TWO-DIMENSIONAL ELECTRON GAS IN A PERIODIC POTENTIAL AND EXTERNAL MAGNETIC FIELD: STATES OF PAIRS AND THREE-PARTICLE SYSTEMS

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The group-theoretical classification of states of identical particle pairs is presented. Then obtained states are coupled with those of an antiparticle to construct states of a three-particle system. Investigations are performed using products of irreducible projective representations of the 2D translation group. For a given Born–von Kármán period N degeneracy of pair states is N, whereas three-particle states are N^2-fold degenerated. It has to be underlined that the case of even N is more complicated since pair states are labelled by four inequivalent irreducible projective representations. The problem of symmetry properties with respect to particles transposition is briefly discussed.

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1. Introduction

The quantum Hall effect and high temperature superconductivity have raised interest in properties of the two-dimensional electron gas subjected to electric and magnetic fields. The observation of (negatively) charged excitons [1] has recalled a forty-years old concept of excitons, "trions" or "charged excitons", introduced by Lampert in 1958 [2]. Recently, such excitons, consisting of two holes and an electron or two electrons and a hole (denoted $X^{\pm}$, respectively), have been investigated both experimentally and theoretically [3–5]. In the earlier paper [6] the trion states have been discussed; they have been classified using a direct coupling of three one-particle states and in a two-step procedure: (i) coupling of particle and antiparticle states to obtain states of an electrically neutral system and then (ii) coupling with hole (electron) states to determine states of a trion $X^{\pm}$. The aim of this work is to present classification of three-particle states (strictly speaking, states of particle–particle–antiparticle systems) when at first step states of a pair of identical (charged) particles are constructed. The antiparticle states are taken into account in the second step. Such an approach allows to discuss pair states, which may be important in considerations of high-T_c superconductors, where the Cooper pairs are confined to Cu–O planes.

The classification presented in this paper, like in the previous one [6], is based on translational symmetry in the presence of a periodic potential and an external, constant and homogeneous, magnetic field. To perform this task the so-called magnetic translation operators, introduced by Brown [7] and Zak [8], are used. These operators commute with the standard Hamiltonian of an electron in the magnetic field $\boldsymbol{H} = \nabla \times \boldsymbol{A}$ and a periodic potential $V(\boldsymbol{r})$:

$$\mathcal{H} = \frac{1}{2m}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2 + V(\boldsymbol{r}). \tag{1}$$

The Born–von Kármán (BvK) periodic conditions imposed on a system lead to a finite-dimensional projective representation of the 2D translation group formed by the magnetic translation operators. The Kronecker products of irreducible projective representations can be applied to describe multiparticle states [9, 10], which has been done lately for trions [6]. It seems that investigations of pairs should be much easier. However, trions, consisting of two electrons and a hole or two holes and an electron, have the total charge $\pm e$ and, therefore, behave in a similar way as a single electron and a hole, respectively. On the other hand, a pair of electrons (holes) have the charge $\mp 2e$ and one has to take into account parity of the Born–von Kármán period N. It means that cases of odd and even period should be considered separately.

Investigating problems, which involve the magnetic field $\boldsymbol{H}$ determined by the vector potential $\boldsymbol{A}$, one has to keep in mind that some results may depend on a chosen gauge, though physical properties should be gauge-independent. Two gauges are most frequently used in description of the 2D electron systems: the Landau gauge with $\boldsymbol{A} = [0, xH, 0]$ and the antisymmetric one with $\boldsymbol{A} = (\boldsymbol{H} \times \boldsymbol{r})/2$. The relations between these gauges was discussed in the earlier article [11], therefore this problem is left out in the present considerations. However, it should be underlined that a form of representation matrices depends on chosen gauge and, moreover, obtained representations are inequivalent, which means, among others, that their bases are not related by a unitary transformation. Since it is a symmetry adapted basis which results from the presented material, then it is important to stress that the Landau gauge is assumed and the obtained results cannot be immediately applied to other gauges. For the sake of simplicity the considerations are limited to this case and the presented results correspond to the limit of high magnetic fields, i.e. there is no Landau level mixing.

2. Pairs of identical particles

The movement of two-dimensional charged particles in a periodic potential and, like in the Hall effect, subjected to a magnetic field is described by the magnetic translation operators introduced by Brown [7] and Zak [8]. These operators form a projective representation of the two-dimensional translation group

$T = \{\boldsymbol{R} = [n_1, n_2] | n_1, n_2 \in \boldsymbol{Z}\}$ [8]; a series of articles on such representations were presented by Backhouse and Bradley [12]. Possible applications to multiparticle states were presented by the author in some earlier works [9, 13], where products of projective representations have been considered, but the irreducible basis has not been determined. The first attempt to determine trion states was based on two approaches: (i) a direct coupling of three one-particle states and (ii) construction of particle–antiparticle states and then coupling with one-particle state [6].

Investigating movement of charged particles in a magnetic field, one has to distinguish the periodic (Born–von Kármán) conditions related to the periodicity of a crystal lattice and the magnetically periodic conditions introduced by Brown [7, 11]. Let D be a projective representation of T with basis vectors labelled by $|w\rangle$. Then, in short, the magnetic period $N_{\rm mag}$ is determined by the equality $D(N_{\rm mag}\boldsymbol{R})|w\rangle = {\rm e}^{{\rm i}\varphi}|w\rangle$, whereas the Born–von Kármán period N requires ${\rm e}^{{\rm i}\varphi} = 1$. It means, for example, that standard (one-dimensional) irreducible vector representations $\Gamma^{k_1,k_2}[n_1, n_2]|w\rangle = \exp[2\pi{\rm i}(k_1 n_1 + k_2 n_2)/N]|w\rangle$ correspond to $N_{\rm mag} = 1$. To simplify the present considerations it is assumed that the magnetic field and the Born–von Kármán period N are such that single particle states are related to an N-dimensional projective representation (i.e. $N_{\rm mag} = N$):

$$D^l_{jk}[n_1, n_2] = \delta_{j,k-n_2}\omega_N^{ln_1 j}, \tag{2}$$

where l is mutually prime with $N, \omega_N = \exp(2\pi{\rm i}/N), j, k, n_1, n_2 = 0, 1, \ldots, N-1$ and all expressions are calculated modulo N [11]. This formula can be also expressed by the action of D^l on basis vectors $|s\rangle$, $s = 0, 1, 2, \ldots, N-1$:

$$D^l[n_1, n_2]|s\rangle = \omega_N^{ln_1(s-n_2)}|s - n_2\rangle. \tag{3}$$

It is assumed that a single particle in question is a hole (or any particle with the charge $q > 0$) but this assumption does not lead to any loss of generality.

2.1. Representations with odd dimension

The case of N odd is very simple. Since N is odd and l is mutually prime with N, then also $2l$ is mutually prime with N. Therefore (cf. [9, 10])

$$D^l \otimes D^l = ND^{2l}, \tag{4}$$

i.e. two-particle states are N-fold degenerated and related to the projective representation D^{2l}. One of the possible forms of the irreducible basis is

$$|s\rangle_2^r = |s + r\rangle|s - r\rangle, \qquad s, r = 0, 1, 2, \ldots, N-1, \tag{5}$$

where the subscript "2" indicates states of a pair and r is the repetition index. For the sake of simplicity, here and thereafter the symbol "$\otimes$" is omitted in the tensor product of vectors. This result can be easily verified using Eq. (3):

$$(D^l \otimes D^l)[n_1, n_2]|s\rangle_2^r = D^l[n_1, n_2]|s + r\rangle D^l[n_1, n_2]|s - r\rangle$$

$$= \omega_N^{2ln_1(s-n_2)}|s + r - n_2\rangle|s - r - n_2\rangle = D^{2l}[n_1, n_2]|s\rangle_2^r.$$

2.2. Representations with even dimension

The case $N = 2M$ leads to the product of representations in the following form [9]:

$$D^l \otimes D^l = M(D^{l;00} \oplus D^{l;10} \oplus D^{l;01} \oplus D^{l;11}), \tag{6}$$

where each of four representations on the right-hand side is M-dimensional and they form a complete set of inequivalent irreducible projective representations of $\boldsymbol{Z}_N \otimes \boldsymbol{Z}_N$ (or, in the other words, the magnetic translation operators with $N_{\rm mag} = M$ and the Born–von Kármán period N). The first representation, labelled by "00", is analogous to that given by Eqs. (2) and (3)

$$D^l_{jk}[n_1, n_2] = \delta_{j,k-\eta_2}\omega_M^{l\eta_1 j}, \tag{7}$$

$$D^l[n_1, n_2]|s\rangle = \omega_M^{l\eta_1(s-\eta_2)}|s-\eta_2\rangle, \tag{8}$$

where $s = 0, 1, 2, \ldots, M-1$ now and $\eta_i = n_i \bmod M,\ i = 1, 2$. All representations can be expressed by a general formula [12]

$$D^{l;k_1k_2}[n_1, n_2]|s\rangle = \omega_N^{ln_1(2s-2\eta_2+k_1)}(-1)^\alpha|s-\eta_2\rangle, \tag{9}$$

where $k_1, k_2 = 0, 1$ and $\alpha = 1$ for $1 \le n_2 - s \le M$ and $\alpha = 0$ otherwise. The irreducible bases can be chosen in the following way (vectors are not normalized throughout this paper):

$$|s\rangle_2^{r;00} = |s+r\rangle|s-r\rangle + |M+s+r\rangle|M+s-r\rangle, \tag{10}$$

$$|s\rangle_2^{r;10} = |s+r\rangle|s-r+1\rangle + |M+s+r\rangle|M+s-r+1\rangle, \tag{11}$$

$$|s\rangle_2^{r;01} = |s+r\rangle|s-r\rangle - |M+s+r\rangle|M+s-r\rangle, \tag{12}$$

$$|s\rangle_2^{r;11} = |s+r\rangle|s-r+1\rangle - |M+s+r\rangle|M+s-r+1\rangle, \tag{13}$$

where $s, r = 0, 1, 2, \ldots, M-1$. These formulae can be verified by simple calculations.

3. Trion states

In the presented approach trion states are constructed from states of a pair, discussed in the previous section, and antiparticle states related to the N-dimensional representation D^{-l}. Therefore, we start from investigating the products $D^{2l} \otimes D^{-l}$ and $D^{l;k_1k_2} \otimes D^{-l}$, for odd and even N, respectively. In both cases, the resultant representation is a multiplicity of D^l [6], namely

$$D^{2l} \otimes D^{-l} = ND^l, \tag{14}$$

$$D^{l;k_1k_2} \otimes D^{-l} = MD^l. \tag{15}$$

The first case (N odd) is again very simple. Let $|w\rangle_t$ denote a trion state and $|u\rangle_-$ — a state of an antiparticle, then one of possible choices of basis vectors is as follows:

$$|w\rangle_t^v = |w+v\rangle_2|w+2v\rangle_-, \qquad w = 0, 1, 2, \ldots, N-1,$$

where $v = 0, 1, 2, \ldots, N-1$ is the repetition index. Since pair states have additional repetition index r, then the trion states in this case are labelled by $w, r, v = 0, 1, 2, \ldots, N-1$, where w is a vector index and r, v are repetition indices. Taking into account Eq. (5), one obtains

$$|w\rangle_t^{r,v} = |w+v\rangle_2^r |w+2v\rangle_- = |w+v+r\rangle |w+v-r\rangle |w+2v\rangle_- . \tag{16}$$

The states

$$|w\rangle_t^{pq} = |w+p\rangle |w+q\rangle |w+p+q\rangle_- \tag{17}$$

obtained previously [6] correspond to $p = v+r$ and $q = v-r$ (calculated mod N, of course). Note that for $N = 2M$ this relations cannot be inverted since $v = (p+q)/2$ and $r = (p-q)/2$ have no solutions mod N for odd $p+q$ and $p-q$, respectively.

In the case $N = 2M$, the considerations are a bit more difficult but since one knows the final results, states $|w\rangle^{pq}$, then they may serve as a useful hint. There are $N^2 = 4M^2$ different bases labelled by $p, q = 0, 1, 2, \ldots, N-1$, however if $p' = p + M$ and $q' = q + M$ then states $|w\rangle^{pq}$ and $|w\rangle^{p'q'}$ have the same third element in the tensor products because (see Eq. (17))

$$|w\rangle^{p'q'} = |w+p+M\rangle |w+q+M\rangle |w+p+q\rangle_- . \tag{18}$$

Therefore, they can be gathered into $2M^2 = NM$ pairs

$$|w\rangle^{pq} \quad \text{and} \quad |w\rangle^{(p+M)(q+M)}.$$

The ranges of the indices have to be chosen in such a way that pairs pq and $(p+M)(q+M)$ run over two separate sets. This problem will be discussed below in more detail. For each pair p, q one can form two new bases, labelled by "+" and "−", respectively,

$$|w\rangle^{pq\pm} = |w\rangle^{pq} \pm |w\rangle^{(p+M)(q+M)}, \tag{19}$$

therefore

$$|w\rangle^{pq\pm} = (|w+p\rangle |w+q\rangle \pm |w+p+M\rangle |w+q+M\rangle) |w+p+q\rangle_- . \tag{20}$$

These vectors have the first part (in parentheses) in the form resembling Eqs. (10)–(13). Therefore, one has to relate the repetition index r and a label u of a vector $|u\rangle_-$ of the representation D^{-l} with the repetition indices p, q and $\pm$ in Eq. (20). Moreover, due to the relation (15) the repetition index $v = 0, 1, \ldots, M-1$ has to be taken into account. The similarity of Eq. (20) and Eqs. (10)–(13) suggests that the irreducible basis of the product $D^{l;k_1 k_2} \otimes D^{-l}$ can be chosen as tensor products $|s\rangle_2 |u\rangle_-$. The simplest to solve is the case of representations with $k_2 = 0$, i.e. $D^{l;k_1 0}$:

$$|w\rangle_t^v = |s+v\rangle_2^{00} |w+2v\rangle_- , \tag{21}$$

$$|w\rangle_t^v = |s+v\rangle_2^{10} |w+2v+1\rangle_- , \tag{22}$$

with $s = w$ mod M; $s + v$ is calculated mod M, whereas $w + 2v$ and $w + 2v + 1$ are calculated mod N. For example, when $N = 6$ and $l = 1$ one obtain bases in the following form:

$$\{|0\rangle_2^{00}|0\rangle_-, |1\rangle_2^{00}|1\rangle, |2\rangle_2^{00}|2\rangle_-, |0\rangle_2^{00}|3\rangle, |1\rangle_2^{00}|4\rangle_-, |2\rangle_2^{00}|5\rangle\},$$

$$\{|1\rangle_2^{00}|2\rangle_-, |2\rangle_2^{00}|3\rangle, |0\rangle_2^{00}|4\rangle_-, |1\rangle_2^{00}|5\rangle, |2\rangle_2^{00}|0\rangle_-, |0\rangle_2^{00}|1\rangle\},$$

$$\{|2\rangle_2^{00}|4\rangle_-, |0\rangle_2^{00}|5\rangle, |1\rangle_2^{00}|0\rangle_-, |2\rangle_2^{00}|1\rangle, |0\rangle_2^{00}|2\rangle_-, |1\rangle_2^{00}|3\rangle\},$$

$$\{|0\rangle_2^{10}|1\rangle_-, |1\rangle_2^{10}|2\rangle, |2\rangle_2^{10}|3\rangle_-, |0\rangle_2^{10}|4\rangle, |1\rangle_2^{10}|5\rangle_-, |2\rangle_2^{10}|0\rangle\},$$

$$\{|1\rangle_2^{10}|3\rangle_-, |2\rangle_2^{10}|4\rangle, |0\rangle_2^{10}|5\rangle_-, |1\rangle_2^{10}|0\rangle, |2\rangle_2^{10}|1\rangle_-, |0\rangle_2^{10}|2\rangle\},$$

$$\{|2\rangle_2^{10}|5\rangle_-, |0\rangle_2^{10}|0\rangle, |1\rangle_2^{10}|1\rangle_-, |2\rangle_2^{10}|2\rangle, |0\rangle_2^{10}|3\rangle_-, |1\rangle_2^{10}|4\rangle\}.$$

These results can be used to determine the set over which the pairs of indices pq run. However, in each case one has two possibilities, pq and $(p+M)(q+M)$, and the appropriate choice can be done after considering representations $D^{l;k_1 1}$. Equations (10) and (12) with formulae (21) and (22) give us bases $|w\rangle_t^{r,v;00}$ and $|w\rangle_t^{r,v;10}$, respectively (as above $s = w \bmod M$)

$$(|s+r+v\rangle|s-r+v\rangle + |M+s+r+v\rangle|M+s-r+v\rangle)|w+2v\rangle_-, \tag{23}$$

$$(|s+r+v\rangle|s-r+v+1\rangle$$
$$+|M+s+r+v\rangle|M+s-r+v+1\rangle)|w+2v+1\rangle_-. \tag{24}$$

Therefore, the admissible values of indices (pq) are $(v+r)(v-r)$ and $(v+r)(v+1-r)$ for $v, r = 0, 1, \ldots, M-1$ and all values are calculated mod N. In the considered example $N = 6$ one obtains the following 18 pairs: "00", "11", "22", "15", "20", "31", "24", "35", "40", "01", "12", "23", "10", "21", "32", "25", "30", "41". In a general case the two following arrays can be constructed:

v \ r	0	1	2	...	$M-1$
0	00	$1(N-1)$	$2(N-2)$	...	$(M-1)(M+1)$
1	11	20	$3(N-1)$	...	$M(M+2)$
2	22	31	40	...	$(M+1)(M+3)$
...	...	...	...	...	
$M-1$	$(M-1)(M-1)$	$M(M-2)$	$(M+1)(M-3)$	...	$(N-2)0$

v \ r	0	1	2	...	$M-1$
0	01	10	$2(N-1)$	...	$(M-1)(M+2)$
1	12	21	30	...	$M(M+3)$
2	23	32	41	...	$(M+1)(M+4)$
...	...	...	...	...	
$M-1$	$(M-1)M$	$M(M-1)$	$(M+1)(M-2)$	...	$(N-2)1$

It can be easily observed that the index p has $N-1$ values $0, 1, \ldots, N-2$ and the index q has the same $p+1$ values for p and $p' = N-2-p$. These values are $N-p,\ N-p+2,\ \ldots,\ p$ in the first case and $N-p+1,\ N-p+3, \ldots, p+1$ for the representation $D^{l;10}$.

To complete the discussion one has to consider representations $D^{l;k_1 1}$. The irreducible bases of products $D^{l;k_1 1} \otimes D^{-l}$ have a form similar to those given by Eqs. (21) and (22):

$$|w\rangle_t^v = (-1)^\alpha |s+v\rangle_2^{01} |w+2v\rangle_-, \tag{25}$$

$$|w\rangle_t^v = (-1)^\alpha |s+v\rangle_2^{11} |w+2v+1\rangle_-, \tag{26}$$

where α is an integer part of $(w+v)/M$. Since $w+v = 0,1,2,\ldots,N+M-2$, then $(-1)^\alpha = -1$ for $w+v = M,\ M+1,\ \ldots,\ N-1$. The same example as in the previous case ($N=6,\ l=1$) leads to the following bases:

$$\{|0\rangle_2^{01}|0\rangle_-, |1\rangle_2^{01}|1\rangle, |2\rangle_2^{01}|2\rangle_-, -|0\rangle_2^{01}|3\rangle, -|1\rangle_2^{01}|4\rangle_-, -|2\rangle_2^{01}|5\rangle\},$$

$$\{|1\rangle_2^{01}|2\rangle_-, |2\rangle_2^{01}|3\rangle, -|0\rangle_2^{01}|4\rangle_-, -|1\rangle_2^{01}|5\rangle, -|2\rangle_2^{01}|0\rangle_-, |0\rangle_2^{01}|1\rangle\},$$

$$\{|2\rangle_2^{01}|4\rangle_-, -|0\rangle_2^{01}|5\rangle, -|1\rangle_2^{01}|0\rangle_-, -|2\rangle_2^{01}|1\rangle, |0\rangle_2^{01}|2\rangle_-, |1\rangle_2^{01}|3\rangle\},$$

$$\{|0\rangle_2^{11}|1\rangle_-, |1\rangle_2^{11}|2\rangle, |2\rangle_2^{11}|3\rangle_-, -|0\rangle_2^{11}|4\rangle, -|1\rangle_2^{11}|5\rangle_-, -|2\rangle_2^{11}|0\rangle\},$$

$$\{|1\rangle_2^{11}|3\rangle_-, |2\rangle_2^{11}|4\rangle, -|0\rangle_2^{11}|5\rangle_-, -|1\rangle_2^{11}|0\rangle, -|2\rangle_2^{11}|1\rangle_-, |0\rangle_2^{11}|2\rangle\},$$

$$\{|2\rangle_2^{11}|5\rangle_-, -|0\rangle_2^{11}|0\rangle, -|1\rangle_2^{11}|1\rangle_-, -|2\rangle_2^{11}|2\rangle, |0\rangle_2^{11}|3\rangle_-, |1\rangle_2^{11}|4\rangle\}.$$

Formulae (21) and (22) have to be modified accordingly. However, the elements $|v+r\rangle|v-r\rangle$ and $|v+r\rangle|v-r+1\rangle$, corresponding to $w=s=0$, always have coefficient $+1$, since $v < M$ and $w = 0$ when one calculates α in Eqs. (25) and (26). Therefore, the set of indices pq determined in the previous cases does not need to be changed. If one wants to replace a pair pq by its counterpart $(p+M)(q+M)$, for example 35 by 02 in the considered example, then the sign in the corresponding formula, (25) or (26), has to be changed, i.e. α is replaced by $\alpha+1$.

4. Symmetrization of states

In the previous section the relation between the basis $|w\rangle_t^{pq}$, determined in [6] by a direct evaluation of a product of three representations $D^l \otimes D^l \otimes D^{-l}$, and the basis $|w\rangle_t^{r,v;k_1k_2}$, presented in this paper, has been established. Since the symmetrization of the basis $|w\rangle_t^{pq}$ and its relation with the basis related to the product $D^l \otimes (D^l \otimes D^{-l})$ are considered in the earlier paper [6], then the problem stated in the title of this section may be left out. However, having at hand the basis $|s\rangle_2$ for pairs of identical particles it is natural and obvious to investigate the symmetry properties related to the transposition of particles.

As in all above problems the case of odd $N = 2M+1$ is quite easy. It follows from Eq. (5) that $r=0$ leads to symmetric states

$$|0\rangle|0\rangle, |1\rangle|1\rangle, \ldots, |N-1\rangle|N-1\rangle.$$

The other $N^2 - N$ vectors are grouped in $N-1 = 2M$ representations labelled by $r = 1,\ldots,N-1$. To construct symmetric (antisymmetric) states it is enough to take combinations

$$|s\rangle_2^{r\pm} = |s+r\rangle|s-r\rangle \pm |s-r\rangle|s+r\rangle, \tag{27}$$

where $r = 1, 2, \ldots, M$ now.

In the case of even $N = 2M$ the product $D^l \otimes D^l$ decomposes into M copies of four M-dimensional projective representations. Since the symmetrization of states has a slightly different form in each of these cases, then they are considered separately.

The non-symmetrized basis of the representation $D^{l;00}$ is given by Eq. (10) as

$$|s\rangle_2^{r;00} = |s+r\rangle|s-r\rangle + |M+s+r\rangle|M+s-r\rangle.$$

The case $r = 0$ again gives M symmetric states. To consider the other $M-1$ representations one has to check the parity of M. If $M = 2\mu$, then for $r = \mu$ we have $M-r = r$ and $M+r = N-r$, therefore in this case all vectors are symmetric. The other vectors form symmetric and antisymmetric states in the standard way and it also concerns the case of odd M.

The "additional" symmetric state found above for $N = 4\mu$ is "lost" considering $D^{l;10}$. Its irreducible basis (11):

$$|s\rangle_2^{r;10} = |s+r\rangle|s-r+1\rangle + |M+s+r\rangle|M+s-r+1\rangle$$

has no symmetric states for $M = 2\mu$, but for odd $M = 2\mu - 1$ and $r = \mu$ the above formula reads

$$|s\rangle_2^{\mu;10} = |s+\mu\rangle|s-\mu+1\rangle + |s-\mu+1\rangle|s+\mu\rangle, \tag{28}$$

therefore these states are symmetric.

The similar considerations have to be performed for representations $D^{l;k_1 1}$ with special attention to the fact that

$$|s-r\rangle|s+r\rangle - |M+s-r\rangle|M+s+r\rangle$$
$$= -(|s+r'\rangle|s-r'\rangle + |M+s+r'\rangle|M+s-r'\rangle),$$

where $r = 0, 1, \ldots, M-1$ and $r'' = M - r$.

5. Final remarks

The presented considerations have shown that the parity of the BvK period strongly influences the obtained results, whereas free trions behave in a similar way as free electrons or holes and, therefore, their states do not depend on the parity. Of course, coupling of three (in general, n) identical particles will lead to problems when $N = 3M$ ($N = nM$, in general case). It should be stressed that the degeneracy of the obtained states is very high and there are many possibilities to construct states $|s\rangle_2$ of pairs and $|w\rangle_t$ of three-particle systems. Therefore we have to keep in mind that only a few possible choices of the irreducible bases have been presented in this paper. For example, formulae (5)–(13) and following them Eqs. (16), (23), (24) are asymmetric with respect to the transposition of identical particles. Symmetrization of such states has been briefly discussed in Sec. 4. In these simplified considerations there are no interactions between trions or the Landau level mixing and, moreover, the spin or angular momentum numbers. Taking into account spins would allow a construction of states completely antisymmetric with respect to the permutational symmetry. Such a problem has been discussed lately by Dzyubenko et al. [5] in the absence of a periodic potential $V(\boldsymbol{r})$, so there is no discrete translational symmetry. The relations for the total angular moment projections he obtained are the same as those presented in this paper for indices of vectors (taking into account the sign of charges). For example, in Eq. (16) one obtains

$$w = (w + v + r) + (w + v - r) - (w + 2v).$$

It is interesting that Dzyubenko et al. obtained their results in the antisymmetric gauge $\boldsymbol{A} = (\boldsymbol{H} \times \boldsymbol{r})/2$, whereas in the presented considerations the Landau gauge has been used. It confirms that the physical properties are gauge-independent. On the other hand, the actual form of wave functions is not discussed here, but only the relations between representations and their product are taken into account. These relations are independent of the matrix representations and, similarly, the form of resultant basis is independent of the function form: for a given BvK period N and any gauge irreducible projective representations are N-dimensional and their action on basis vectors are similar (up to a factor system) [7, 8, 11, 14].

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