}[\rho^2] < \mathrm{Tr}[\rho] = 1$, the state cannot be a pure state. Example (37) presented in Sect. 3.1 is an example of a separable but non-pure state.

The reason for considering the density matrices is clear. Even though we work out the time evolution of the system in the Schrödinger representation, when we need to make predictions, we have to compute the expected values. We do this in terms of traces with respect to the density matrices. Since the system evolves in a random environment, first we compute the trace with respect to the density matrix for each possible random evolution, and then average over all possible random evolutions.

The aim of this paper is to determine the time evolution of a single qubit subjected to a random perturbation, where the effect of the perturbation manifests itself as a flipping of the unperturbed basis states; the switching occurs at exponentially distributed time intervals. Once we find the explicit time evolution for any random sequence of shocks and any initial state in the Schrödinger representation, we transport it to the Heisenberg representation to determine the corresponding time evolution of density matrices.

After that, we consider two identical, non-interacting, two level sub-systems (say, qubits), in an arbitrary initial state (density matrix), immersed in the heat bath described above. We show that in this model of heat bath, the system evolves into a trivial product state.

For applications in quantum information theory and quantum computer design, a worrisome issue is the presence of thermal noise, which destroys the entanglement necessary for the quantum codes. Therefore, a simple model displaying this phenomenon might be interesting.

Let us now detail the basic assumptions of the model. We consider two identical, non-interacting two-level systems coupled through their initial density matrix. The time evolution of each system is described by the same Hamiltonian H, and the effect of shocks on the states are represented by the following matrices on the canonical basis

$$H = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

In (3), H represents the Hamiltonian of each component, and V the unitary operator describing the effect of the shocks produced by the heat bath. The basis vectors for each state will be denoted by the customary $|1\rangle$ and $|2\rangle$, such that $H|i\rangle = \epsilon_i\,|i\rangle$ for $i = 1, 2$. And the effect of V on the same basis vectors is given by $V|1\rangle = |2\rangle$ and $V|2\rangle = |1\rangle$.

The model of the heat bath considered here is similar to the one derived by Clauser and Blume in [3]. The system starts at some initial state Ψ (or $|\Psi\rangle$ depending of the notation convenience), evolves according to H for a random time τ_1, at which the perturbation given by V acts instantaneously and flips the spin of the system. After the shock, the system again evolves according to H for another random time τ_2, at the end of which another instantaneous state flip occurs according to V. This is repeated infinitely many times. The instantaneous perturbation may be a more elaborate operator that changes the state of the system instantaneously, but the model we considered here allows for full analytical computations and suffices to convey the essence of the thermalization process.

The proposal underlying the paper [3] can be extended to the theory of random quantum evolutions of systems with a finite number of levels, as in [4]. The assumptions underlying the model are the following:

(i) The two sub-systems do not interact. That is, each evolves according to the Hamiltonian in (3).
(ii) The heat bath affects each sub-system independently. That is, the shock events each subsystem experiences are independent of each other (we may assume that they are equally distributed).

The assumption (ii) is natural if we think of the subsystems as spatially separated. Thus, even if they respond similarly to the heat bath, their responses are independent of each other.

Now let us recall some statistical properties of the noise. We assume that the times between shocks are independent and exponentially distributed, with frequency λ (or the mean inter arrival time $1/\lambda$). If we write $T_n = \sum_k \tau_k$ to denote the time of occurrence of the n-th shock, the shock counting process $N(t)$ can be modelled by $\{N(t) = n\} \Leftrightarrow \{T_n \le t \le T_{n+1}\}$. One sets $T_0 = 0$ to ensure that when the experiment is started, no shocks occur.

Statistically speaking, the process is characterized by saying that $N(0) = 0$, i.e., that for any t and $s \ge 0$, the random variables $N(t+s) - N(s)$ have the same probability distribution as $N(t)$, and that for any $t_1 < t_2 < t_3 < t_4$, the increments $N(t_4) - N(t_3)$ and $N(t_2) - N(t_1)$ are statistically independent. Furthermore, for any $t \ge 0$,

$$P\big(N(t){=}n\big) = \frac{(\lambda\, t)^n}{n!}\, \mathrm{e}^{t\,\lambda}. \tag{4}$$

Not only that, if $0 < T_1 < T_2 < ... < T_n < hl \dots$ denote the (random) jump times of $N(t)$, then times $\tau_1 = T_1$ and $\tau_n = T_n - T_{n-1}$ are independent and exponentially distributed with parameter λ. Besides, $\{N(t) = 0\} = \{t < T_1\}$ and $\{N(t) = n\} = \{T_n \le t < T_{n+1}\}$. To finish, the conditional distribution of $T_1, T_2, ..., T_n$, given $\{N(t) = n\}$, is uniform in the sector $0 < t_1 < t_2 < ... < t_n$ of the positive orthant in $\mathbb{R}^n$ with the density $n!/t^n$. This means that for any continuous function, say $h(T_1, T_2, ..., T_n)$, we have

$$E\Big[h(T_1, T_2, ..., T_n)\big|N(t) = n\Big] =$$
$$\frac{n!}{t^n}\int_0^t \mathrm{d}t_n \int_0^{t_n} \mathrm{d}t_{n-1} \int_0^{t_n-2} ... \int_0^{t_2} \mathrm{d}t_H(t_1, t_2, ..., t_n), \tag{5}$$

We shall put this to a good use below.

The stochastic nature of this thermal bath model is simple but intuitive, which — combined with the spin flipping perturbation — makes the model analytically tractable. Many properties and applications of the Poisson process are discussed in [5] or [6], and a more physical flavor in [7] or [8]. For our needs, it suffices to say that the process $N(t)$ has right-continuous trajectories with left limits, that it is constant between jumps, and that all jumps are of size 1.

At this point, we pass a quick review of the literature. The literature about disentanglement is rather large. To put it into context, for several reviews on entanglement, see [2, 9, 10], and for the experimental determination of disentanglement, consider [11]. Overlapping themes in the list of references include the relationship to decoherence, blind source separation, Bell's inequalities, the nature of a two-state system, and the nature of the thermal bath. To begin with, see [12] for the problem of entanglement, and consider [13, 14] for the creation and stability of entangled states.

Other topics are also worth mentioning:

- For the possible existence of disentanglement-free states, see [15].
- For different theoretical issues related to disentangling, see [16–24].
- For the dynamics of disentangling, see [12, 25].
- For disentanglement after spontaneous emission, see [26].
- For different models of the thermal bath, see [27–31].
- For a review of both theoretical and experimental results and several models of thermal bath, see [32].
- For the thermalization of a photonic qubit, see [33].
- For thermalization of two-level quantum systems, see [34].
- For systems interacting with a bosonic heat bath, see [35], but also see [36], where the authors replace the heat bath by successive scattering with a moving particle.
- For thermalization and information scrambling in quantum superprocessors, see [37, 38].
- For the relationships between disentanglement decoherence, and dephasing, consider [39–44].
- For disentanglement and teleportation, see [45, 46].
- For relationship to Bell's inequalities, see [47].
- For potential application to blind source separation, see [48].

2. The Blume–Clauser model for the time evolution of the density matrix

The starting point is the time evolution operator in the Schrödinger picture. Before a random shock happens, the time evolution operator $U_0(t) = \exp(-\mathrm{i}tH/h)$ describes the time evolution of the unperturbed system. (We use $U(t)$ to denote the time evolution operator of the perturbed system.) Suppose that at random time T_1 a random shock occurs; since before the first shock the evolution was according to U_0, by continuity we have

$$U(T_1-) = \lim_{t\uparrow T_1} U(t) = \lim_{t\uparrow T_1} U_0(t) = U_0(T_1). \qquad (6)$$

Or, if $\Psi(0)$ denotes the initial state, then right before the collision the state is

$$\Psi(T_1-) = U_0(T_1-)\,\Psi(0). \qquad (7)$$

Right after the (instantaneous) shock at T_1, the time evolution operator of the random system is

$$U(T_1) = V\,U(T_1-) = V\,U_0(T_1). \qquad (8)$$

And in terms of the state vector, it is as follows

$$\Psi(T_1) = V\,\Psi(T_1-) = VU_0(T_1-)\,\Psi(0). \qquad (9)$$

These equations capture the essence of the mathematical description of the discontinuity of the state vector and the evolution operator. To continue, during the time interval $T_1 \le t < T_2$, the system again evolves according to U_0, i.e.,

$$U(t) = U_0(t-T_1)U(T_1) = U_0(t-T_1)\,V\,U_0(T_1). \qquad (10)$$

When the next shock occurs at T_2, then

$$U(T_2-) = \lim_{t\uparrow T_2} U(t) = \lim_{t\uparrow T_2} U_0(t-T_1)\,V\,U_0(T_1) = U_0(T_2-T_1)\,V\,U_0(T_1) \qquad (11)$$

Right after the shock, $U(T_2) = VU(T_2-) = V\,U_0(T_2-T_1)\,V\,U_0(T_1)$. And so on and on.

These comments can be summarized as follows

$$U(t) = \begin{cases} \text{Evolution operator} & \text{Number of shocks} & \text{Probability} \\ U_0(t), & \{N(t)=0\}, & \mathrm{e}^{-\lambda t}, \\ U_0(t-T_n)\,V\,U_0(T_n-T_{n-1})V...V\,U_0(T_1), & \{N(t)=n\}, & \mathrm{e}^{-\lambda t}\dfrac{(\lambda t)^n}{n!}. \end{cases} \qquad (12)$$

If we introduce the notations $\delta_{\{N(t),n\}}$ to denote the binary random Kronecker delta, which assumes the value 1 when the event $\{N(t) = n\}$ happens and 0 when it does not, the previous definition can be written in a more convenient form

$$U(t) = \sum_{n=0}^{\infty} U_0(t-T_n)\, V\, U_0(T_n-T_{n-1})\, V \ldots \ldots V\, U_0(T_1)\delta_{\{N(t),n\}}. \tag{13}$$

Notice that the first term in the sum, corresponding to $n=0$, is simply $U_0(t)\delta_{\{N(t),0\}}$. When considering states (density matrices) in the Heisenberg representation, it is proven convenient to introduce the super-operator notation,

$$V^\sharp(A) = V\, A\, V^\dagger, \qquad U_0^\sharp(t)(A) = U_0(t)\, A\, U_0^\dagger(t),$$
$$H^\times(A) = [H, A], \tag{14}$$

where A stands for any 2×2 matrix. Omitting the argument of the operators, and keeping in mind that $\delta_{\{N(t),n\}}\delta_{\{N(t),m\}} = 0$ when $n \neq m$, the time evolution in the Heisenberg representation is given by

$$U^\sharp(t) = \sum_{n=0}^{\infty} U_0^\sharp(t-T_n) V^\sharp U_0^\sharp(T_n-T_{n-1}) V^\sharp \ldots \ldots V^\sharp U_0^\sharp(T_1)\delta_{\{N(t),n\}}. \tag{15}$$

In each random state of the environment, the sum reduces to one term determined by the number of shocks $N(t)$ occurring. For each state of the heat bath, i.e., for each random sequence of shocks (for each initial state $\rho(0)$), the (random) expected value of the observable A is given by

$$\mathrm{Tr}\left[\rho(t)A\right] = \mathrm{Tr}\left[A\, U^\sharp(t)(\rho(0))\right] = \mathrm{Tr}\left[A\, U(t)\rho(0)U(t)^\dagger\right]. \tag{16}$$

We put $\rho(t) = U^\sharp\rho(0)$ for the random density matrix at time t. If we denote by $\langle\rho(t)\rangle_{av}$ the average value of $\rho(t)$ over all states of the heat bath, then the expected value of any observable is obtained by averaging (16) over all states of the heat bath (that is over all possible random sequences of shock times). Thus, this is equal to

$$\left\langle \mathrm{Tr}\left[\rho(t)A\right]\right\rangle_{ave} = \mathrm{Tr}\left[A\langle U^\sharp(t)\rangle_{ave}(\rho(0)\right]. \tag{17}$$

Let us assume $u(t) = \langle U^\sharp(t)\rangle_{ave}$ and use (15) and (5) to write $u(t)$ as

$$u(t) = \mathrm{e}^{-t\lambda}\sum_{n=0}^{\infty}\lambda^n \int_0^t \mathrm{d}t_n\, U_0^\sharp(t-t_n)\, V^\sharp \int_0^{t_n} \mathrm{d}t_{n-1}\, U_0^\sharp(t_n-t_{n-1})\, V^\sharp \int_0^{t_n-2} \ldots\, V^\sharp \int_0^{t_2} \mathrm{d}t_1\, U_0^\sharp(t_1) =$$
$$\mathrm{e}^{-t\lambda}\, U_0^\sharp + \int_0^t \mathrm{d}s\, \lambda\, \mathrm{e}^{-s\lambda}\, U_0^\sharp(s)\, V^\sharp u(t-s) = \mathrm{e}^{-t\lambda}\, U_0^\sharp + \int_0^t \mathrm{d}s\, \lambda\, \mathrm{e}^{-(t-s)\lambda}\, U_0^\sharp(t-s)\, V^\sharp u(s). \tag{18}$$

The expression shown in the first line of (18) is quite useful when (19) below can not be explicitly integrated. Obtaining the first line of (18) from the second requires iterating the latter and writing the result as an infinite sum. In the last expression in (18), the first term asserts that when no shocks occur up to time t (with probability $\mathrm{e}^{-t\lambda}$), the evolution is unperturbed. The second term follows from the application of the Markov property at the time of occurrence of the first shock, which happens with probability $\lambda\mathrm{e}^{-s\lambda}\mathrm{d}s$ in the interval $(s, s+\mathrm{d}s)$. Now apply $\mathrm{i}\hbar\partial/\partial t$ to the second term of the last expression of (18), we obtain

$$\mathrm{i}\hbar\frac{\partial u}{\partial t} = H^\times u + \mathrm{i}\hbar\,\lambda\big(V^\sharp - \mathbb{I}\big)u. \tag{19}$$

To see how this comes about, consider first

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\big(\mathrm{e}^{-t\lambda}\, U_0^\sharp\big) = -\mathrm{i}\lambda\mathrm{e}^{-t\lambda}\, U_0^\sharp + \mathrm{e}^{-t\lambda}\, H^\times U_0^\sharp. \tag{20}$$

Next, consider

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\int_0^t \mathrm{d}s\, \lambda\, \mathrm{e}^{-(t-s)\lambda}\, U_0^\sharp(t-s)\, V^\sharp u(s) =$$
$$\mathrm{i}\hbar\, V^\sharp u(t) + H^\times \int_0^t \mathrm{d}s\, \lambda\, \mathrm{e}^{-(t-s)\lambda}\, U_0^\sharp(t-s)\, V^\sharp u(s)$$
$$+\mathrm{i}\hbar\,\lambda \int_0^t \mathrm{d}s\, \lambda\, \mathrm{e}^{-(t-s)\lambda} U_0^\sharp(t-s) V^\sharp u(s). \tag{21}$$

Combining these two steps together, we obtain (19). This was mentioned but not derived in [4].

2.1. The integration of (19)

To avoid using so many subscripts, let us write $\rho(t) = u(t)(\rho(0)) = \langle U^\sharp(t)\rangle\rho(0)$ as a 2×2 matrix

$$\rho(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{22}$$

whose coefficients we have to determine by solving (19). This is a Hermitian matrix of trace 1. A simple computation shows that the actions of $H^\times$ and $V^\sharp$ on $\rho(t)$ are given by

$$H^\times\rho(t) = (\epsilon_1-\epsilon_2)\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix},$$
$$\big(V^\sharp - \mathbb{I}\big)\rho = \begin{pmatrix} d-a & c-b \\ b-c & a-d \end{pmatrix}. \tag{23}$$

Equating the terms on both sides of (13), we obtain two equation systems. The equation system for a and b is

$$\tfrac{\partial a}{\partial t} = \lambda(d-a), \qquad \tfrac{\partial d}{\partial t} = -\lambda(d-a). \tag{24}$$

In the equation system for c and b, we put $\omega = (\epsilon_1-\epsilon_2)/\hbar$. As above, we obtain

$$\tfrac{\partial b}{\partial t} = -\mathrm{i}\omega b - \lambda(b-c), \qquad \tfrac{\partial c}{\partial t} = \mathrm{i}\omega c + \lambda(b-c). \tag{25}$$

The perturbation matrix V is such that the dynamics diagonal and non-diagonal elements are separated. That system is easy to integrate. Note that by adding and subtracting the two equations, we obtain

$$\frac{\partial(a+d)}{\partial t} = 0 \quad \Longrightarrow \quad a(t)+d(t) = a(0)+d(0).$$
$$\frac{\partial(a-d)}{\partial t} = -2\lambda(a-d) \quad \Longrightarrow \quad \Longrightarrow \quad a(t)-d(t) = (a(0)-d(0))\,\mathrm{e}^{-2t\lambda}. \tag{26}$$

It is not hard to verify that by adding and subtracting, and differentiating the system (25) again, we obtain

$$\frac{\partial^2(b+c)}{\partial t^2} + 2\lambda\frac{\partial(b+c)}{\partial t} + \omega^2(b+c) = 0.$$
$$\frac{\partial^2(b-c)}{\partial t^2} + 2\lambda\frac{\partial(b-c)}{\partial t} + \omega^2(b-c) = 0. \tag{27}$$

The initial condition for the derivatives is obtained by evaluating (26) at $t=0$,

$$\frac{\partial(b+c)}{\partial t}(t=0) = -\mathrm{i}\omega(b-c)(0).$$
$$\frac{\partial(b-c)}{\partial t}(t=0) = -\mathrm{i}\omega(b+c)(0) - 2\lambda(b-c). \tag{28}$$

If we write $x(t)$ for the unknown in any of (28), then the standard procedure leads to

$$x(t) = \frac{\dot{x}(0)-r_-x(0)}{r_+-r_-}\,\mathrm{e}^{r_+t} + \frac{-\dot{x}(0)+r_+x(0)}{r_+-r_-}\,\mathrm{e}^{r_-t}, \tag{29}$$

where $r_\pm = -\lambda \pm \sqrt{\lambda^2-\omega^2}$ are the characteristic roots of the equations.

Now, undoing all the changes of variables, the explicit time dependence of $a(t)$, $b(t)$, $c(t)$, $d(t)$ turns out to be, respectively,

$$a(t) = \frac{a(0)}{2}\big(1+\mathrm{e}^{-2t\lambda}\big) + \frac{d(0)}{2}\big(1-\mathrm{e}^{-2t\lambda}\big),$$
$$d(t) = \frac{a(0)}{2}\big(1-\mathrm{e}^{-2t\lambda}\big) + \frac{d(0)}{2}\big(1+\mathrm{e}^{-2t\lambda}\big),$$
$$b(t) = \frac{\big[\dot{b}(0)-r_-b(0)\big]\,\mathrm{e}^{t\,r_+} + \big[-\dot{b}(0)+r_+b(0)\big]\,\mathrm{e}^{t\,r_-}}{r_+-r_-},$$
$$c(t) = \frac{\big(\big[\dot{c}(0)-r_-c(0)\big]\,\mathrm{e}^{t\,r_+} + \big[-\dot{c}(0)+r_+c(0)\big]\,\mathrm{e}^{t\,r_-}}{r_+-r_-}. \tag{30}$$

We have already mentioned that the initial conditions $\dot{b}(0)$ and $\dot{c}(0)$ are to be obtained from (25). Notice that regardless of the sign of the discriminant $\lambda^2-\omega^3$, the long-time behavior of the exponential function decays at the rate λ. Therefore, taking $t\to\infty$, we obtain that the non-diagonal terms of $\rho(t)$ asymptotically vanish, and since $a(0)+d(0)=1$, if the trace of $\rho(0)=1$, then

$$\rho(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \to \begin{pmatrix} \frac{(a(0)+d(0))}{2} & 0 \\ 0 & \frac{(a(0)+d(0))}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{31}$$

The last step follows from $a(0)+b(0)=1$. An interesting twist appears when we notice that in the passage from (19) and (23) to (30) no specific form of the initial matrix was used. We therefore proceed to integrate the system assuming that the initial conditions correspond to the Pauli matrices σ_i (for $i=1,2,3$). These are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{32}$$

We add $\sigma_0 = \mathbb{I}$ and, with an obvious abuse of notation, denote by $\sigma_\mu(t)$ (with $\mu = 0,1,2,3$) the result of adapting (30) to the case where the initial condition $\rho(0) = \sigma_m u$.

Applying (30) successively to each Pauli matrix, after some computations we obtain

$$\sigma_0(t) = \sigma_0(0) = \mathbb{I}, \tag{33}$$

$$\sigma_1(t) = \frac{\omega\big[\mathrm{e}^{t\,r_+}-\mathrm{e}^{t\,r_-}\big]}{r_+-r_-}\,\sigma_2 - \frac{\big[r_+\,\mathrm{e}^{t\,r_-}-r_-\,\mathrm{e}^{t\,r_+}\big]}{r_+-r_-}\,\sigma_1, \tag{34}$$

$$\sigma_2(t) = \frac{\omega\big[\mathrm{e}^{t\,r_-}-\mathrm{e}^{t\,r_+}\big]}{r_+-r_-}\,\sigma_1 + \frac{(r_++2\lambda)\,\mathrm{e}^{t\,r_-}}{r_+-r_-}\,\sigma_2 - \frac{(r_-+2\lambda)\,\mathrm{e}^{t\,r_+}}{r_+-r_-}\,\sigma_2, \tag{35}$$

$$\sigma_3(t) = \mathrm{e}^{-2\lambda t}\sigma_3. \tag{36}$$

The fact that the identity matrix is constant simply means that random evolution preserves unitarity when the average is taken over all states of the environment. These identities may prove useful when studying the time evolution of a state like (1) if we translate from the bra–ket notation to the Bloch representation $\rho = \frac{1}{2}(\mathbb{I} + \boldsymbol{r}\cdot\boldsymbol{\sigma})$ for pure states.

Of course, the same applies to (2) if the initial state of the two-qubit system is a classical correlation of non-pure states.

3. Examples of asymptotic decorrelation

We begin with a simple example of a separable but not pure state becoming asymptotically pure. After that, we consider the general state and examine under what conditions it becomes asymptotically uncorrelated, or perhaps pure.

3.1. Simple separable state

The following state is separable (in the Heisenberg representation)

$$\rho(0) = \frac{1}{2}\big(|1\rangle\langle 1| \otimes |2\rangle\langle 2| + |2\rangle\langle 2| \otimes |1\rangle\langle 1|\big), \tag{37}$$

In matrix form, the state is given by

$$\rho(0) = \frac{1}{2}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right). \tag{38}$$

The left factor of the tensor product stands for a density matrix of the first sub-system, and the right factor stands for a density matrix of the second sub-system. If we work out the tensor products, we obtain the 4×4 density matrix

$$\rho(0) = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & \\ 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{39}$$

This is the proper density matrix describing a separable, non-pure state of a composite system, and it is also invariant under exchange of the sub-systems, which is required for systems composed of identical sub-systems. Since $\mathrm{Tr}[\rho^2] = \frac{1}{2}$, this ρ is not a pure state. To examine its time evolution under the action of the random environment, let us first notice that

- $|1\rangle\langle 1|$ in the Bloch representations is $\frac{1}{2}(\mathbb{I}+\sigma_3)$;
- $|2\rangle\langle 2|$ in the Bloch representations is $\frac{1}{2}(\mathbb{I}-\sigma_3)$.

Therefore, (37) can be rewritten as

$$\rho(0) = \frac{1}{8}\Big((\mathbb{I}+\sigma_3)\otimes(\mathbb{I}-\sigma_3) + (\mathbb{I}-\sigma_3)\otimes(\mathbb{I}+\sigma_3)\Big). \tag{40}$$

As the perturbation acts independently on each qubit, invoking (36), we obtain that

$$\rho(t) = \frac{1}{8}\big(\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{I}\big) = \frac{1}{4}\mathbb{I}\otimes\mathbb{I}. \tag{41}$$

In this simple example, both states of the two-qubit system have the same probability of occurring.

3.2. General case

Here we work out the same example in two different ways, differing in the form used to express the initial state $\rho(0)$. The first way relies on the fact that $\sigma_0 \equiv \mathbb{I}$ and the Pauli matrices σ_i (for $i = 1, 2, 3$) form a basis for the space of 2×2-Hermitian matrices. The second way uses the representation of separable states introduced in (2). Furthermore, we assume that the two particles (or subsystems) are identical, and the representation should be invariant under exchange of particles (or labels). These two representations are

$$\rho(0) = \frac{1}{4}\sum_{\mu,\nu=0}^{3} C_{\mu,\nu}\sigma_\mu\otimes\sigma_\nu, \tag{42}$$

$$\rho(0) = \sum_{k=1}^{K} p_k\rho_{1,k}\otimes\rho_{2,k}. \tag{43}$$

The first representation stands for the expansion of an arbitrary density matrix in terms of the basis for the density matrices of the bipartite system. The second representation may be a separable state if $\rho_{i,k}$ is a pure state; otherwise $\rho(0)$ is a classically correlated state. Next, we examine each asymptotic state for each representation.

The factor $\frac{1}{4}$ has been introduced to set $C_{0,0} = 1$ henceforward. We use the assumption of symmetry with respect to exchange of particles to simplify the representation. In Sect. 1, we assumed that the environment acts independently on each particle. Therefore, each element in the tensor products evolves in time according to (33)–(36), and so the asymptotic form of the state (42) is

$$\rho(\infty) = \frac{1}{4}\big(\mathbb{I}\otimes\mathbb{I}\big) = \frac{1}{2}\mathbb{I}\otimes\frac{1}{2}\mathbb{I}. \tag{44}$$

To consider the representation (43), we again use the fact that (33)–(36) imply that any initial single qubit $\rho(0)$ as time evolves tends to $\mathbb{I}/2$. Therefore,

$$\rho(t) = \sum_{k=1}^{K} p_k\rho_{1,k}(t)\otimes\rho_{2,k}(t) \longrightarrow$$
$$\longrightarrow \rho(\infty) = \Big(\sum_{k=1}^{K} p_k\Big)\frac{1}{2}\mathbb{I}\otimes\frac{1}{2}\mathbb{I} = \frac{1}{2}\mathbb{I}\otimes\frac{1}{2}\mathbb{I}. \tag{45}$$

A side remark is the following. Suppose that the system is initially in a state like (43), and that the density matrices correspond to pure states, but the second component is isolated from the thermal bath. Write the pure states of the second component as $\rho_{2,k} = \mathbb{I} + \boldsymbol{n}_{2,k}\cdot\boldsymbol{\sigma} = \mathbb{I} + \sum_{j=1}^{3} n_{2,k}(j)\sigma_j$, with $||\boldsymbol{n}_{2,k}|| = 1$. Asymptotically, this state will be

$$\rho(t) \approx \frac{1}{2}\mathbb{I}\otimes\Big[\mathbb{I} + \Big(\sum_k \mu_k\boldsymbol{n}_{2,k}\Big)\cdot\boldsymbol{\sigma}\Big]. \tag{46}$$

The time dependence of the second factor is oscillatory for the coefficients σ_1 and σ_2 and constant for the coefficient of σ_3. The thermal bath destroys the initial correlation between the two subsystems. Note that an isolated system may not be in a pure state unless $||\sum_k \mu_k\,\boldsymbol{n}_{2,k}|| = 1$.

3.3. The entropy change due to thermalization

In the particular case of the first example, the entropy change due to thermalization is simple to compute. Using $S(\rho(t)) = -\mathrm{Tr}\big[\rho(t)\ln(\rho(t))\big]$, with the conventional $0\ln(0) = 0$, we see from (37) and (41) that $(\rho(0)) = \ln(2)$, whereas in the upper limit, $S\big(\rho(\infty)\big) = \ln(4) = 2\ln(2)$. Since a system in a thermal bath is far from isolated, we can interpret the increase in entropy as related to the work done by the environment to decorrelate the system.

4. Conclusions

The model developed above is simple enough to allow for a full analytical treatment. Its features contain basic ingredients that confirm the physical intuition of the situation, i.e., if the environment affects the qubits independently, the system evolves to a trivial equilibrium state.

A possible explanation for the fact that, regardless of the initial sate of the qubit, the two levels are equally populated in equilibrium, lies in the nature of the environment effect. All that the random shock does is to flip the microscopical state. So, in the long run, its effect is to evenly spread the initial probability among the two levels.

The model also applies to any number of non-interacting qubits, arbitrarily coupled through their initial state. Therefore, in a system of N identical qubits, in equilibrium the total number of qubits in the upper level follows the binomial distribution $B(N, 1/2)$.

Notice that the representation (42) applies to any density matrix for a two-qubit system, whether entangled or separable, thus in the Blume–Clauser model of thermal bath, the results in (44) or (45) provide a transparent way of understanding how a pair of entangled qubits becomes decorrelated under the action of the thermal bath.

Acknowledgments

I thank the reviewer and editors for their numerous comments, suggestions for improving the presentation of the results in this work, and a short list of extra references. I want to thank as well the copy-proofing team for further improving the presentation.

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