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IRREDUCIBLE BASIS FOR PERMUTATION REPRESENTATIONS

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For a given finite group G its permutation representation P, i.e. an action on an n-element set, is considered. Introducing a vector space L as a set of formal linear combinations of $|j\rangle$, $1 \leq j \leq n$, the representation P is linearized. In general, the representation obtained is reducible, so it is decomposed into irreducible components. Decomposition of L into invariant subspaces is determined by a unitary transformation leading from the basis $\{|j\rangle\}$ to a new, *symmetry adapted* or *irreducible*, basis $\{|\Gamma r \gamma\rangle\}$. This problem is quite generally solved by means of the so-called Sakata matrix. Some possible physical applications are indicated.

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1. Introduction

Permutation representations are well known in solid state physics as they appear in the induction procedure leading to irreducible representations of space groups [1]. On the other hand, they are also useful in the investigation of phase transitions, especially when the disordered and ordered phases correspond to a group–subgroup pair [2]. These two applications formed the foundation for the discussions presented by T. Lulek et al. [3, 4] in several papers. An equivalent definition of permutation representations can be based on the notion of group action on a finite set [5]. Such an approach is especially useful in the case of n identical objects "carrying" finite many internal states, e.g. spin projections in magnetic models, colors in quantum chromodynamics, energy levels in boson systems. In this paper the first application is considered as an example (see also [6]).

In the cases mentioned above a basis space of states is spanned by orthonormal vectors $|s_1\, s_2\, \dots\, s_n\rangle$, where $1 \leq s_j \leq m$ is one of m possible one-particle states of the j-th object (particle). For finite n and m this basis is also finite and, for identical objects, a finite-action of a subgroup G of the symmetric group Σ_n can be

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defined [5, 6]. This action can be linearized, however, in a general case, the vector representation obtained is reducible. This means that for a Hamiltonian invariant under G the space of states decomposes into eigenspaces labeled by irreducible representations of G, even for a transitive action of G. The aim of this paper is to determine a unitary transformation from the basis $\{|s_1, s_2, \ldots, s_n\rangle\}$ of a tensor product of single-particle spaces to an irreducible (or symmetry adapted) basis, in which a Hamiltonian matrix has a block-diagonal form. This diagonalization can be accompanied by classification by other good quantum numbers, e.g. by the total magnetization and the total spin in the case of spin models [7, 8].

This paper is organized as follows. The definition of permutation representations in Sec. 2 is followed by a brief discussion on properties of the transformation matrix. In the next section the so-called projection matrix is introduced and, applying the results of Sakata [9], the problem stated is solved. As an example the dihedral group $D_8 \subset \Sigma_8$ is considered (Sec. 4) and the formulae obtained are applied to a spin model in the next section. Some general remarks are gathered in Sec. 6.

2. Permutation representations

A permutation representation can be defined as an action of a group G on its left cosets determined by a subgroup H [3, 5] by the left multiplication, i.e.

$$G \ni g : rH \longrightarrow (gr)H.$$

This representation will be denoted hereafter as $G : H$ and, in fact, is transitive, i.e. a set of cosets forms one orbit of this action. In a general case, the action of a group G on a given set decomposes it into pairwise disjoint subsets, the so-called orbits [5]. An action restricted to one orbit is called transitive and each group action (permutation representation) can be decomposed into transitive ones.

Let us label the cosets $g_j H$ by integers $j = 1, 2, \ldots, n$, where $n = |G|/|H|$ is an index of H in G. Elements $g_j \in G$ are representatives of these cosets with $g_1 = e \in G$ being the unit of G. The subgroup H is the stabilizer of the first coset $hH \equiv H$ for all $h \in H$, whereas the stabilizer of a coset $g_j H$ is the conjugate

$$G_j := g_j H g_j^{-1}$$

of H by the representative g_j, since for $h \in H$ and $G_j \ni g = g_j h g_j^{-1}$ we have

$$g(g_j H) = (g g_j)H = (g_j h)H = g_j H.$$

A permutation representation describes an action of G on a finite set of cosets. It can be treated as a *linear* action by introducing, in a formal way, a linear space L with a basis $\mathcal{B} = \{|1\rangle, |2\rangle, \ldots, |n\rangle\}$, so

$$L = \mathrm{lc}_{\mathbb{C}}\mathcal{B} = \left\{ |\phi\rangle = \sum_{j=1}^{n} \alpha_j |j\rangle \middle| \alpha_j \in \mathbb{C} \right\}. \tag{1}$$

This space is equipped with a unitary structure in a natural way by putting

$\langle j|k\rangle = \delta_{jk}$. The linearization of the permutation representation is given by a vector representation $T\colon G \to U(n)$ determined as

$$T(g)|j\rangle = \sum_{k=1}^{n} T_{kj}(g)|k\rangle := |gj\rangle, \tag{2}$$

where gj labels the coset $(gg_j)H$. Therefore,

$$T_{kj}(g) = \delta_{k,gj} = \delta_{g^{-1}k,j}. \tag{3}$$

In a general case, this linear representation is a reducible one, so it can be decomposed into a sum of unitary irreducible representations $\Gamma \in \widetilde{G}$ ($\widetilde{G}$ is a set of all inequivalent unitary irreducible representations of G)

$$T = \bigoplus_{\Gamma\in\widetilde{G}} m(T,\Gamma)\Gamma, \tag{4}$$

where $m(T,\Gamma)$ is a multiplicity of Γ in T. The space L is decomposed in a similar way into a direct sum of invariant (irreducible) subspaces $L_{\Gamma r}$, where $r = 1,2,\ldots,m(T,\Gamma)$ is a repetition index labeling copies of the carrier space L_Γ of the irreducible representation Γ. Therefore, there exists an orthonormal basis

$$\mathcal{B}^{\text{irr}} = \{|\Gamma r\gamma\rangle | \Gamma \in \widetilde{G}, m(T,\Gamma) > 0,\ 1 \le r \le m(T,\Gamma),\ 1 \le \gamma \le [\Gamma]\} \tag{5}$$

($[\Gamma]$ denotes a dimension of Γ), in which the representation T is quasi-diagonal, i.e. is represented by a block matrix

$$T(g)|\Gamma v\gamma\rangle = \sum_{\gamma'=1}^{[\Gamma]} \Gamma_{\gamma'\gamma}(g)|\Gamma r\gamma'\rangle. \tag{6}$$

Let $\boldsymbol{B} \in U(n)$ be a such transformation that $\boldsymbol{B}(\mathcal{B}) = \mathcal{B}^{\text{irr}}$, i.e.

$$|\Gamma r\gamma\rangle = \sum_{j=1}^{n} \boldsymbol{B}_{j,\Gamma r\gamma}|j\rangle. \tag{7}$$

Our task is to determine the matrix $\boldsymbol{B}$ in the most efficient and general way.

For the sake of simplicity we limit hereafter ourselves to only one copy of a given irreducible representation Γ and below we will consider the matrix $\boldsymbol{B}$ of dimension $n \times [\Gamma]$ with rows labeled by j and columns labeled by γ. Hence, indices Γ and r will be hereafter omitted. Columns of the matrix $\boldsymbol{B}$ form a symmetry adapted basis of the representation T for a given $\Gamma \in \widetilde{G}$ and $1 \le r \le m(T,\Gamma)$.

Lemma: *Let $\langle b_j|$ be the j-th row of the matrix $\boldsymbol{B}$, i.e. $\langle b^j| = [B_{j1}, B_{j2}, \ldots, B_{j[\Gamma]}]$. For all $j > 1$ we have*

$$\langle b^j| = \langle b^1|\Gamma(g^{-1}), \qquad \textit{where} \qquad g \in g_jH, \tag{8}$$

i.e. $g = g_jh$ with $h \in H$.

Proof: The (co-)vector $\langle b^1|$ is a part of the vector $\langle 1| = |1\rangle^\dagger$, where $|1\rangle$ can be determined from the inverse of the transformation (7), i.e.

$$|1\rangle = \sum_{\Gamma\in\widetilde{G}} \sum_{r=1}^{m(T,\Gamma)} \sum_{\gamma=1}^{[\Gamma]} B^*_{1,\Gamma r\gamma}|\Gamma r\gamma\rangle.$$

For any $g \in g_j H$ we have $T(g)|1\rangle = |j\rangle$, so

$$|j\rangle = \sum_{\Gamma \in \widetilde{G}} \sum_{r=1}^{m(T,\Gamma)} \sum_{\gamma=1}^{[\Gamma]} B^*_{1,\Gamma r \gamma} T(g) |\Gamma r \gamma\rangle$$

and from Eq. (6) one obtains

$$|j\rangle = \sum_{\Gamma \in \widetilde{G}} \sum_{r=1}^{m(T,\Gamma)} \sum_{\gamma,\gamma'=1}^{[\Gamma]} B^*_{1,\Gamma r \gamma} \Gamma_{\gamma'\gamma}(g) |\Gamma r \gamma'\rangle .$$

On the other hand

$$|j\rangle = \sum_{\Gamma \in \widetilde{G}} \sum_{r=1}^{m(T,\Gamma)} \sum_{\gamma'=1}^{[\Gamma]} B^*_{j,\Gamma r \gamma'} |\Gamma r \gamma'\rangle .$$

Therefore, the coefficients $B_{j,\Gamma r \gamma}$ have to satisfy

$$\sum_{\gamma=1}^{[\Gamma]} B_{1,\Gamma r \gamma} \Gamma_{\gamma\gamma'}(g^{-1}) = B_{j,\Gamma r \gamma'} ,$$

which is another formulation of Eq. (8) and, henceforth, ends the proof.

Therefore, to determine the matrix $\boldsymbol{B}$ we have to find one, for example the first, of its rows. This problem can be solved immediately in some very simple cases. For example, if Γ is the unit representation $\Gamma(g) = 1$ then each b^j is a complex number and $b^j = b^1$ for $1 < j \leq n$. Normalizing this vector the well-known result is obtained: $b^j = n^{-1/2}$.

3. Projection matrix

Theorem: *The first row of a matrix $\boldsymbol{B}$ (for a given unitary irreducible representation Γ and a given repetition index r) can be calculated as*

$$\langle b^1 | = A F^{(\Gamma)}(H), \tag{9}$$

where

(i) A is an arbitrary row vector $A = [A_1, A_2, \ldots, A_{[\Gamma]}]$ with $A_\gamma \in \mathbb{C}$;

(ii) $F^{(\Gamma)}(H)$ is a sum of matrices $\Gamma(h)$ over the whole stabilizer $H \subseteq G$, i.e.

$$F^{(\Gamma)}(H) := \sum_{h \in H} \Gamma(h).$$

This matrix will be called hereafter a *projection matrix*.

Proof: Sakata [9] showed that for any unitary representation T a symmetry adapted basis (for a given Γ) consists of normalized columns of a matrix

$$B^{(\Gamma)}(G) = \sum_{g \in G} T(g^{-1}) S \Gamma(g), \tag{10}$$

where S is an arbitrary $n \times [\Gamma]$ complex matrix (we calculate $\mathcal{B}^{\mathrm{irr}}$ for only one representation Γ). The matrix $B^{(\Gamma)}(G)$ is sometimes called the *Sakata matrix* for the group G and the representation Γ. Moreover, for any subgroup $G' \subseteq G$ and decomposition of G into left cosets

$$G = \bigcup_{m=1}^{|G|/|G'|} g_m G',$$

where elements g_m are representatives of cosets, the matrix $B^{(\Gamma)}(G)$ can be calculated as

$$B^{(\Gamma)}(G) = \sum_{m=1}^{|G|/|G'|} T(g_m^{-1}) B^{(\Gamma)}(G') \Gamma(g_m).$$

Taking $G' = H$ and elements g_j as coset representatives it can be shown that the first row of the Sakata matrix can be written as

$$B_{1\gamma} = \sum_{j,k=1}^{n} \sum_{h \in H} \sum_{\gamma'=1}^{[\Gamma]} T_{1k}\left((g_j h)^{-1}\right) S_{k\gamma'} \Gamma_{\gamma'\gamma}(g_j h).$$

Since $T_{1k}((g_j h)^{-1}) = \delta_{j,k}$ then

$$\begin{aligned} B_{1\gamma} &= \sum_{j=1}^{n} \sum_{h \in H} \sum_{\gamma',\gamma''=1}^{[\Gamma]} S_{j\gamma'} \Gamma_{\gamma'\gamma''}(g_j) \Gamma_{\gamma''\gamma}(h) \\ &= \sum_{\gamma''=1}^{[\Gamma]} \left(\sum_{j=1}^{n} \sum_{\gamma'=1}^{[\Gamma]} S_{j\gamma'} \Gamma_{\gamma'\gamma''}(g_j) \right) \left(\sum_{h \in H} \Gamma_{\gamma''\gamma}(h) \right). \end{aligned}$$

Introducing a projection matrix

$$F^{(\Gamma)}(H) = \sum_{h \in H} \Gamma(h) \tag{11}$$

and a row vector

$$A = \sum_{j=1}^{n} a^j \Gamma(g_j), \tag{12}$$

where $a^j = [S_{j1}, S_{j2}, \ldots, S_{j[\Gamma]}]$ is the j-th row of (an arbitrary) matrix S the results obtained can be written in a compact form as

$$b^1 = A F^{(\Gamma)}(H). \tag{13}$$

From Eq. (12) it follows that

$$A_\gamma = S_{1\gamma} + \sum_{j=2}^{n} \sum_{\gamma'=1}^{[\Gamma]} S_{j\gamma'} \Gamma_{\gamma'\gamma}(g_j).$$

The complex numbers $S_{1\gamma'}$, $1 \le \gamma' \le [\Gamma]$, are arbitrary or, in the other words, independent, so we cannot find any relations between them. The sum on the right-hand side does not contain the elements $S_{1\gamma'}$, so the numbers A_γ are independent, i.e. they can be chosen in an arbitrary way. This means we need not calculate A but we can take any arbitrary row-matrix $A = [A_1, A_2, \ldots, A_{|\Gamma|}]$, which completes the proof.

Summarizing the results of the proven lemma and theorem we can write that the rows of the transformation matrix $\boldsymbol{B}$ (for fixed Γ and r) are given as

$$\langle b^j| = \langle b^1|\Gamma(g^{-1}) = AF^{(\Gamma)}(H)\Gamma(g^{-1}), \tag{14}$$

where $g \in g_j H$. Therefore, the elements of the transformation matrix $\boldsymbol{B}$ can be determined according to the following relation:

$$B_{j\gamma} = \left(AF^{(\Gamma)}(H)\Gamma(g_j^{-1})\right)_\gamma,$$

where, for the sake of simplicity, we have assumed $g = g_j$. Substituting it to Eq. (7) one obtains

$$|\gamma\rangle = \sum_{j=1}^{n} \sum_{\alpha,\beta=1}^{[\Gamma]} \sum_{h\in H} A_\alpha \Gamma_{\alpha\beta}(h)\Gamma_{\beta\gamma}(g_j^{-1})|j\rangle. \tag{15}$$

Taking into account Eq. (3) one can calculate $T(g)|\gamma\rangle$ as

$$T(g)|\gamma\rangle = \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{[\Gamma]} \sum_{h\in H} A_\alpha \Gamma_{\alpha\beta}(h)\Gamma_{\beta\gamma}(g_{g^{-1}k}^{-1})|k\rangle.$$

The element $g_{g^{-1}k}$ transforms the first coset eH into the coset $g_l G_j$ with $l = (g^{-1}g_k)1$ so for an appropriate h_1 it has to satisfy

$$g_{g^{-1}k}h_1 = g^{-1}g_k.$$

With this result the action $T(g)|\gamma\rangle$ is given as

$$\begin{aligned} T(g)|\gamma\rangle &= \sum_{\gamma'=1}^{[\Gamma]} \Gamma_{\gamma'\gamma}(g) \left(\sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{[\Gamma]} \sum_{h\in H} A_\alpha \Gamma_{\alpha\beta}(hh_1)\Gamma_{\beta\gamma'}(g_k^{-1})|k\rangle \right) \\ &= \sum_{\gamma'=1}^{[\Gamma]} \Gamma_{\gamma'\gamma}(g)|\gamma'\rangle. \end{aligned} \tag{16}$$

Therefore, the vectors $|\gamma\rangle$ given by Eq. (15) transform as vectors of the irreducible representation Γ and, after normalization, they form a unitary irreducible (symmetry adapted) basis $\mathcal{B}^{\mathrm{irr}}$.

3.1. Some remarks on the projection matrix

At first let us note that the irreducible basis for a given Γ is determined by $[\Gamma]$ arbitrary parameters A_γ. If one considers the regular representation (i.e. $H = \{e\}$) then all of them are independent and, as a result, one obtains $[\Gamma]$ different sets of vectors $|\gamma\rangle$, which corresponds to the multiplicity of Γ in the regular representation. To begin with, these sets can be calculated for $A_\gamma = \delta_{\alpha,\gamma}0$ for a chosen index $1 \le \alpha \le [\Gamma]$. This labeling scheme, i.e. when the repetition index is identified with the representation vector, is consistent with a choice of repetition indices for induced representations proposed by B. Lulek and T. Lulek [4]. In the general case, the vectors $|\Gamma\alpha\gamma\rangle$ obtained are neither normalized nor orthogonal but an orthonormal basis can be obtained by means of standard procedures. On the other hand, Eq. (15) yields the following result for the regular representation (j can be replaced by $g1$ since each coset consists of one element):

$$|\gamma\rangle = \sum_{g\in G}\left(\sum_{\alpha=1}^{[\Gamma]} A_\alpha \Gamma_{\alpha\gamma}(g^{-1})\right)|g1\rangle = \sum_{g\in G} c_g|g1\rangle. \tag{17}$$

The sum over all group elements can be performed in two steps: over a given subgroup G' and then over representatives g_m of its left cosets. So the above formula can be rewritten as

$$|\gamma\rangle = \sum_{m=1}^{|G|/|G'|}\left(\sum_{g'\in G'} c_{g_m g'}|g_m g'1\rangle\right). \tag{18}$$

If we identify kets $|g'1\rangle$, i.e. we put $|g'1\rangle = |1\rangle_{G'}$, then, in fact, we consider the action of G on the left cosets $\{g_m G'\}$. It is consistent with another definition of a permutation representation: it is a representation obtained by the induction $\Xi_0 \uparrow G$, where Ξ_0 is the unit representation of G' [2, 3, 10]. In this case Eq. (18) gives

$$|\gamma\rangle_{G'} = \sum_{m=1}^{|G|/|G'|}\left(\sum_{g'\in G'} c_{g_m g'}\right)|g_m 1\rangle_{G'} = \sum_{m=1}^{|G|/|G'|} c_{g_m}^{(G')}|g_m 1\rangle'_G. \tag{19}$$

This means that by a summation of coefficients $c_{g_m g'}$ for the regular representation we obtain coefficients $c_{g_m}^{(G')}$ for a given permutation representation $\Xi_0 \uparrow G$. However, the resulting vector $|\gamma\rangle'_G$ is not normalized. Moreover, a similar procedure can be performed for any group–subgroup pair $G'' \subset G'$.

The projection matrix F can be used to calculate the multiplicities $m(T,\Gamma)$. Indeed we have

$$m(T,\Gamma) = \frac{1}{|G|}\sum_{j=1}^{n}\sum_{g\in G} T_{jj}(g)\chi^{\Gamma}(g) = \frac{1}{|G|}\sum_{j=1}^{n}\sum_{g\in G}\delta_{j,gj}\chi^{\Gamma}(g).$$

Therefore, for fixed $j = 1,2,\ldots,n = |G|/|H|$, nonzero terms occur only for $g \in G_j = g_j H g_j^{-1}$. So, the sum over $g \in G$ can be replaced by the sum over $g \in G_j$, which, for a chosen representative g_j, can be replaced by the sum over $h \in H$, whereas g is substituted by $g_j h g_j^{-1}$. Therefore,

$$m(T,\Gamma) = \frac{1}{|G|}\sum_{j=1}^{n}\sum_{h\in H}\chi^{\Gamma}(g_j h g_j^{-1}) = \frac{1}{|H|}\mathrm{Tr}F^{(\Gamma)}(H). \tag{20}$$

Moreover, applying the Schur lemma it can be shown that for Γ not appearing in the decomposition of T, i.e. $m(T,\Gamma) = 0$, $F = \mathbf{0}$, where $\mathbf{0}$ is the $[\Gamma]\times[\Gamma]$ zero matrix. From this expression one obtains that the unit representation Γ_0 is always contained once in the decomposition, since

$$F^{(\Gamma_0)}(H) = \sum_{h\in H}\Gamma_1(h) = |H|.$$

It is also worth noting that the summation over all $h \in H$ in the definition of $F^{(\Gamma)}(H)$ can be done in steps, when a chain of subgroups $\{e\} = H_0 \subset H_1 \subset \ldots \subset H_m = H$ is introduced. If $g_k^{(i)}$ are coset representatives in decomposition

$$H_i = \bigcup_{k=1}^{n_i} g_k^{(i)} H_{i-1},$$

where $n_i = |G_i|/|G_{i-1}|$ then

$$F^{(\Gamma)}(H_i) = \left(\sum_{k=1}^{n_i} \Gamma(g_k^{(i)}) \right) F^{(\Gamma)}(H_{i-1}). \tag{21}$$

4. An example: $G = D_8$

Let us consider the dihedral group $D_8 = \{C_8^k, \sigma_k \mid 0 \leq k \leq 7\}$, where C_8^k is the $k\pi/4$ rotation about the z-axis, $\sigma_k = C_8^k \sigma_0$ with σ_0 being the reflection $(x, y, z) \to (x, -y, z)$. There are 11 classes of conjugated subgroups (it suffices to consider only a representative of each class of conjugated subgroups [5]) represented by the following subgroups:

$$H_1 = C_1 = \{e = C_8^0\}, \qquad H_2 = C_2 = \{e, C_8^4\},$$

$$H_3 = D_1^{(0)} = \{e, \sigma_0\}, \qquad H_4 = D_1^{(1)} = \{e, \sigma_1\},$$

$$H_5 = C_4 = \{e, C_8^2, C_8^4, C_8^6\},$$

$$H_6 = D_2^{(0)} = \{e, C_8^4, \sigma_0, \sigma_4\}, \qquad H_7 = D_2^{(1)} = \{e, C_8^4, \sigma_1, \sigma_5\},$$

$$H_8 = C_8 = \{e, C_8, C_8^2, C_8^3, C_8^4, C_8^5, C_8^6, C_8^7\},$$

$$H_9 = D_4^{(0)} = \{e, C_8^2, C_8^4, C_8^6, \sigma_0, \sigma_2, \sigma_4, \sigma_6\},$$

$$H_{10} = D_4^{(1)} = \{e, C_8^2, C_8^4, C_8^6, \sigma_1, \sigma_3, \sigma_5, \sigma_7\},$$

$$H_{11} = D_8 = \{e, C_8, C_8^2, C_8^3, C_8^4, C_8^5, C_8^6, C_8^7, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}.$$

There are seven inequivalent unitary irreducible representations. Their matrices for the generators C_8 and σ_0 of D_8 are given as

$$A_1(C_8) = A_2(C_8) = -B_1(C_8) = -B_2(C_8) = (1),$$

$$A_1(\sigma_0) = -A_2(\sigma_0) = B_1(\sigma_0) = -B_2(\sigma_0) = (1),$$

$$E_1(C_8) = \begin{pmatrix} a & -a \\ a & a \end{pmatrix}, \qquad E_2(C_8) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$E_3(C_8) = \begin{pmatrix} -a & -a \\ a & -a \end{pmatrix}, \qquad E_1(\sigma_0) = E_2(\sigma_0) = E_3(\sigma_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $a = \sqrt{2}/2$. For each pair $H \subseteq D_8$ and $\Gamma \in \widetilde{D}_8$ a matrix F has been determined and the results are gathered in Table I, which enables us to determine decompositions of permutation representations $G : H$, which are presented below:

$$D_8 : C_1 = A_1 \oplus A_2 \oplus B_1 \oplus B_2 \oplus 2E_1 \oplus 2E_2 \oplus 2E_3,$$

$$D_8 : C_2 = A_1 \oplus A_2 \oplus B_1 \oplus B_2 \oplus 2E_2,$$

$$D_8 : D_1^{(0)} = A_1 \oplus B_1 \oplus E_1 \oplus E_2 \oplus E_3,$$

$$D_8 : D_1^{(1)} = A_1 \oplus B_2 \oplus E_1 \oplus E_2 \oplus E_3,$$

$$D_8 : C_4 = A_1 \oplus A_2 \oplus B_1 \oplus B_2, \qquad D_8 : D_2^{(0)} = A_1 \oplus B_1 \oplus E_2,$$

$$D_8 : D_2^{(1)} = A_1 \oplus B_2 \oplus E_2, \qquad D_8 : C_8 = A_1 \oplus A_2,$$

$$D_8 : D_4^{(0)} = A_1 \oplus B_1, \qquad D_8 : D_4^{(1)} = A_1 \oplus B_2, \qquad D_8 : D_8 = A_1.$$

Note that the representation A_2 appears only in decompositions for H being a cyclic subgroup of D_8. This is a characteristic feature of the dihedral groups and as a result this representation rarely appears in physical problems, since it is difficult to construct a state with a cyclic stabilizer (see the next section or [11]).

TABLE I

Matrices F for all pairs (H, Γ) for $G = D_8$; $\mathbf{0}$ denotes 2×2 zero matrix, $\mathbf{1}$ — the unit matrix, $a = \sqrt{2}/2$.

	A_1	A_2	B_1	B_2	E_1	E_2	E_3
C_1	1	1	1	1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
C_2	2	2	2	2	$\mathbf{0}$	$\mathbf{21}$	$\mathbf{0}$
$D_1^{(0)}$	2	0	2	0	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$
$D_1^{(1)}$	2	0	0	2	$\begin{pmatrix} 1+a & a \\ a & 1-a \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1-a & a \\ a & 1+a \end{pmatrix}$
C_4	4	4	4	4	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$D_2^{(0)}$	4	0	4	0	$\mathbf{0}$	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathbf{0}$
$D_2^{(1)}$	4	0	0	4	$\mathbf{0}$	$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$	$\mathbf{0}$
C_8	8	8	0	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$D_4^{(0)}$	8	0	8	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$D_4^{(1)}$	8	0	0	8	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
D_8	16	0	0	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

Let us consider the stabilizer $H = D_2^{(0)}$. Then there are four cosets represented by C_8^j, $j = 0,1,2,3$. Using the matrices F presented in Table I we can easily determine the irreducible basis:

$$|A_1a_1\rangle = \frac{1}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle), \qquad |B_1b_1\rangle = \frac{1}{2}(|0\rangle - |1\rangle + |2\rangle - |3\rangle),$$

$$|E_2c\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |2\rangle), \qquad |E_2s\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle).$$

In these formulae $|j\rangle$ represents a coset $C_8^j D_2^{(0)}$ and all vectors were normalized.

5. Application to spin models

A very illustrative example of the application of permutation representations is provided by spin models. It is assumed that nodes carrying spins form an orbit of a group G — in the case of one-dimensional models with the periodic boundary conditions $G = D_n$, where n denotes the lattice period. Magnetic (or spin) configurations can be considered as mappings

$$f\colon \{1,2,\ldots,n\} \to \{-s,-s+1,\ldots,s\}$$

with s being a spin number [11, 12]. The dihedral group D_n permutes all configurations according to the following formula:

$$gf(j) = f(g^{-1}j),$$

where $g^{-1}j$ is determined by the action of D_n on the set of nodes. These mappings are in a one-to-one correspondence with the Ising states $|f(1)f(2),\ldots,f(n)\rangle$, which form a basis space of the states. The anisotropic Heisenberg Hamiltonian commutes with this action, so the irreducible basis is very useful in solving eigenproblems (see Ref. [11] for details).

Let us consider one orbit for the case $n = 8$ and $s = 1/2$. A configuration $f(1) = 1/2$, $f(2) = \ldots = f(8) = -1/2$ represents an eight-element orbit consisting of configurations:

$$|1\rangle = |+-------\rangle, \qquad |2\rangle = |-+------\rangle,$$

$$|3\rangle = |--+-----\rangle, \qquad |4\rangle = |---+----\rangle,$$

$$|5\rangle = |----+---\rangle, \qquad |6\rangle = |-----+--\rangle,$$

$$|7\rangle = |------+-\rangle, \qquad |8\rangle = |-------+\rangle,$$

where symbols "+" and "−" stand for $\pm 1/2$, respectively. The nodes can be arranged in such a way that $G_j = \{e, \sigma_{2j-2}\}$ is a stabilizer of $|j\rangle$ ($2j-2$ is calculated mod 8) and

$$C_8^{j-1}|1\rangle = |j\rangle.$$

Therefore, the permutation representation of D_8 in the space spanned by the orthonormal vectors $|1\rangle, |2\rangle, \ldots, |8\rangle$ can be decomposed as

$$A_1 \oplus B_1 \oplus E_1 \oplus E_2 \oplus E_3.$$

To determine the irreducible basis at first we consider the regular representation $D_8 : C_1$. This case is very important since most orbits of spin configurations lead to this case [11, 12]. Vectors of this representation can be labelled by the elements of D_8 (see Eq. (17)) and vectors of the irreducible basis for one-dimensional irreducible representations can be written as

$$|A_1a_1\rangle = \frac{1}{4}\sum_{g\in D_8}|g1\rangle, \qquad |A_2a_2\rangle = \frac{1}{4}\sum_{j=0}^{7}\left(|C_8^j1\rangle - |\sigma_j1\rangle\right),$$

$$|B_1b_1\rangle = \frac{1}{4}\sum_{j=0}^{3}\left(|C_8^{2j}1\rangle - |C_8^{2j+1}1\rangle + |\sigma_{2j}1\rangle - |\sigma_{2j+1}1\rangle\right),$$

$$|B_2b_2\rangle = \frac{1}{4}\sum_{j=0}^{3}\left(|C_8^{2j}1\rangle - |C_8^{2j+1}1\rangle - |\sigma_{2j}1\rangle + |\sigma_{2j+1}1\rangle\right).$$

In the case of two-dimensional irreducible representations, E_1, E_2, E_3, one has to be careful since each of them appears twice in the decomposition. We choose an arbitrary vector A in a general form as $[\alpha, \beta]$ and multiplying it by appropriate matrices $E_k(g^{-1})$, $k = 1, 2, 3$, $g \in D_8$, we obtain the coefficients c_g gathered in Table II. To obtain the irreducible basis for orbits with a stabilizer G_j we have

TABLE II

Coefficients c_g for the two-dimensional irreducible representations of $G = D_8$; $a = \sqrt{2}/2$.

$\lvert g1\rangle$	$\lvert E_1c\rangle$	$\lvert E_1s\rangle$	$\lvert E_2c\rangle$	$\lvert E_2s\rangle$	$\lvert E_3c\rangle$	$\lvert E_3s\rangle$
$\lvert 1\rangle$	α	β	α	β	α	β
$\lvert C_8 1\rangle$	$a(\alpha-\beta)$	$a(\alpha+\beta)$	$-\beta$	α	$-a(\alpha+\beta)$	$a(\alpha-\beta)$
$\lvert C_8^2 1\rangle$	$-\beta$	α	$-\alpha$	$-\beta$	β	$-\alpha$
$\lvert C_8^3 1\rangle$	$-a(\alpha+\beta)$	$a(\alpha-\beta)$	β	$-\alpha$	$a(\alpha-\beta)$	$a(\alpha+\beta)$
$\lvert C_8^4 1\rangle$	$-\alpha$	$-\beta$	α	β	$-\alpha$	$-\beta$
$\lvert C_8^5 1\rangle$	$-a(\alpha-\beta)$	$-a(\alpha+\beta)$	$-\beta$	α	$a(\alpha+\beta)$	$-a(\alpha-\beta)$
$\lvert C_8^6 1\rangle$	β	$-\alpha$	$-\alpha$	$-\beta$	$-\beta$	α
$\lvert C_8^7 1\rangle$	$a(\alpha+\beta)$	$-a(\alpha-\beta)$	β	$-\alpha$	$-a(\alpha-\beta)$	$-a(\alpha+\beta)$
$\lvert \sigma_0 1\rangle$	α	$-\beta$	α	$-\beta$	α	$-\beta$
$\lvert \sigma_1 1\rangle$	$a(\alpha+\beta)$	$a(\alpha-\beta)$	β	α	$-a(\alpha-\beta)$	$a(\alpha+\beta)$
$\lvert \sigma_2 1\rangle$	β	α	$-\alpha$	β	$-\beta$	$-\alpha$
$\lvert \sigma_3 1\rangle$	$-a(\alpha-\beta)$	$a(\alpha+\beta)$	$-\beta$	$-\alpha$	$a(\alpha+\beta)$	$a(\alpha-\beta)$
$\lvert \sigma_4 1\rangle$	$-\alpha$	β	α	$-\beta$	$-\alpha$	β
$\lvert \sigma_5 1\rangle$	$-a(\alpha+\beta)$	$-a(\alpha-\beta)$	β	α	$a(\alpha-\beta)$	$-a(\alpha+\beta)$
$\lvert \sigma_6 1\rangle$	$-\beta$	$-\alpha$	$-\alpha$	β	β	α
$\lvert \sigma_7 1\rangle$	$a(\alpha-\beta)$	$-a(\alpha+\beta)$	$-\beta$	$-\alpha$	$-a(\alpha+\beta)$	$-a(\alpha-\beta)$

to add the coefficients corresponding to the cosets $\{C_8^{j-1}, \sigma_{j-1}\}$, $j = 1, 2, \ldots, 8$; they will give the coefficients of the kets $|j\rangle$. Comparing the decompositions of the regular representation and $D_8 : D_1^{(0)}$ we see that the representations A_2 and B_2 should be removed and all two-dimensional representations appear once, so the corresponding vectors should be determined by one parameter. In the first case (i.e. for the one-dimensional representation) we see that C_8^j and σ_j have the same sign for A_1 and B_1, whereas they differ in sign for the other representations. It is easy to notice that for the two-dimensional representations the parameter β has a different sign for C_8^j and σ_j, so this confirms our prediction. The vectors obtained are presented in Table III.

TABLE III

Coefficients c_g for the permutation representation $D_8 : D_1^{(0)}$; normalized vectors of the irreducible basis are obtained for $\alpha = \sqrt{2}/4$ and $\beta = 1/2$ $(a = \sqrt{2}/2)$.

$\vert j\rangle$	$\vert A_1a_1\rangle$	$\vert B_1b_1\rangle$	$\vert E_1c\rangle$	$\vert E_1s\rangle$	$\vert E_2c\rangle$	$\vert E_2s\rangle$	$\vert E_3c\rangle$	$\vert E_3s\rangle$
$\vert 1\rangle$	α	α	β	0	β	0	β	0
$\vert 2\rangle$	α	$-\alpha$	$a\beta$	$a\beta$	0	β	$-a\beta$	$a\beta$
$\vert 3\rangle$	α	α	0	β	$-\beta$	0	0	$-\beta$
$\vert 4\rangle$	α	$-\alpha$	$-a\beta$	$a\beta$	0	$-\beta$	$a\beta$	$a\beta$
$\vert 5\rangle$	α	α	$-\beta$	0	β	0	$-\beta$	0
$\vert 6\rangle$	α	$-\alpha$	$-a\beta$	$-a\beta$	0	β	$a\beta$	$-a\beta$
$\vert 7\rangle$	α	α	0	$-\beta$	$-\beta$	0	0	β
$\vert 8\rangle$	α	$-\alpha$	$a\beta$	$-a\beta$	0	$-\beta$	$-a\beta$	$-a\beta$

For $H = C_2$ we can calculate the transformation matrix $\boldsymbol{B}$ applying Eq. (14). Since the representation E_2 appears twice, then we determine vectors $|E_2r\gamma\rangle$, $\gamma = c, s$ for two different (row) vectors A. Taking into account the note at the beginning of Sec. 3.1 we use the vectors $[\alpha, 0]$ and $[0, \alpha]$, $\alpha \in \mathbb{C}$. The resulting vectors of $\mathcal{B}^{\text{irr}}$ will be denoted by $|\Gamma c\gamma\rangle$ and $|\Gamma s\gamma\rangle$, respectively. In that case no orthogonalization procedure is necessary for such a choice. Labeling the cosets C_2, C_8C_2, $C_8^2C_2$, $C_8^3C_2$, σ_0C_2, σ_1C_2, σ_2C_2, σ_3C_2 by $j = 1, 2, \ldots, 8$ the vectors of the irreducible basis are given as

$$|A_1a_1\rangle = \frac{\sqrt{2}}{4}(|1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle + |8\rangle),$$

$$|A_2a_2\rangle = \frac{\sqrt{2}}{4}(|1\rangle + |2\rangle + |3\rangle + |4\rangle - |5\rangle - |6\rangle - |7\rangle - |8\rangle),$$

$$|B_1b_1\rangle = \frac{\sqrt{2}}{4}(|1\rangle - |2\rangle + |3\rangle - |4\rangle + |5\rangle - |6\rangle + |7\rangle - |8\rangle),$$

$$|B_2b_2\rangle = \frac{\sqrt{2}}{4}(|1\rangle - |2\rangle + |3\rangle - |4\rangle - |5\rangle + |6\rangle - |7\rangle + |8\rangle),$$

$$|E_2cc\rangle = \frac{1}{2}\left(|1\rangle - |3\rangle + |5\rangle - |7\rangle\right), \qquad |E_2cs\rangle = \frac{1}{2}\left(|2\rangle - |4\rangle + |6\rangle - |8\rangle\right),$$

$$|E_2sc\rangle = -\frac{1}{2}\left(|2\rangle - |4\rangle - |6\rangle + |8\rangle\right), \qquad |E_2ss\rangle = \frac{1}{2}\left(|1\rangle - |3\rangle - |5\rangle + |7\rangle\right).$$

6. Final remarks

The proposed procedure is an application of the Sakata method [9] to the case of a transitive permutation representation. Therefore, this approach reveals its properties. It is another version of the standard projection methods [13], however the vectors obtained have the same phase, i.e. the procedure ensures that the condition

$$T(g)|\Gamma r\gamma\rangle = \sum_{\gamma'=1}^{[\Gamma]} \Gamma_{\gamma'\gamma}(g)|\Gamma r\gamma'\rangle$$

is satisfied, see Eqs. (5) and (16). On the other hand, the relations determined can be considered as a special case of those given by B. Lulek and T. Lulek [4] for induced representations.

The importance of the results is relevant to possible applications and can ease numerical calculations. There are many packages with symbolic algebra, especially these dedicated to group theory, which can be used to determine the symmetry adapted bases in such cases. Combining these results with some (algebraic) combinatorics leads to a significant reduction of eigenproblem dimension for many, e.g. spin, Hamiltonians [7, 11].

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