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ON THE TRANSMISSION COEFFICIENT FOR THE DOUBLE δ'-FUNCTION POTENTIAL

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The unbound-state solution of the Schrödinger equation is examined for the potential that is defined as the sum of two δ'-functions of non-equal strengths. The analytical expression for the transmission coefficient is derived from the solution. The transmission coefficient has an absolute maximum at the zero wave number. Further, the transmission coefficient is shown to exhibit relative maxima and minima. Moreover, it is proved that the transmission coefficient in its relative maxima is larger and in its relative minima is smaller than the transmission coefficient for the corresponding single δ'-function potential.

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1. Introduction

Transmissions through a one-dimensional double-barrier structure have extensively been studied from the viewpoint of both the design technology [1] and the fundamental physics [2]. With the progress of the fabrication technique, the transmission through such structures has become a physical reality. Modern computers now allow problems of the transmission through realistic potentials to be solved numerically with a relative ease. However, an analytical solution of the transmission problem is still of an instructive value, since it enables one to get an insight into phenomena which typically take place. There exists a very simple archetype of a double-barrier structure, namely a double delta barrier. The transmission through a symmetrical double delta barrier was thoroughly studied by Galindo and Pascual [3]. The study of the transmission through an asymmetrical double delta barrier was carried out by the author [4].

The δ-function potential belongs to a family of point interactions in one-dimensional quantum mechanics [5]. A point interaction is such that it is zero everywhere except at a given point $x = x_0$, where x represents the spatial variable. The most familiar of the point interactions is the δ interaction that can be written in the form of the potential $V(x) = g\delta(x - x_0)$, where $\delta(x)$ is the Dirac delta function and g is its strength. The wave function $\psi(x)$ of a particle that has the mass m and that moves in the given potential $V(x)$ is subject to the two boundary conditions at the point $x = x_0$, namely $\psi(x_0 + 0) = \psi(x_0 - 0)$ and $\psi'(x_0 + 0) - \psi'(x_0 - 0) = \kappa\psi(x_0)$, where $\kappa = 2mg/\hbar^2$ and $\hbar$ is the reduced Planck constant [3, 5, 6]. The first boundary condition expresses the continuity of the wave function, the second one the discontinuity of its first derivative $\psi'(x) = \mathrm{d}\psi(x)/\mathrm{d}x$ at the point $x = x_0$.

Another of these point interactions is the so-called δ' interaction. It has, however, little resemblance to the first derivative of the Dirac delta function $\delta'(x - x_0) = \mathrm{d}\delta(x - x_0)/\mathrm{d}x$. With the δ' interaction at the point $x = x_0$, it is understood that the wave function $\psi(x)$ satisfies this set of the boundary conditions $\psi(x_0 + 0) - \psi(x_0 - 0) = \lambda\psi'(x_0)$ and $\psi'(x_0 + 0) = \psi'(x_0 - 0)$, where $\lambda = 2mg/\hbar^2$ and g is the strength of the δ' interaction [5]. This means that, while the derivative $\psi'(x)$ is continuous, the wave function itself $\psi(x)$ is discontinuous. These two boundary conditions are to be interpreted as the device that leads to the definition of the δ' interaction of the strength g at the point $x = x_0$. A positive value of the strength corresponds to a barrier, while a negative value corresponds to a well.

The purpose of this paper is to present a pedestrian treatment of the transmission through a potential that is formed of the two δ' interactions. For this purpose, the Schrödinger equation is solved for such a potential and some attributes of its transmission coefficient are presented briefly.

2. Solution of the Schrödinger equation

Thus, the solution of the Schrödinger equation is to be obtained for a potential that is defined as the sum of the two δ' interactions of the different strengths g_1 and g_2. The δ' interactions are supposed to be situated at the points $x = -a$ and $x = a$, where $2a$ is the distance between them. Evidently, the unbound-state solution of the Schrödinger equation has the form of the plane waves moving from the left to the right and vice versa, i.e. $\psi(x) = A_{\mathrm{I}}\mathrm{e}^{+\mathrm{i}kx} + B_{\mathrm{I}}\mathrm{e}^{-\mathrm{i}kx}$ if $x < -a$, $\psi(x) = A_{\mathrm{II}}\mathrm{e}^{+\mathrm{i}kx} + B_{\mathrm{II}}\mathrm{e}^{-\mathrm{i}kx}$ if $-a < x < a$ and $\psi(x) = A_{\mathrm{III}}\mathrm{e}^{+\mathrm{i}kx} + B_{\mathrm{III}}\mathrm{e}^{-\mathrm{i}kx}$ if $a < x$, where A_{I}, B_{I}, A_{II}, B_{II}, A_{III} and B_{III} are the amplitudes of the two waves in three different regions. The positive wave number k is introduced by the relation $E = \hbar^2k^2/2m$, where E is the energy of the particle. The boundary conditions can be written in the transfer-matrix form

$$\begin{bmatrix} A_{\mathrm{II}} \\ B_{\mathrm{II}} \end{bmatrix} = M(\mathrm{II},\mathrm{I}) \begin{bmatrix} A_{\mathrm{I}} \\ B_{\mathrm{I}} \end{bmatrix}, \quad \begin{bmatrix} A_{\mathrm{III}} \\ B_{\mathrm{III}} \end{bmatrix} = \mathrm{M}(\mathrm{III},\mathrm{II}) \begin{bmatrix} A_{\mathrm{II}} \\ B_{\mathrm{II}} \end{bmatrix},$$

$$M_{11}(\mathrm{II},\mathrm{I}) = 1 + \frac{\mathrm{i}\lambda_1 k}{2} = M^*_{22}(\mathrm{II},\mathrm{I}), \quad M_{12}(\mathrm{II},\mathrm{I}) = -\frac{\mathrm{i}\lambda_1 k}{2}\mathrm{e}^{+2\mathrm{i}ka} = M^*_{21}(\mathrm{II},\mathrm{I}),$$

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where $\lambda_1 = 2mg_1/\hbar^2$ and $\lambda_2 = 2mg_2/\hbar^2$. It is easy to obtain that det M(II,I) = 1 and det$M$(III,II) = 1. The two consecutive transfers are to be joined into one

$$\begin{bmatrix} A_{\rm III} \\ B_{\rm III} \end{bmatrix} = M({\rm III,II})M({\rm II,I}) \begin{bmatrix} A_{\rm I} \\ B_{\rm I} \end{bmatrix} = M({\rm III,II}) \begin{bmatrix} A_{\rm I} \\ B_{\rm I} \end{bmatrix},$$

$$M_{11}({\rm III,I}) = \frac{(2+\mathrm{i}\lambda_2 k)(2+\mathrm{i}\lambda_1 k) + \lambda_2\lambda_1 k^2 \mathrm{e}^{-4\mathrm{i}ka}}{4} = M^*_{22}({\rm III,I}) = \frac{1}{t^*(k)},$$

$$M_{12}({\rm III,I}) = -\frac{\mathrm{i}\lambda_2 k(2-\mathrm{i}\lambda_1 k)\mathrm{e}^{-2\mathrm{i}ka} + \mathrm{i}\lambda_1 \mathrm{k}(2+\mathrm{i}\lambda_2 k)\mathrm{e}^{+2\mathrm{i}ka}}{4}$$

$$= M^*_{21}({\rm III,I}) = -\frac{r^*(k)}{t^*(k)}.$$

It is easy to see that det M(III,I) $= [1 - r(k)r^*(k)]/t(k)t^*(k) =$ det M(III,II) $\times$det M(II,I) = 1.

The two new-introduced quantities $t(k)$ and $r(k)$ enable one to express the relation between the amplitudes in the first and third region in this simple form

$$\begin{bmatrix} A_{\rm III} \\ B_{\rm I} \end{bmatrix} = \begin{bmatrix} t(k) & -\frac{t(k)r^*(k)}{t^*(k)} \\ r(k) & t(k) \end{bmatrix} \begin{bmatrix} A_{\rm I} \\ B_{\rm III} \end{bmatrix}.$$

Thus, the wave function in the first and third region is, respectively, of the form

$$\psi_{\rm I}(x) = A_{\rm I}\mathrm{e}^{+\mathrm{i}kx} + [r(k)A_{\rm I} + t(k)B_{\rm III}]\mathrm{e}^{-\mathrm{i}kx},$$

$$\psi_{\rm III}(x) = \left[t(k)A_{\rm I} - \frac{t(k)r^*(k)}{t^*(k)}B_{\rm III}\right]\mathrm{e}^{+\mathrm{i}kx} + B_{\rm III}\mathrm{e}^{-\mathrm{i}kx}.$$

Obviously, the quantities $t(k)$ and $r(k)$ represent the transmission and reflection amplitude, respectively (actually, $r(k)$ is the reflection amplitude from the left and $-t(k)r^*(k)/t^*(k)$ is the reflection amplitude from the right, they differ only in a phase). The transmission coefficient is defined by $T(k) = t(k)t^*(k)$. The reflection coefficient can be obtained from the well-known relation $R(k) = r(k)r^*(k) = 1 - t(k)t^*(k) = 1 - T(k)$. Its validity for the transmission through the double δ'-function potential has been verified here.

3. Transmission coefficient

After the straightforward algebra, one obtains the transmission coefficient for the double δ'-function potential in a close form

$$T(k) = \frac{4}{4 + k^2(\lambda_2-\lambda_1)^2\sin^2(2ka) + k^2[(\lambda_2+\lambda_1)\cos(2ka) - \lambda_2\lambda_1 k\sin(2ka)]^2}.$$

In the limit $a \to 0$, the two δ' interactions join into the single δ' interaction. Thus, $\lim_{a\to 0} T(k) = 4/[4 + k^2(\lambda_2+\lambda_1)^2] = T_0(k)$, where $T_0(k)$ is the transmission coefficient for the single δ'-function potential of the strength $g_2 + g_1$. It is understood that for $\lambda_2 = 0$ or $\lambda_1 = 0$, the expression for $T(k)$ is also reduced to the transmission coefficients for the corresponding single δ'-function potential.

The transmission coefficient for the δ'-function potential has an interesting attribute, namely $\lim_{k\to 0} T(k) = \lim_{k\to 0} T_0(k) = 1$. Thus, the δ'-function potential is ideally transmittable for a particle with the infinitesimally small wave number k. On the other hand, $\lim_{k\to\infty} T_0(k) = 0$. Thus, the single δ'-function potential is untransmittable for a particle with the large wave number k.

When the two δ' interactions are of the same strength g, i.e. when $g_2 = g_1 = g$, the expression for $T(k)$ is transformed into the transmission coefficient for the symmetrical double δ'-function potential

$$T_{\mathrm{S}}(k) = \frac{4}{4 + k^2\lambda^2[2\cos(2ka) - \lambda k \sin(2ka)]^2},$$

where $\lambda = 2mg/\hbar^2$. Evidently, the transmission coefficient $T_{\mathrm{S}}(k)$ has an infinite number of absolute maxima and relative minima. The maxima take place at all the positive values of the wave number k that obey the following resonance condition: $2\cos(2ka) - \lambda k \sin(2ka) = 0$.

Apparently, the transmission coefficient for the asymmetrical double δ'-function potential $T(k)$ has also an infinite number of maxima and minima. It is also clear from Fig. 1 that there do not exist ideal transmissions through the asymmetrical double δ' barrier or the asymmetrical double δ' well for a particle with the non-zero wave number k. Generally, transmissions through asymmetrical double-barrier structures or asymmetrical double-well structures are not ideal. A transmission through the asymmetrical double δ'-function potential would be ideal, if two independent resonance conditions were satisfied simultaneously. One of them is the maximum condition for the peak value $(\lambda_2+\lambda_1)\cos(2ka)-\lambda_2\lambda_1 k\sin(2ka) = 0$. It strictly depends on the strengths of the δ' interactions. The other is the so-called phase-difference condition $\sin(2ka) = 0$, thus $2ka = n\pi$ and $n = 1, 2, 3, \ldots$ It has nothing to do with the shape of the two obstacles a particle transmits through (in the present case, an obstacle is represented either by a δ' barrier or a δ' well). The phase-difference condition is identical with the eigenenergy condition of a particle in an infinite rectangular well of the width $2a$ [3, 6]. It requires that the wave phase after the second reflection in the region between the obstacles is identical to the phase of the wave just transmitted through the first obstacle. Then, the amplitudes of the forward and backward waves in the region between the obstacles may be emphasised to reach a maximum value. These two resonance conditions can be satisfied simultaneously only in the case of the transmission through the skew symmetrical double δ'-function potential, i.e. when $\lambda_2 = -\lambda_1$. Thus, only transmissions through the symmetrical ($\lambda_2 = \lambda_1$) and skew symmetrical ($\lambda_2 = -\lambda_1$) double δ'-function potential can be ideal for a particle with the non-zero wave number k.

The exact positions of extremes of the function $T(k)$ can be found only by numerical computations. However, it is possible to find two equations that approximately determine their positions. To find them one has to differentiate the function $T(k)$,

$$\frac{\mathrm{d}T(k)}{a\mathrm{d}k} \approx k^2[(\lambda_2 + \lambda_1)\cos(2ka) - \lambda_2\lambda_1 k \sin(2ka)]$$
$$\times[(\lambda_2 + \lambda_1)\sin(2ka) + \lambda_2\lambda_1 k\cos(2ka)]T(k).$$

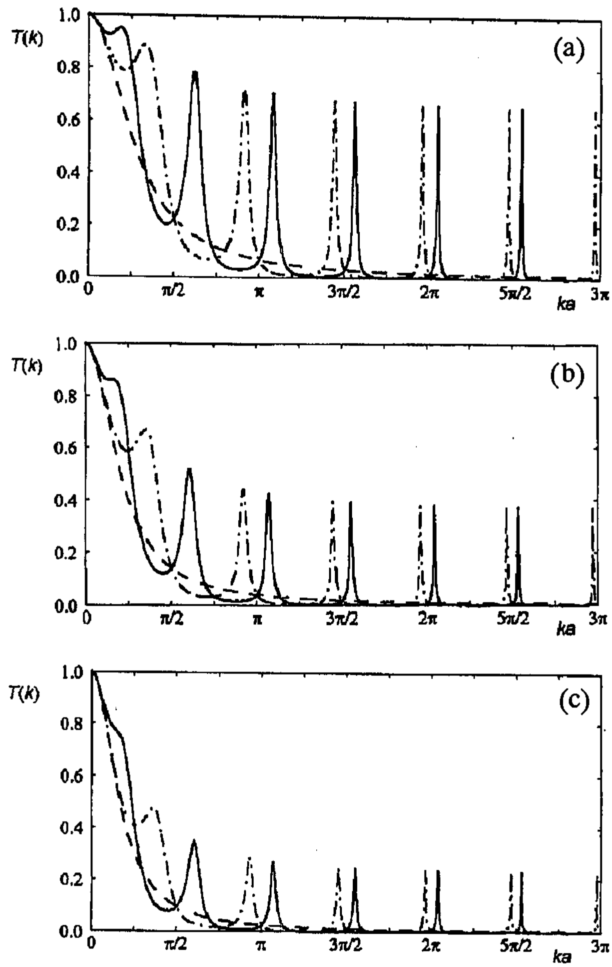

Fig. 1. The transmission coefficient for the asymmetrical double δ' barrier (the full curve) and the asymmetrical double δ' well (the dashed-dotted curve): (a) $|\lambda_1| = \pi a/4$, $|\lambda_2| = \pi a/2$; (b) $|\lambda_1| = \pi a/4$, $|\lambda_2| = 3\pi a/4$; (c) $|\lambda_1| = \pi a/4$, $|\lambda_2| = \pi a$. The dashed curve is the transmission coefficient for the single δ' barrier and for the single δ' well with the parameter $\lambda_1 + \lambda_2$.

Thus, the relative maxima of the function $T(k)$ approximately occur at all the positive values of the wave number k obeying the maximum condition for the peak value. When this condition is satisfied, the function $T(k)$ takes the form

$$T_{\max}(k) = \frac{4}{4 + \frac{k^2(\lambda_2^2 - \lambda_1^2)^2}{(\lambda_2+\lambda_1)^2 + \lambda_2^2\lambda_1^2 k^2}}.$$

One easily sees that the function $T_{\max}(k)$ is identically equal to unity when the transmitted structure is symmetrical or skew symmetrical. The asymmetry always decreases peaks in the transmission coefficient. Further, one obtains that $T_0(k) < T_{\max}(k)$, if $0 < \lambda_2\lambda_1(4 + \lambda_2\lambda_1 k^2)$. The latter condition is always satisfied in the case of the transmission through the double δ' barrier and the double δ' well, i.e. when $0 < \lambda_2\lambda_1$. If $\lambda_2\lambda_1 < 0$ (a well–barrier structure), the latter condition requires $4/(-\lambda_2\lambda_1) < k^2$.

The minima of the function $T(k)$ should approximately occur at all the positive values of the wave number k obeying the minimum condition for the valley value $(\lambda_2 + \lambda_1)\sin(2ka) + \lambda_2\lambda_1 k\cos(2ka) = 0$. Provided that it is satisfied, the function $T(k)$ can be arranged into this form

$$T_{\text{min}}(k) = \frac{4}{4 + k^2(\lambda_2+\lambda_1)^2 + \lambda_2^2\lambda_1^2k^4 + \frac{\lambda_2^2\lambda_1^2k^4(\lambda_2-\lambda_1)^2}{(\lambda_2+\lambda_1)^2+\lambda_2^2\lambda_1^2k^2}}.$$

It is easy to see that $T_{\text{min}}(k) < T_0(k)$.

Summarising, the transmission coefficient for the double δ'-function potential in its minima is smaller and in its maxima is larger than the transmission coefficient $T_0(k)$, i.e. $T_{\text{min}}(k) < T_0(k) < T_{\text{max}}(k)$, where $T_0(k)$ is the transmission coefficient for the single δ'-function potential arisen by joining the two δ' interactions of the given double δ'-function potential. Thus, there exists some constructive as well as some destructive interference between the waves just transmitting through the first δ' obstacle and those being reflected off the second one. Both the functions $T(k)$ and $T_0(k)$ have the same value when one of the following intersection conditions is satisfied:

$$(\lambda_2\lambda_1k^2 - 4)\sin(2ka) - 2k(\lambda_2+\lambda_1)\cos(2ka) = 0 \quad \text{or} \quad \sin(2ka) = 0.$$

They determine abscissae of the intersections of the curves that represent the functions $T(k)$ and $T_0(k)$. The second equation is the familiar phase-difference condition.

In the case of large values of the wave number k, i.e. when $1 \ll ka$, the maximum condition for the peak value reduces to the phase-difference condition $\sin(2ka) = 0$. The minimum condition for the valley value reduces to $\cos(2ka) = 0$. Both the conditions are independent of the strengths of the δ' interactions. In this case, the maxima of the function $T(k)$ are approximately at the values $k = n\pi/2a$ and its minima are approximately at the values $k = (n+1/2)\pi/2a$, where n's are large positive integers. Thus, the function $T(k)$ finally oscillates between the two limit values

$$\lim_{k\to\infty} T_{\text{max}}(k) = 4\lambda_2^2\lambda_1^2/[4\lambda_2^2\lambda_1^2 + (\lambda_2^2 - \lambda_1^2)^2]$$

and

$$\lim_{k\to\infty} T_{\text{min}}(k) = 0.$$

The first intersection condition also reduces to the phase-difference condition at the large values of the wave number. Therefore, the curves depicting the functions $T(k)$ and $T_0(k)$ finally intersect just at the points, where the function $T(k)$ has its maxima. This means that there are very sharp peaks in the function $T(k)$. In general, the double δ'-function potential is also untransmittable for a particle with the large wave number k. Only particles with the wave numbers $k = n\pi/2a$ can transmit through the double δ'-function potential.

In Fig. 1, the transmission coefficient $T(k)$ is drawn as a function of the wave number k for different values of the dimensionless parameters $\lambda_1 a$ and $\lambda_2 a$. These

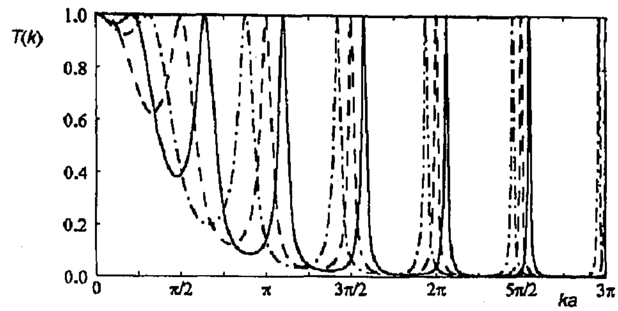

Fig. 2. The transmission coefficient for the symmetrical double δ' barrier (the full curve), the symmetrical double δ' well (the dashed-dotted curve) and the symmetrical well–barrier structure (the dashed curve). All the curves are drawn for $|\lambda_1| = |\lambda_2| = \pi a/4$.

curves demonstrate the existence of relative maxima and minima in the transmission through an asymmetrical double δ'-function potential. It is clearly seen that peaks become narrower as the wave number increases since the transmission probability through each of the δ' obstacles decreases with the increase in the wave number. The sharpness of a peak also increases with the simultaneous growth of both the parameters. When only one of the parameters grows, peaks become lower. Generally, the asymmetry always decreases peaks in the transmission coefficient. However, the effect of the region between the δ' obstacles exists even when one of them is much larger than the other. These are the characteristics proper to the transmission through an asymmetrical double δ' barrier or an asymmetrical double δ' well. In Fig. 2, the transmission coefficients for the symmetrical and skew symmetrical double δ'-function potential are shown.

4. Comments

On the one hand, the δ-function potential is frequently employed in the solid state physics. On the other hand, the δ'-function potential does not seem to be widely known. Unfortunately, it cannot be visualised and is defined only by the boundary condition. However, the transmission coefficient for it has unusual attributes.

As was mentioned, the transmission through the δ'-function potential is ideal for a particle with the infinitesimally small wave number. In general, the transmission coefficient does not equal unity for the zero wave number. Further, the transmission coefficient for a finite potential and also for the δ-function potential approaches unity as the wave number goes to infinity. However, the transmission coefficient for the δ'-function potential rapidly goes to zero at large values of the wave number. Nevertheless, the transmission coefficient for the double δ'-function potential still has well pronounced peaks at the certain values of the wave number. Such transmission peaks occur when an integer number of the half-wavelengths fits into the region between the δ' obstacles. They are to be attributed to the so-called resonant transmissions that are a result of the constructive interference of the waves in the region between the δ' obstacles. Thus, the resonant transmissions through the double δ'-function potential are a plausible quantum-mechanical example of the constructive interference of the waves. A device with such a performance could be used as an effective selector of particles with the given energy, especially in the high-energy region.

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