BRAIDS FOR PRETZEL KNOTS

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The words of braid closures for the pretzel knots, in particular the twist knots relevant to molecular supercoils in biochemistry, are recorded in a standardized form, which enables one to see the regular pattern of the words, and thus to write the braid words representing pretzel knots for general values of the crossing number.

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1. Introduction

The braid closures are useful for representing knots [1–4]. In the published tabulation of Jones [2], of braid words for braid closures representing all knots with up to 10 crossings, the braid word expressions have not been optimized yet. While no method has been proposed so far [1, 2] to find the most economic presentation of braids for knots, there appears a possibility to standardize systematically [5, 6] braid words for the simple knots.

2. The torus-, the twist-, and the pretzel-knots

We refer to the classical knot designation and to the link diagrams of Rolfsen [7]. We define the braid generator b_n so that it produces a positive crossing [2, 8, 9]. As in [2], in a braid word, for the braid generator b_n, a factor n will be written, and n^p for $(b_n)^p$ with an integral power exponent p.

If the natural numbers N and p are relatively prime, the braid word

$$w(N,p) = (123\cdots(N-1))^p, \qquad\qquad N, p, \qquad\qquad (2.1)$$

represents a torus knot [2, 3, 10–13]. The $(C, 2)$ torus knots [7, 10–14] have the number of crossings, or nodes, C, odd. If $C = 2n + 1$, with natural n, the braid word for the $(C, 2)$ torus knot with positive crossings is [2, 3, 11, 12]:

$$w(C_1) = w((2n+1)_1) = 1^{2n+1}, \qquad\qquad 2n + 1. \qquad\qquad (2.2)$$

The next simple knots are the twist knots which have a plectonemic region with two string segments interwoven as in a braid of $C - 2$ crossings, and are locked by an interlock with two crossings [7, 14,15].

A pretzel knot [12, 16–18] is composed of a row of $|b| + m$ parallel two-string braids. In the pretzel knot notation, $P(b; q_1, \ldots, q_m)$, the integer b separated by a semicolon gives the signed number of the one-crossing braids, the integers separated by a comma give the signed number of crossings in the braids [12, 17, 18].

The braid closure words, $w(C_i)$, for twist knots, and several types of pretzel knots, will be written in a standardized form by the use [2, 3, 5, 8, 9, 19–21] of blocks $A(n)$ and $D(n)$, with natural $n > 1$:

$$A(n) = 23 \cdots (n-2)(n-1)n,$$

$$D(n) = n(n-1)(n-2) \cdots 321. \tag{2.3}$$

The Conway notation [14], based on the concept of tangles and indicated in the Rolfsen [7] table of link diagrams, will follow, after a comma, the braid word for each knot, without specifying the sign of the crossings.

3. Knots with an odd number of crossings

In the classical knot labeling [7], when the crossing number C is odd, the $(C, 2)$ torus knot is labeled C_1. The first type of twist knot, labeled C_2, has a plectonemic region interwoven as in the $(C-2, 2)$ torus knot with $C-2$ crossings of the same sign, and is locked by an interlock with two crossings [7, 10–12, 15, 22–24]. A knot having an interlock with b one-crossing braids [7, 12, 14, 17, 18] is a pretzel knot $P(b; C-|b|)$, as seen in Fig. 1. The second type of twist knot which is locked by an interlock with three crossings [7] is a pretzel knot $P(3; C-3)$ and is labeled C_3. Knots with the crossing index $C = 11$ are labeled here by the subscripts used by Conway [14] and by Perko [22]. Knots with larger crossing indices will be labeled by numbers of the Knotscape program [25].
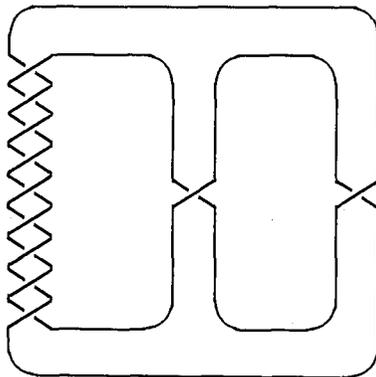

Fig. 1.   The knot 11_5 having an interlock with two crossings [12, 17, 18, 22], presented as a pretzel knot $P(2; 9)$.

For the twist knots C_2 with an odd crossing number $C = 2n + 1$, the braid words [2] can be written, by induction on the number of crossings, for natural $n > 1$, as:

$$w(C_2) = w((2n+1)_2) = 1A(n)nD(n)(A(n))^{-1}, \qquad (2n-1)\ 2. \qquad (3.1)$$

Since the length of each of the blocks $A(n)$ and $D(n)$ increases by one when n increases by one, an increment of the crossing number C by two increases the length of the braid word [2, 3] by three.

The braids for the second type of twist knots C_3 with the odd crossing number C are 3-braids, i.e. have the braid index $N = 3$. The braid words [2, 3], for natural $n > 2$, are:

$$w(C_3) = w((2n+1)_3) = 1^{2n-2}2^212^{-1}, \qquad (2n-2)\ 3. \qquad (3.2)$$

The braid words for knots 7_3 and 9_3 conform in crossing signs to the respective knot diagrams of Rolfsen [7, 10]. Knots represented by 3-braids [1, 5, 6] have been illustrated also by diagrams of Akutsu et al. [3].

The braid words for the particular pretzel knots, see Fig. 1, can be inferred from the table of Jones [2] by induction. For a natural number k we have:

$$w(P(2k;\ 5)) = 1A(3)32^{2k}1(A(3))^{-1}, \qquad\qquad 2k\ 5,$$

$$w(P(2k;\ 7)) = 1A(4)432^{2k}1(A(4))^{-1}, \qquad\qquad 2k\ 7,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.3)$$

$$w(P(2k;\ 9)) = 1A(5)5432^{2k}1(A(5))^{-1}, \qquad\qquad 2k\ 9,$$

$$w(P(2k;\ 11)) = 1A(6)65432^{2k}1(A(6))^{-1}, \qquad\qquad 2k\ 11.$$

For the pretzel knots with odd crossing numbers and with one one-crossing braid the braid words can be expressed:

$$w(P(2n-3,1,3))$$
$$= 1A(n-1)(n-1)nD(n)(A(n))^{-1}, \qquad\qquad 2n-3\ 1\ 3,$$

$$w(P(2n-5,1,5)) \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.4)$$
$$= 1A(n-2)(n-2)(n-1)nD(n)(A(n))^{-1}, \qquad\qquad 2n-5\ 1\ 5,$$

$$w(P(2n-7,1,7))$$
$$= 1A(n-3)(n-3)(n-2)(n-1)nD(n)(A(n))^{-1}, \qquad 2n-7\ 1\ 7,$$

for n such that each braid generator number is > 1.

At partition of the crossing number C into more than three integers, it is convenient to use a natural number k, and to record

$$w(P(2k,3,3,3)) = 1^42^31^{-1}32^33^{2k-1}, \qquad\qquad 2k,3,3,3,$$

$$w(P(2k+1,2,3,3)) = 1^42^{2k+1}1^{-1}32^33, \qquad\qquad 2k+1,2,3,3,$$

$$w(P(2k+1,3,2,3)) = 1^{2k+2}2^31^{-1}32^33, \qquad\qquad 2k+1,3,2,3, \qquad (3.5)$$

$$w(P(2k+1,4,3,3)) = 1^42^{2k+1}1^{-1}3^32^33, \qquad\qquad 2k+1,4,3,3,$$

$$w(P(2k+1,3,4,3)) = 1^{2k+2}2^31^{-1}3^32^33, \qquad\qquad 2k+1,3,4,3.$$

Examples of braid closure words for the pretzel knots with three-crossing braids can be written:

$$w(P(3,3,3)) = 1^2 21^{-1} 32^{-1} 3243^{-1} 23^2 4^{-1},$$

$$= 123^2 1^{-1} 23^{-1} 43^{-1} 213^2 4^{-1}, \qquad\qquad 3,3,3,$$

$$w(P(5,3,3)) = 1234^2 34^{-1} 54^{-1} 3214^2 2^{-1} 3^{-1} 5^{-1}, \qquad 5,3,3,$$

$$w(P(7,3,3)) = 12345^2 45^{-1} 65^{-1} 43215^2 2^{-1} 3^{-1} 4^{-1} 6^{-1}, \qquad 7,3,3,$$

$$(3.6)$$

$$w(P(9,3,3)) = 123456^2 56^{-1} 76^{-1} 543216^2 2^{-1} 3^{-1} 4^{-1} 5^{-1} 7^{-1}, \qquad 9,3,3,$$

$$w(P(7,3,5)) = 1^2 21^{-1} 32^{-1} 45^{-1} 46^2 54^{-1} 6^{-1} 76^{-1} 54326^2 54^{-1} 7^{-1},$$

$$= 12345^{-1} 46^2 54^{-1} 6^{-1} 76^{-1} 543216^2 52^{-1} 3^{-1} 4^{-1} 7^{-1}, \qquad 7,3,5,$$

$$w(P(3,3,3,3,3))$$

$$= 1^2 2 32^{-2} 343^{-2} 5^2 45^{-1} 65^{-1} 43215^2 432^{-1} 6^{-1}, \qquad 3,3,3,3,3.$$

The pretzel knot $P(7,3,5)$, presented in Fig. 2, is a non-invertible knot with the lowest possible values of the indices [16, 26].

15.85213

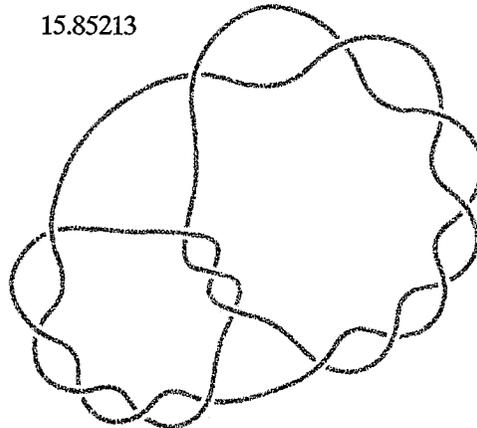Fig. 2.   The non-invertible [16, 26] knot $P(7,3,5)$.

Braid words for the pretzel knots with the five-crossing braids can be written:

$$w(P(5,5,3))$$

$$= 1231^{-1} 4^{-1} 35^2 43^{-1} 5^{-1} 65^{-1} 43215^2 43^{-1} 6^{-1}, \qquad 5,5,3,$$

$$(3.7)$$

$$w(P(5,5,5))$$

$$= 1234^{-1} 35^{-1} 4^{-1} 6^2 54^2 3214^{-1} 6^{-1} 76^{-1} 56^2 542^{-1} 3^{-1} 7^{-1}, \qquad 5,5,5.$$

4. Knots with an even number of crossings

If the crossing number C of knot is even, the first type of twist knot which has a plectonemic region interwoven as in a torus knot with $C - 2$ crossings, and is locked by an interlock with two crossings of the other sign [7, 15, 23, 24], is a pretzel knot $P(2; C - 2)$, and is labeled C_1. The second type of twist knot, locked by an interlock with three crossings [7], is a $P(C - 3, 1, 2)$ knot, and is labeled C_2. The third type of twist knot, labeled C_3, locked by an interlock with four crossings [7], is a $P(4; C - 4)$ knot, see Fig. 3.
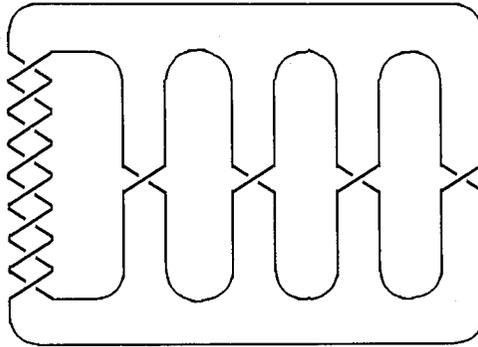

Fig. 3. The knot $P(4; 8)$ has Knotscape No. 12.1166.

The braid words [2] for the twist knots C_1, with an even crossing number $C = 2n$, can be written, by induction on the number of crossings, for natural $n > 2$, as:

$$w(C_1) = w((2n)_1)$$

$$= 1A(n-1)n^{-1}D(n-1)(A(n-1))^{-1}n^{-1}, \qquad (2n-2)\ 2. \qquad (4.1)$$

An increment of the crossing number C by two increases the length of the braid word [2, 3] by three.

The braids for the second type of twist knots C_2, with the even crossing number C, are 3-braids. The braid words [2, 3], for natural $n > 2$, are:

$$w(C_2) = w((2n)_2) = 1^{2n-3}2^{-1}12^{-1}, \qquad (2n-3)\ 1\ 2. \ (4.2)$$

The braid words [2] for the third type of twist knots C_3, with the even crossing number C, can be written, by induction on the number of crossings, for natural $n > 3$, as:

$$w(C_3) = w((2n)_3)$$

$$= 1A(n-1)n^{-1}(n-1)^{-1}D(n-2)(A(n-1))^{-1}n^{-1}, \qquad (2n-4)\ 4. \qquad (4.3)$$

An increment of the crossing number C by two increases the length of the braid word [2, 3] by three.

For the pretzel knots with even crossing numbers and with one one-crossing braid the braid words can be expressed:

$$w(P(2n-4,1,3))$$

$$= 1A(n-2)(n-1)^{-1}D(n-2)(n-1)^{-2}(A(n-1))^{-1}, \quad 2n-4 \; 1 \; 3,$$

$$w(P(2n-6,1,5))$$

$$= 1A(n-3)(n-2)^{-1}D(n-3)(n-2)^{-4}(A(n-2))^{-1}, \quad 2n-6 \; 1 \; 5,$$

$$w(P(2n-8,1,7))$$

$$= 1A(n-4)(n-3)^{-1}D(n-4)(n-3)^{-6}(A(n-3))^{-1}, \quad 2n-8 \; 1 \; 7,$$

(4.4)

for n such that each braid generator number is > 1.

At partition of the crossing number C into three, and more integers, it is convenient to use natural numbers k, l, and to record the words:

$$w(P(2k+1,2l+1,2))$$

$$= 1^{2k+1}2^{-1}1^{2l+1}2^{-1}, \qquad\qquad 2k+1, 2l+1, 2,$$

$$w(P(2k+1,2l+1,4))$$

$$= 1^{2k+1}2^{-1}1^{2l+1}32^{-1}3^{-2}, \qquad\qquad 2k+1, 2l+1, 4,$$

$$w(P(2k+1,2l+1,6))$$

$$= 1^{2k+1}2^{-1}1^{2l+1}432^{-1}3^{-1}4^{-2}3^{-1}, \qquad 2k+1, 2l+1, 6,$$

(4.5)

$$w(P(2k+1,2l+1,8))$$

$$= 1^{2k+1}2^{-1}1^{2l+1}5432^{-1}3^{-1}4^{-1}5^{-2}4^{-1}3^{-1}, \quad 2k+1, 2l+1, 8,$$

$$w(P(2k+1,2l+1,10))$$

$$= 1^{2k+1}2^{-1}1^{2l+1}65432^{-1}3^{-1}4^{-1}5^{-1}6^{-2}5^{-1}4^{-1}3^{-1}, \qquad 2k+1, 2l+1, 10.$$

The diagram of the pretzel knot $P(5,5,6)$ in Fig. 4 has been drawn, like the knot in Fig. 2, by the Knotscape program [25] which allows one to draw knots with up to 16 crossings.

At partition of C into five integers we record:

$$w(P(2k+1,3,2,3,3))$$

$$= 123^32^21^{-1}2^{2k+1}4^{-1}3^{-1}2^334^{-1}, \qquad\qquad 2k+1, 3, 2, 3, 3,$$

$$w(P(2k+1,2,3,3,3))$$

$$= 123^32^21^{-1}2^34^{-1}3^{-1}2^{2k+1}34^{-1}, \qquad\qquad 2k+1, 2, 3, 3, 3,$$

(4.6)

$$w(P(2k+1,3,4,3,3))$$

$$= 123^32^21^{-1}2^{2k+1}4^{-1}3^{-1}2^3354^{-1}5^{-2}, \qquad 2k+1, 3, 4, 3, 3,$$

$$w(P(2k+1,4,3,3,3))$$

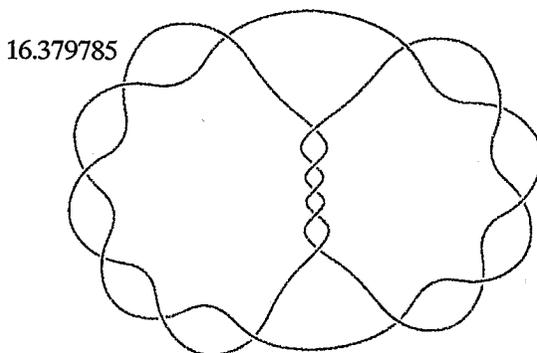$$= 123^32^21^{-1}2^34^{-1}3^{-1}2^{2k+1}354^{-1}5^{-2}, \qquad 2k+1, 4, 3, 3, 3.$$

16.379785

Fig. 4. The knot $P(5,5,6)$.

The braid words recorded for the mirror image of the knot diagrams of Rolfsen [7] will conform in the crossing sign to Rolfsen diagrams if the sign of the power exponent at each braid generator would be reversed, or the braid generator b_n would be defined [1, 3] so that it yields a negative crossing.

5. Concluding remarks

The braid words can be transformed according to the rules known for the closed braid moves [1, 2, 8, 10–13].

The expression of braid words for the pretzel knots in a standardized form [5, 6, 9] enables one to see the regular pattern of the words, and thus to write the braid words of these knots for general values of the crossing number [27–29], and to locate a desired pretzel knot in such tabulation e.g. as provided by the Knotscape program [25].

In the laboratory synthesis of long organic molecular chains, in particular of polyether ladders, a synthesis of the knot 4_1 was discussed [30]. The knotted carbynes, i.e. carbon allotropes with pure *sp*-hybridization, were considered, up to the 6_3 knot, in the *ab initio* calculations [31], to estimate their stability and the
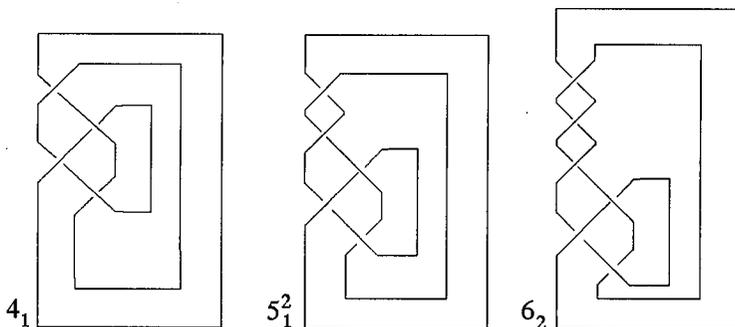

4_1 5_1^2 6_2

Fig. 5. Braid closures relevant to successive rounds of tangle addition in the processive recombination of DNA.

number of their NMR lines. In the processive recombination of a closed DNA ring, products are created in a sequence with an increasing number of supercoil crossings [32–37]. The recombination of the DNA ring by a resolvase enzyme yields in successive rounds of tangle addition: the two-component link 2_1^2, the twist knot 4_1, the two-component link 5_1^2, and the twist knot 6_2, shown by diagrams in the review [34] and in Fig. 5. The review [34] reports also the results of recombination by the Gin system of bacteriophage Mu on a knotted DNA substrate. From the knotted DNA substrates, as products are found, *inter alia*, the knots: 3_1, 4_1, 5_2, 6_1, 7_2, and 7_5. Based on an abundant supply of illustrative examples, the review [34] asserts: "Biologically, the two most important families of knots and catenanes are the torus and twist families". A standardized form of braid words is useful for presentation of these links in a discussion of their recombination processes.

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