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DYNAMICS OF $\chi^{(2)}$ SOLITARY WAVES: A MULTIPLE SCALES APPROACH

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The dynamics of solitary waves in second-order nonlinear materials are discussed using a multiple scales model. After making some comments on the applicability of other perturbation techniques the multiple scales approach is developed with a view to setting up a line of approach that, in principle, permits radiative effects to be modelled. After a closure condition is applied, equations for the evolution of dynamical variables are developed. Applications of these equations to loss and interactions are presented together with confirmation from numerical simulations. It is emphasised that the method is capable of extension to higher-order perturbations and, hence, into the solitary wave fusion region. The established interpretation of quasi-phase-matching fluctuations as loss is discussed and the simple problems of soliton (solitary wave) pair interactions in both loss-free and lossy media are analysed.

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1. Introduction

Although the process of second-harmonic generation in quadratically nonlinear media has been understood for a long time, this process has attracted new attention in the last few years [1]. The interest is driven partly by a better comprehension of how the fundamental wave interacts with the harmonic wave in materials displaying second-order nonlinearity ($\chi^{(2)}$ materials). It is also driven by excellent progress in the material science underpinning this field.

Compared to the third-order Kerr media (displaying $\chi^{(3)}$ nonlinearity), evolution equations in $\chi^{(2)}$ media have more degrees of freedom and describe a richer set of soliton dynamics. For example, in collision problems, fusion of beams is possible. This and other phenomena open the door to a new range of sophisticated all-optical devices, such as logic gates, transistors [2] and memory [3].

Most of the theoretical papers published in soliton dynamics rely ultimately on numerical methods to study the evolution of the beams. Some of papers use analytical methods, of course but are restricted to very specific cases such as simple collisions [4]. It is appropriate, therefore, to ask if a *general* set of equations can be

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set up to discuss the evolution of the as many beam parameters, as desired, such as amplitude, position, velocity and phase. Can this also be performed for an arbitrary perturbation? The aim of this paper is to try to answer this question by setting up such a set of equations for a type I fundamental-harmonic wave generation mechanism. Analytical formulae are found and these are checked against numerical solutions for a range of general perturbations like loss, walk-off, interface and collision problems. Except in few exceptional cases, where violation of a small field overlap assumption occurs, very good agreement is found between the theory and the numerical confirmations.

2. Mathematical development

The problem is to find out how solitary wave parameters evolve when the governing equations are slightly perturbed. The equations are the well-known partial differential equations governing the evolution of the fundamental and the second harmonic waves in what is known in this field as a type I configuration. There are two sets of four parameters, for each wave: amplitude, position, velocity and phase. Also in an asymmetric collision case, it is necessary to deal with four sets of equations.

Several well-known perturbation methods exist and each possesses their own advantages. The adiabatic method and the Lagrangian method if the unperturbed governing equations are integrable — which is not the case here — the perturbed inverse scattering transform method seem to be desirable. A short presentation of these techniques for $\chi^{(3)}$ materials can be found in Ref. [5], for instance.

It is interesting (and reassuring) to note that these methods lead to the same set of evolution equations. This is easy to demonstrate for $\chi^{(3)}$ materials and it is true here as well. It can then be asked: what are the merits of a given method? The real beauty of a particular method lies not only in the analytical formulae but in its internal structure, which can give insight into the soliton dynamics, if properly interpreted. For example, if the perturbation method selected is the *multiple scales method* [6] it can, in principle, give information on the radiative part of the solution. This piece of information is not given by the adiabatic method; by definition, no radiation is possible in this technique.

For a type I interaction the evolution of the ordinary (extraordinary) o(e) polarised fundamental wave, w, is related to the e(o) polarised second-harmonic, v, via a form of the familiar coupled equations that connect the electric fields associated with those frequencies [7]. The coupled equations will be used in the following form:

$$\frac{\partial^2 E_\omega}{\partial z^2} + \frac{\partial^2 E_\omega}{\partial x^2} + \frac{\omega^2}{c^2}\varepsilon_\omega E_\omega + \frac{\omega^2}{c^2}\kappa E_\omega^* E_{2\omega} = 0, \tag{1a}$$

$$\frac{\partial^2 E_{2\omega}}{\partial z^2} + \frac{\partial^2 E_{2\omega}}{\partial x^2} + \frac{4\omega^2}{c^2}\varepsilon_{2\omega} E_{2\omega} + \frac{2\omega^2}{c^2}\kappa E_\omega^2 = 0, \tag{1b}$$

where E_ω and $E_{2\omega}$ are the Fourier amplitude of the <u>scalar</u> electric fields at the fundamental angular frequency ω and the second-harmonic angular frequency 2ω, respectively, c is the velocity of light in a vacuum, κ is the nonlinear coefficient and ε_ω, $\varepsilon_{2\omega}$ are the <u>linear</u> dielectric functions of the material at ω and 2ω, respectively.

The choice of propagation direction is z with the $\partial^2/\partial x^2$ terms which are to be interpreted as diffraction terms. At this stage of the development, the second derivatives $\partial^2/\partial z^2$ must be scrutinised carefully because it contains nonpariaxiality. The difficulty surrounding the introduction of nonparaxiality will be resolved here with the common assumption that E_ω, $E_{2\omega}$ can be written as a product of slowly varying amplitudes and rapidly varying phase functions. In addition, the separation of E_ω, $E_{2\omega}$ into an amplitude, total phase product will be done in one step that takes out the nonlinear phase from the very beginning. This is not the common practice in this field, yet it is much more logical than the two-step process that first separates E_ω, $E_{2\omega}$ into an amplitude and linear phase factor product, and then introduces an assumption that this amplitude is slowly varying. This two-step process ignores the possibility that the nonlinear phase, contained in this definition of amplitude, will lead to a violation of the slowly varying assumption.

In the light of these remarks, E_ω, $E_{2\omega}$ will be written as

$$E_\omega = E_1 \exp\left(\mathrm{i}\frac{\omega}{c}Bz\right), \quad E_{2\omega} = E_2 \exp\left(\mathrm{i}2\frac{\omega}{c}Bz\right), \tag{2}$$

where B is a total nonlinear phase factor.

Hence, after defining

$$E_1 = \left(\frac{B^2-\varepsilon_\omega}{\sqrt{2}\kappa}\right)w, \quad E_2 = \left(\frac{B^2-\varepsilon_\omega}{\kappa}\right)v,$$

$$z = \frac{2B}{(B^2-\varepsilon_\omega)}\frac{c}{\omega}z', \quad x = \frac{1}{\sqrt{B^2-\varepsilon_\omega}}\frac{c}{\omega}x', \tag{3}$$

the basic equations become

$$\mathrm{i}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial x^2} - w + w^* v = 0, \tag{4a}$$

$$2\mathrm{i}\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} - \gamma v + w^2 = 0, \tag{4b}$$

where $\gamma = 4(B^2-\varepsilon_{2\omega})/(B^2-\varepsilon_\omega)$ is a parameter that changes with power, because of the presence of B in its definition.

The stationary solutions are $\overline{w}$, $\overline{v}$ and occur when $\partial w/\partial z = \partial v/\partial z = 0$ are the solutions of

$$\frac{\partial^2 \overline{w}}{\partial x^2} - \overline{w} + \overline{w}^* \overline{v} = 0, \tag{5a}$$

$$\frac{\partial^2 \overline{w}}{\partial x^2} - \gamma\overline{v} + \overline{w}^2 = 0. \tag{5b}$$

These equations have an analytical solution only for $\gamma = 1$, i.e. $B^2 = (4\varepsilon_{2\omega} - \varepsilon_\omega)/3$ which corresponds to a certain level of energy. These solutions are [8]

$$\overline{w} = \overline{v} = \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right). \tag{6}$$

It is extremely interesting that Eqs. (4) are Galilean invariant because only a factor 2 occurs in front of $\mathrm{i}\partial/\partial z$ in (4b). Hence, if $\overline{w}$, $\overline{v}$ is a solution pair of (4) then so is

$$w(x,z) = \overline{w}(x-\xi z)\exp\left(\mathrm{i}\frac{\xi}{2}(x-\xi z) + \mathrm{i}\frac{\xi^2}{4}z\right), \tag{7a}$$

$$v(x,z)=\overline{v}(x-\xi z)\exp\left(\mathrm{i}\xi(x-\xi z)+\mathrm{i}\frac{\xi^2}{2}z\right). \tag{7b}$$

These equations represent a soliton moving at an angle $\tan^{-1}(\xi)$ to the z-axis. Hence, the soliton (solitary wave) centre is, in general, at $x=\overline{x}$ and (7a), (7b) can be written as

$$w(x,z)=\overline{w}(\theta)\exp(\mathrm{i}\alpha), \tag{8a}$$

$$v(x,z)=\overline{v}(\theta)\exp(2\mathrm{i}\alpha), \tag{8b}$$

where $\theta=x-\overline{x}$, $\alpha=\frac{\xi}{2}(x-\overline{x})+\overline{\alpha}$, $\mathrm{d}\overline{x}/\mathrm{d}z=\xi$, $\mathrm{d}\overline{\alpha}/\mathrm{d}z=\xi^2/4$, $\mathrm{d}\xi/\mathrm{d}z=0$.

Suppose now that small perturbations exist, then the evolution equations (4) become

$$\mathrm{i}\frac{\partial w}{\partial z}+\frac{\partial^2 w}{\partial x^2}-w+w^*v=\sigma P, \tag{9a}$$

$$2\mathrm{i}\frac{\partial v}{\partial z}+\frac{\partial^2 v}{\partial x^2}-\gamma v+w^2=\sigma Q, \tag{9b}$$

where, $\sigma\ll 1$, is introduced to keep track of the order of the perturbation. Following Kaup [9], the general equations (2) can be developed into the singular perturbation series

$$w=\eta\overline{w}(\theta)\exp(\mathrm{i}\alpha)+\sigma w_1+\ldots=w_0+\sum_{i=1}^{\infty}\sigma^i w_i, \tag{10a}$$

$$v=\eta\overline{v}(\theta)\exp(\mathrm{i}2\alpha)+\sigma v_1+\ldots=v_0+\sum_{i=1}^{\infty}\sigma^i v_i, \tag{10b}$$

where the subscript i represents the order of the perturbation.

It is clear that the zeroth order is just the solution of Eqs. (5) which are for the unperturbed system. In the absence of perturbations then, the evolution of the parameters is simply

$$\eta=1,\quad \frac{\mathrm{d}\overline{x}}{\mathrm{d}z}=\xi,\quad \frac{\mathrm{d}\overline{\alpha}}{\mathrm{d}z}=\frac{\xi^2}{4},\quad \frac{\mathrm{d}\xi}{\mathrm{d}z}=0. \tag{11}$$

The introduction of the multiple scales, $\tau_n=\sigma^n z$ permits a spatial or time scale to be attached to each physical process. For example, from (11) it is already clear that the position $\overline{x}$ and the phase $\overline{\alpha}$ of the beam are changing directly with z and so have a leading scale τ_0, before any higher order scale is considered. On the other hand, for the amplitude variation η and the velocity ξ, the first order or higher order scales are more appropriate. In other words, the position and phase possess a fast dependence, whereas the other parameters vary on a slower scale. In summary of these points, η, ξ, $\overline{x}$, α can be written, therefore, as

$$\eta=\eta(\tau_1,\tau_2,\ldots),\qquad \xi=\xi(\tau_1,\tau_2,\ldots),$$

$$\overline{x}=\overline{x}_0(\tau_0)+\overline{x}_1(\tau_1,\tau_2,\ldots),\qquad \alpha=\overline{\alpha}_0(\tau_0)+\overline{\alpha}_1(\tau_1,\tau_2,\ldots), \tag{12}$$

σ^0: zeroth order;

$$\frac{\mathrm{d}\overline{x}_0}{\mathrm{d}\tau_0} = \xi, \quad \frac{\mathrm{d}\overline{\alpha}_0}{\mathrm{d}\tau_0} = \frac{\xi^2}{4}, \tag{13}$$

σ^1: first-order.

The first-order results can be written in the compact form

$$\mathrm{i}\frac{\partial \psi}{\partial z} + L\psi = F, \tag{14}$$

where

- $\psi = (\psi_w, \psi_w^*, \psi_v, \psi_v^*)^{\mathrm{T}}$
- $F = (F_w, -F_w^*, F_v, -F_v^*)^{\mathrm{T}}$
- $\psi_w = w_1 \exp(-\mathrm{i}\alpha), \quad \psi_v = v_1 \exp(-\mathrm{i}2\alpha)$
- $L = \begin{bmatrix} \left(\frac{\partial^2}{\partial\theta^2} - 1\right) & \overline{v} & \overline{w} & 0 \\ -\overline{v} & -\left(\frac{\partial^2}{\partial\theta^2} - 1\right) & 0 & -\overline{w} \\ \overline{w} & 0 & \frac{1}{2}\left(\frac{\partial^2}{\partial\theta^2} - \gamma\right) & 0 \\ 0 & -\overline{w} & 0 & -\frac{1}{2}\left(\frac{\partial^2}{\partial\theta^2} - \gamma\right) \end{bmatrix}$
- F_w, F_v are functions of P, Q, α, η, $\overline{w}$, $\overline{v}, \ldots$ that are yet to be specified.

The main idea now is, given some perturbation (or value of F), ψ can then be expanded in eigenstates of L, where the latter are defined through the eigenvalue equation

$$L\phi = \lambda\phi. \tag{15}$$

Obviously ϕ is a column vector and λ is an eigenstate. This part of the argument (15) is quite elementary. Indeed, the bound states have an eigenvalue equal to zero. Furthermore, these bound states have associated eigenvectors that are a linear combination of the functions

$$\phi_{\mathrm{e}} = \begin{pmatrix} \overline{w}/2 \\ -\overline{w}/2 \\ \overline{v} \\ -\overline{v} \end{pmatrix}, \quad \phi_{\mathrm{o}} = \begin{pmatrix} \dot{\overline{w}} \\ \dot{\overline{w}} \\ \dot{\overline{v}} \\ \dot{\overline{v}} \end{pmatrix}. \tag{16}$$

The eigenvectors (16) can be found by inspection by checking that $L\phi_{\mathrm{e}} = 0$ and $L\phi_{\mathrm{o}} = 0$. Here the subscript e or o denotes an even or odd part of the solution and the dot is the usual way to denote the first derivative with respect to θ.

It is interesting that there are also derivative states [6] defined as

$$Ld_{\mathrm{e}} = -\phi e, \qquad Ld_{\mathrm{o}} = \phi_{\mathrm{o}}, \tag{17}$$

where d_{o} is, for this problem,

$$d_{\mathrm{o}} = \begin{pmatrix} -\frac{1}{2}\theta\overline{w} \\ \frac{1}{2}\theta\overline{w} \\ -\theta\overline{v} \\ \theta\overline{v} \end{pmatrix} \tag{18}$$

and

$$d_e(\gamma) = -\frac{1}{4}\begin{pmatrix} \theta\dot{\overline{w}} + 2\overline{w} + (8-2\gamma)\partial w/\partial\gamma \\ \theta\dot{\overline{w}} + 2\overline{w} + (8-2\gamma)\partial w/\partial\gamma \\ \theta\dot{\overline{v}} + 2\overline{v} + (8-2\gamma)\partial v/\partial\gamma \\ \theta\dot{\overline{v}} + 2\overline{v} + (8-2\gamma)\partial v/\partial\gamma \end{pmatrix} = \begin{pmatrix} d_w \\ d_w \\ d_v \\ d_v \end{pmatrix} \tag{19}$$

and d_w, d_v are assumed to be even functions of θ.

Since w_1 and v_1 represent the radiative part of the solution, the expansion for ψ should contain the continuous eigenfunctions of L. Because of this, ψ is orthogonal to the four states ϕ_e, ϕ_o, d_e, d_o. The closure condition, then, is

$$\int_{-\infty}^{\infty} (F_w\phi_1 + F_w^*\phi_2 + F_v\phi_2 + F_v\phi_3 + F_v^*\phi_4)\,\mathrm{d}\theta = 0, \tag{20}$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are elements in the appropriate column vectors ϕ_e, ϕ_o, d_e, d_o. For example,

$$\phi_e = \begin{pmatrix} \overline{w}/2 \\ -\overline{w}/2 \\ \overline{v} \\ \overline{v} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \tag{21}$$

and Eq. (20) becomes

$$\int_{-\infty}^{\infty} \left(F_w\frac{\overline{w}}{2} - F_w^*\frac{\overline{w}}{2} + F_v\overline{v} - F_v^*\overline{v}\right)\mathrm{d}\theta = 0. \tag{22}$$

Now, to first order in σ,

$$F_w = P\exp(-\mathrm{i}\alpha) + \mathrm{i}\frac{\partial\overline{x}_1}{\partial\tau_1}\eta\dot{\overline{w}}$$

$$+\left(\eta\frac{\theta}{2}\frac{\partial\xi}{\partial\tau_1} - \eta\frac{\xi}{2}\frac{\partial\overline{x}_1}{\partial\tau_1} + \eta\frac{\partial\overline{\alpha}_1}{\partial\tau_1} - \mathrm{i}\frac{\partial\eta_1}{\partial\tau_1}\right)\overline{w} + \left(\frac{1-\eta}{\sigma}\right)\eta\overline{w}\,\overline{v}, \tag{23a}$$

$$F_v = \frac{Q}{2}\exp(-2\mathrm{i}\alpha) + \mathrm{i}\frac{\partial\overline{x}_1}{\partial\tau_1}\eta\dot{\overline{v}}$$

$$+\left(\eta\theta\frac{\partial\xi}{\partial\tau_1} - \eta\xi\frac{\partial\overline{x}_1}{\partial\tau_1} + 2\eta\frac{\partial\overline{\alpha}_1}{\partial\tau_1} - \mathrm{i}\frac{\partial\eta_1}{\partial\tau_1}\right)\overline{v} + \left(\frac{1-\eta}{2\sigma}\right)\eta\overline{w}^2. \tag{23b}$$

Hence,

$$F_w - F_w^* = 2\mathrm{i}\ \mathrm{Im}[P\exp(-\mathrm{i}\alpha)] + 2\mathrm{i}\eta\frac{\partial\overline{x}_1}{\partial\tau_1}\dot{\overline{w}} - 2\mathrm{i}\frac{\partial\eta}{\partial\tau_1}\overline{w}, \tag{24a}$$

$$F_v - F_v^* = \mathrm{i}\ \mathrm{Im}[Q\exp(-2\mathrm{i}\alpha)] + 2\mathrm{i}\eta\frac{\partial\overline{x}_1}{\partial\tau_1}\dot{\overline{v}} - 2\mathrm{i}\frac{\partial\eta}{\partial\tau_1}\overline{\mathrm{v}} \tag{24b}$$

and Eq. (22) becomes

$$\mathrm{i}\,\mathrm{Im}\int_{-\infty}^{\infty}[P\exp(-\mathrm{i}\alpha)\overline{w}+Q\exp(-2\mathrm{i}\alpha)\overline{v}]\,\mathrm{d}\theta$$

$$=\mathrm{i}\frac{\partial\eta}{\partial\tau_1}\int_{-\infty}^{\infty}\left(\overline{w}^2+2\overline{v}^2\right)\mathrm{d}\theta-\mathrm{i}\eta\frac{\partial\overline{x}}{\partial\tau_1}\int_{-\infty}^{\infty}\left(\overline{w}\,\dot{\overline{w}}+2\overline{v}\,\dot{\overline{v}}\right)\mathrm{d}\theta. \tag{25}$$

The last integral vanishes for a localised states. For ϕ_{e}, ϕ_{o}, respectively, therefore,

$$\frac{\partial\eta}{\partial\tau_1}\int_{-\infty}^{\infty}(\overline{w}^2+2\overline{v}^2)\mathrm{d}\theta=\mathrm{Im}\int_{-\infty}^{\infty}[P\exp(-\mathrm{i}\alpha)\overline{w}+Q\exp(-2\mathrm{i}\alpha)\overline{v}]\mathrm{d}\theta, \tag{26a}$$

$$-\eta\frac{\partial\xi_1}{\partial\tau_1}\int_{-\infty}^{\infty}(\theta\overline{w}\,\dot{\overline{w}}+2\theta\overline{v}\,\dot{\overline{v}})\mathrm{d}\theta$$

$$=\mathrm{Re}\int_{-\infty}^{\infty}[2P\exp(-\mathrm{i}\alpha)\dot{\overline{w}}+Q\exp\left(-2\mathrm{i}\alpha\right)\dot{\overline{v}}]\mathrm{d}\theta. \tag{26b}$$

The corresponding d_{e}, d_{o} cases are

$$-2\eta\left(\frac{\partial\overline{\alpha}_1}{\partial\tau_1}-\frac{\xi}{2}\frac{\partial\overline{x}_1}{\partial\tau_1}\right)\int_{-\infty}^{\infty}(\overline{w}d_w+2\overline{v}d_v)\mathrm{d}\theta$$

$$=\mathrm{Re}\int_{-\infty}^{\infty}[2P\exp(-\mathrm{i}\alpha)d_w+Q\exp(-2\mathrm{i}\alpha)d_v]\mathrm{d}\theta$$

$$+\frac{(1-\eta)}{\sigma}\eta\int_{-\infty}^{\infty}(2\overline{w}\,\overline{v}d_w+\overline{w}^2d_v)\mathrm{d}\theta, \tag{27a}$$

$$-\eta\frac{\partial\overline{x}_1}{\partial\tau_1}\int_{-\infty}^{\infty}(\theta\overline{w}\,\dot{\overline{w}}+2\theta\overline{v}\,\dot{\overline{v}})\mathrm{d}\theta$$

$$=\mathrm{Im}\int_{-\infty}^{\infty}[P\exp(-\mathrm{i}\alpha)\theta\overline{w}+Q\exp(-2\mathrm{i}\alpha)\theta\overline{v}]d\theta. \tag{27b}$$

The following useful quantities can now be defined

$$M_1=\int_{-\infty}^{\infty}\left(\overline{w}^2+2\overline{v}^2\right)\mathrm{d}\theta, \tag{28a}$$

$$M_2=-\int_{-\infty}^{\infty}\left(\theta\overline{w}\,\dot{\overline{w}}+2\theta\overline{v}\,\dot{\overline{v}}\right)\mathrm{d}\theta, \tag{28b}$$

$$M_3=-\int_{-\infty}^{\infty}\left(\overline{w}d_w+2\overline{v}d_v\right)\mathrm{d}\theta \tag{28c}$$

and

$$P\exp(-\mathrm{i}\alpha)P_{\mathrm{r}}+\mathrm{i}P_{\mathrm{i}}, \tag{29a}$$

$$Q\exp(-2\mathrm{i}\alpha)=Q_{\mathrm{r}}+\mathrm{i}Q_{\mathrm{i}}. \tag{29b}$$

Note that M_2 and M_3 can be simplified and expressed as a function of M_1, i.e.

$$M_3 = \frac{1}{4}\int_{-\infty}^{\infty}\left[\theta\overline{w}\,\dot{w} + 2\overline{w}^2 + (8-2\gamma)\overline{w}\frac{d\overline{w}}{d\gamma} + 2\theta\overline{v}\,\dot{v} + 4\overline{v}^2 + 2(8-2\gamma)\overline{v}\frac{\partial\overline{v}}{\partial\gamma}\right]\mathrm{d}\theta$$

$$= \frac{3}{8}M_1 + \left(1-\frac{\gamma}{4}\right)\frac{\partial\pi_1}{\partial\gamma}, \tag{30a}$$

$$M_2 = \frac{M_1}{2}. \tag{30b}$$

The net result is that the parameter evolution equations are

$$\frac{\mathrm{d}\eta}{\mathrm{d}z} = \frac{\sigma}{M_1}\int_{-\infty}^{\infty}(P_\mathrm{i}\overline{w} + Q_\mathrm{i}\overline{v})\mathrm{d}\theta, \tag{31a}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}z} = \frac{\sigma}{\eta M_2}\int_{-\infty}^{\infty}(2P_\mathrm{r}\dot{\overline{w}} + Q_\mathrm{r}\dot{\overline{v}})\mathrm{d}\theta, \tag{31b}$$

$$\frac{\mathrm{d}\overline{x}}{\mathrm{d}z} = \xi + \frac{\sigma}{\eta M_2}\int_{-\infty}^{\infty}(P_\mathrm{i}\theta\overline{w} + Q_\mathrm{i}\theta\overline{v})\mathrm{d}\theta, \tag{31c}$$

$$\frac{\mathrm{d}\overline{\alpha}}{\mathrm{d}z} = \frac{\xi^2}{4} + \frac{\sigma\xi}{2\eta M_2}\int_{-\infty}^{\infty}(P_\mathrm{i}\theta\overline{w} + Q_\mathrm{i}\theta\overline{v})\mathrm{d}\theta + \frac{\sigma}{2\eta \mathrm{M}_3}\int_{-\infty}^{\infty}[2\mathrm{P_r d_w} + \mathrm{Q_r d_v}]\mathrm{d}\theta$$

$$+\frac{(1-\eta)}{2M_3}\int_{-\infty}^{\infty}\left[2\overline{w}\,\overline{v}d_w + \overline{w}^2 d_v\right]\mathrm{d}\theta. \tag{31d}$$

3. Applications of the multiple-scales theory

3.1. Loss or fluctuating phase matching

One of the simplest effects to study is the linear loss introduced by the crystal. This is rather more interesting than it looks because it turns that fluctuating phase mismatch is entirely equivalent to a loss coefficient [10]. Now since quasi-phase-matching is needed to get as close to linear phase-matching as possible in order to keep the power down in potential $\chi^{(2)}$ devices then loss must be assessed from this point of view. In this case σP and σQ take simple forms and

$$\mathrm{i}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial x^2} - w + w^* v = \mathrm{i}\sigma\sigma_1 w, \tag{32a}$$

$$2\mathrm{i}\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} - \gamma v + w^2 = \mathrm{i}\sigma\sigma_2 v, \tag{32b}$$

where σ is a common factor and σ_1, σ_2 are the loss coefficients.

To illustrate this degradation mechanism, Fig. 1 shows the evolution of the normalised amplitude η for a lithium niobate crystal at a wavelength, for the fundamental, of $\lambda_w = 1.065\ \mu$m. The beam half-width is 15 μm and the loss coefficients are respectively 35 and 17 dB/m [10]. It is immediately apparent that the analytical and numerical solutions are in good agreement. It appears from this that the multiple scales approach is validated, at least for the loss mechanism. Further examples are needed, however, and these will be discussed below.

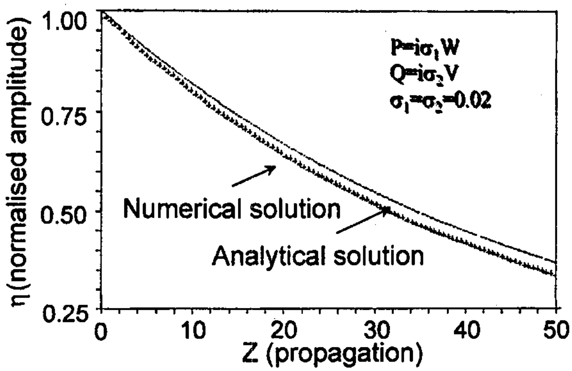

Fig. 1. Evolution of the normalised amplitude η. Data are given in the text.

3.2. Soliton interactions

Under this heading the old problem of soliton interaction modelled by an investigation of their overlap region will be set up and analysed. A schematic representation of the interaction is given in Fig. 2.

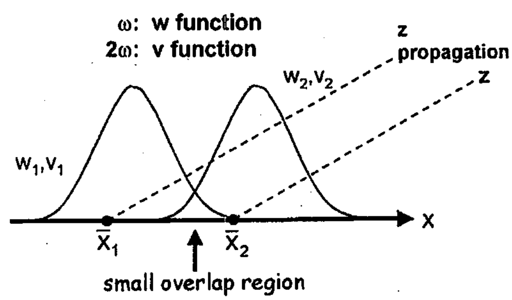

Fig. 2. Schematic representation of two interacting beams.

Using the method of Karpman and Solovev [11]

$$\text{fundamental}: \quad P_{1(2)} = -w^*_{1(2)} v_{2(1)}, \tag{32c}$$

$$\text{second harmonic}: \quad Q_{1(2)} = -w_1 w_2, \tag{32d}$$

where $P_{1(2)}$ and $Q_{1(2)}$ represent the perturbation of the first (second) solitary wave.

The perturbed equation for the first solitary wave (more imprecisely, a soliton)

$$\mathrm{i}\frac{\partial w_1}{\partial z} + \frac{\partial^2 w_1}{\partial x^2} - w_1 + w_1^* v_1 = -w_1^* v_2, \tag{33a}$$

$$2\mathrm{i}\frac{\partial v_1}{\partial z} + \frac{\partial^2 v_1}{\partial x^2} - \gamma v_1 + w_1^2 = -w_1 w_2. \tag{33b}$$

One outcome is a knowledge of the evolution of the beam centre, $\overline{x}_1$, given by

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}z} = \xi_1 + \frac{\sigma}{\eta M_2}\int_{-\infty}^{\infty}(P_1\theta\overline{w}_1 + Q_1\theta\overline{v}_1)\mathrm{d}\theta. \tag{34}$$

Assuming the following Gaussian profile for the stationary solutions:

$$w_i = A_i \exp\left[-\frac{(x-\overline{x}_i)^2}{\rho_i^2}\right] \exp\left[\mathrm{i}\frac{\xi_i}{2}(x-\overline{x}_i)+\mathrm{i}\overline{\alpha}_i\right], \tag{35a}$$

$$v_i = B_i \exp\left[-\frac{(x-\overline{x}_i)^2}{\rho_{(2+i)}^2}\right] \exp\left[\mathrm{i}\xi_i(x-\overline{x}_i)+2\mathrm{i}\overline{\alpha}_i\right], \tag{35b}$$

in which $i = 1, 2$ and $\overline{x}_i$, ξ, $\overline{\alpha}_i$, ρ_i depend upon the propagation direction z, leads to

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}z} = \xi_1 - \frac{\sqrt{\pi}}{\eta_1 M_2}[\alpha_1 f_1 \exp(r_1)\sin(S_1) + \alpha_2 f_2 \exp(r_2)\sin(S_2)], \tag{36}$$

where, for example,

$$\alpha_1 = \frac{A_1^2 A_2 \rho_1 \rho_2}{\sqrt{2\rho_4^2+\rho_1^2}},$$

$$f_2 = \frac{\rho_1^2\rho_4^2}{2\rho_4^2+\rho_1^2}\left[\frac{\xi_1}{2} - \frac{\xi_2}{z} - \frac{2\overline{x}_1}{\rho_1^2} - \frac{\overline{x}_2}{\rho_4^2} + \overline{x}_1\right],$$

$$S_2 = \frac{\overline{x}_1\rho_3^2\rho_2^2 + \overline{x}_1\rho_1^2\rho_2^2 + \overline{\alpha}_2\rho_1^2\rho_3^2}{\rho_3^2\rho_1^2 + \rho_3^2\rho_2^2 + \rho_1^2\rho_2^2}\left[\frac{\xi_1}{2} - \frac{\xi_2}{2} - \frac{\xi_1\overline{x}_1}{2} + \frac{\xi_2\overline{x}_2}{2} + \overline{\alpha}_1 - \overline{\alpha}_2\right].$$

Figure 3a shows a direct numerical simulation of two interacting beams and their behaviour is plotted on the (x, z) plane in Fig. 3b. As expected, the multiple

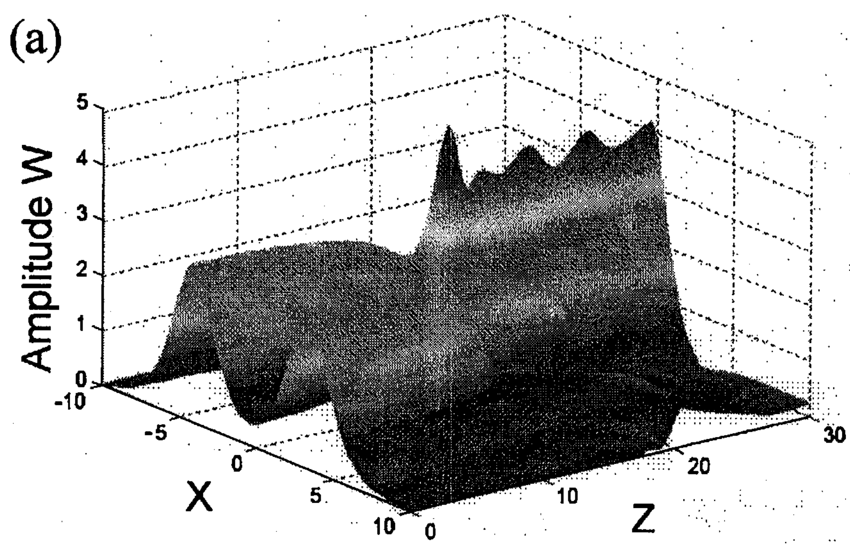

Fig. 3(a). Numerical simulation of the evolution of the fundamental beam at phase-matching. Note the break-up of the multiple scale method after the fusion point. Units: $x \approx 15$ μm, $z \approx 2.7$ mm.

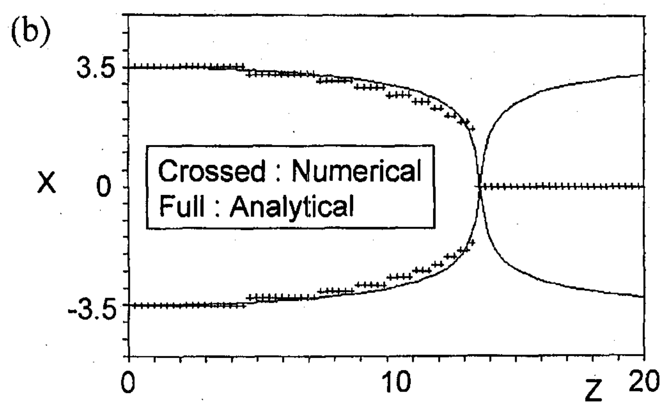

Fig. 3(b). Evolution of the beam centres for the above case.

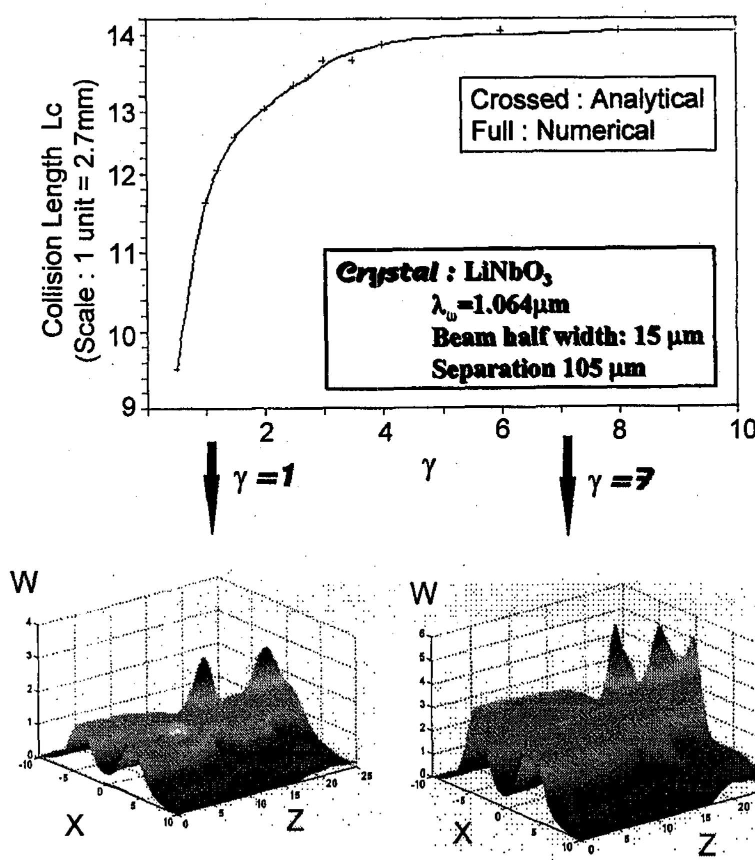

Fig. 4. Relation between the collision length L_c and the phase-matching coefficient γ. The units for the 3D inserts are $x \approx 15$ μm, $z \approx 2.7$ mm.

scales approach fails to give a description in the fusion region where the overlap between the two solitons is no longer small. Here higher order perturbations cannot therefore be neglected and radiative effects become important.

Figure 4 shows the relation between the collision length L_c and the phase matching coefficient γ. Again, good agreement between the two approaches is found for a large range of values γ. Note that for a high value of γ, the collision length L_c reaches a saturation value. It is interesting that this value corresponds to the collision length for two $\chi^{(3)}$ (or Kerr) solitons.

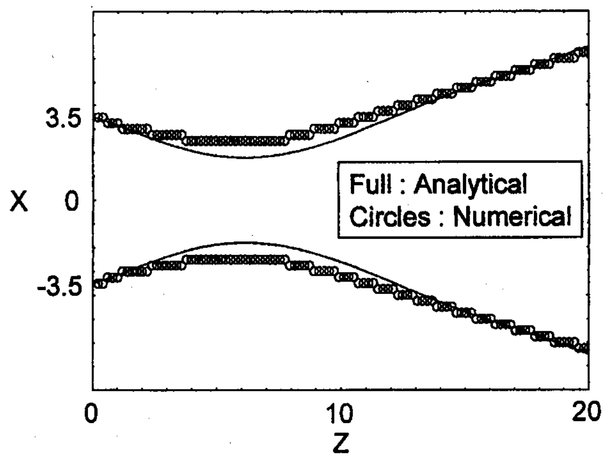

Fig. 5. Evolution, at phase-matching, of the beam centres when the initial phase difference is π. The other parameters are given in the text. Units: $x \approx 15$ μm, $z \approx 2.7$ mm.

Figure 5 shows a case where the two solitons have an initial phase difference of π. Initial attraction is due to the small initial velocity $\xi_{1,2} = 0.3$. Here the overlap between the solitons stays relatively small compared to the width of the solitons themselves and therefore good agreement is found for all values of z.

3.3. Soliton interaction in lossy media

One of the advantages of the approach given in this paper is the ability to deal with arbitrary small perturbations. It is then possible to mix two or more effects — providing that the perturbation term stays small. For example, in the case of interaction in lossy media, the perturbation of the first soliton is

$$P_1 = w_1^* v_2 + i\sigma_1 w_1, \tag{37a}$$

$$Q_1 = w_1 w_2 + i\sigma_2 v_1. \tag{37b}$$

Figure 6 shows the evolution of the beam centres (analytical) for both lossless and lossy media. The seemingly peculiar fact that the loss reduces the collision length can be qualitatively explained by the broadening of the beams (due to the loss) and therefore the bigger overlap. The bigger overlap is responsible, in turn, for an earlier stronger attraction force between the two solitons.

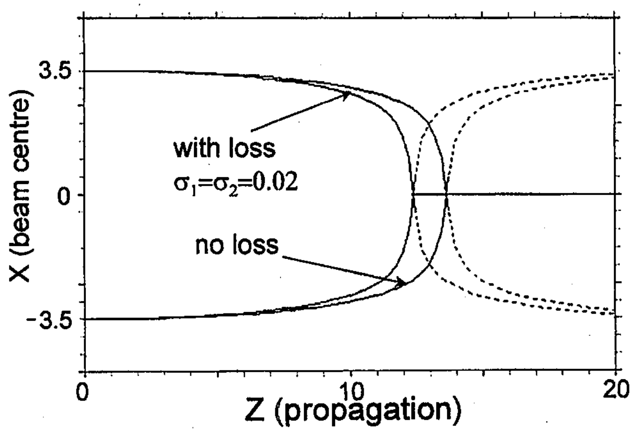

Fig. 6. Evolution, at phase-matching, of the beam centres in the case of ideal and lossy media. Trajectories are obtained analytically. Units: $x \approx 15$ μm, $z \approx 2.7$ mm.

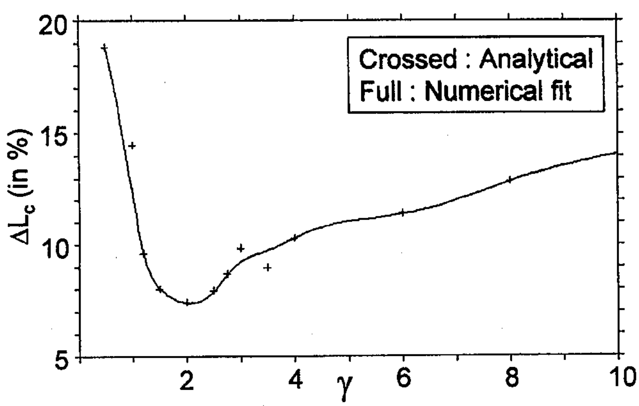

Fig. 7. Evolution of ΔL_c (see definition in the text) with the phase-matching coefficient γ.

The multiple scales approach can be used to investigate the evolution of $\Delta L_c = [L_c(\text{loss}) - L_c(\text{ideal})]/L_c(\text{ideal})$ (in %) with the phase matching coefficient γ. As seen in Fig. 7, this difference can be quite important for small γ, for high second-harmonic content in the wave structure.

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