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ON THE DIPOLE-INDUCED RESONANCE MOTION IN A PAUL TRAP

C.M. NICULAE*

Bucharest University, Department of Physics
POB MG11, 76900 Bucharest-Magurele, Romania

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The problem of resonant motion induced by the dipole component of the antisymmetric potential function in a radiofrequency Paul trap is investigated here. We show that, at low-level additional rf-voltage, the line-shape and the increasing rate of its amplitude, for the ion-excitation phenomenon, can be derived from the dipole approximation of the antisymmetric term in the potential energy. This approximation does not take into account the observed line-shift.

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1. Introduction

An increasing number of high-precision experiments performed on charged particles trapped both in Paul or Penning traps, has proved that such kinds of devices are valuable tools for precision spectroscopy, time and frequency standards and for experiments in quantum optics. Many of these experiments, but particularly the frequency standards using stored ions in a Paul trap, need an electronic loop to control the ion number. This is usually done by monitoring the image currents induced in the trap electrodes by the motion of the ion cloud [1–3]. Some of the active schemes use an rf pulse tuned to the secular frequency ω_z and applied across the caps of the trap. Under these circumstances the ion cloud exhibits an unbounded motion known as ion-excitation phenomenon. At the end of the rf exciting pulse, a receiver may watch the free decay of the ion cloud oscillation damped by the buffer gas or other effects. If the rf pulse is too long all the ions are lost in the excitation process. Since in such a resonant motion the ion cloud manifests itself as almost like a single charge, we can estimate the timescale of the phenomenon from the single-ion motion analysis [4].

*e-mail: cnic@fpce4.fizica.unibuc.ro

2. Theoretical background

The single-ion motion into a Paul trap exhibits two fundamental frequencies ω_r and ω_z (known as secular frequencies) corresponding to the r and z axis, respectively. The main idea of inducing ion-excitation phenomenon is to apply an additional ac voltage across the end caps of the trap tuned at the secular frequency ω_z (Fig. 1). Under these circumstances, the amplitude of an ion cloud increases in the z direction, and after some time, all the stored ions are lost by hitting one of the trap electrodes. This unbounded ion-motion on the z direction defines the ion-excitation phenomenon. In applications, it is of importance to control the number of the stored ions, since the intensity of the fluorescence light spread by the ion cloud depends on this number, for example. There are two electronic ways to detect the ion number in the trap both based on monitoring the image currents induced into a resonant circuit applied across the end caps of the trap. The resonance frequency of this circuit is just the secular frequency of the single-ion motion in the z direction ω_z.

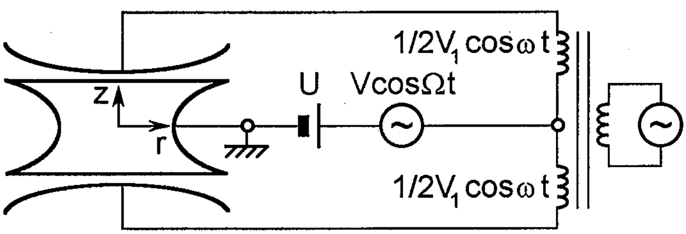

Fig. 1. The principle scheme of inducing ion-excitation phenomenon.

In the passive schemes, the image currents induced into the resonant circuit are monitored by a receiver tuned at the secular frequency ω_z. Of course, the magnitude of the received signal depends on the number of the stored ions.

The active schemes use rf pulses tuned to the secular frequency ω_z. In this case, the duration of such a pulse must not exceed a certain value in order that the number of stored ions should be conserved. During the time elapsed between two consecutive pulses a receiver watches the image currents induced by the free decay of the ion cloud oscillation damped mainly by the buffer gas.

The additional voltage (of amplitude V_1 and frequency ω) applied across the end caps of the trap causes the appearance of an antisymmetric term of the potential function [5] which may be expressed in cylindrical coordinates as

$$\Phi(r,z,t) = (U + V\cos\Omega t)\psi_s(r,z) + \frac{V_1\cos(\omega t+\varphi)}{2}\psi_a(r,z), \tag{1}$$

where the spatial parts satisfy the boundary conditions: $\psi_s(r,z) = 1$ and $\psi_a(r,z) = \pm 1$ at the end caps, and $\psi_s(r,z) = \psi_a(r,z) = 0$ at the ring electrode, respectively. The symmetric spatial part of the potential has in cylindrical coordinates, the well-known expression

$$\psi_s(r,z) = (2z^2 - r^2)/4z_0^2, \tag{2}$$

while the antisymmetric potential may be chosen as

$$\psi_a(r,z) = \sum_{k=1}^{n} A_k H_k(r,z), \qquad k = 1,3,5,\ldots \tag{3}$$

where $H_k(r,z)$ are the usual harmonic polynomials. In the pursuing numerical example we considered the next harmonic polynomials

$$\begin{aligned}
&H_1(r,z) = z/z_0,\\
&H_3(r,z) = z(-2z^2+3r^2)/z_0^3,\\
&H_5(r,z) = z(8z^4-40z^2r^2+15r^4)/z_0^5,\\
&H_7(r,z) = z(-16z^6+168z^4r^2-210z^2r^4+35r^6)/z_0^7,\\
&H_9(r,z) = z(128z^8-2304z^6r^2+6048z^4r^4-336z^2r^6+315r^8)/z_0^9.
\end{aligned} \tag{4}$$

For instance, in the ninth order approximation of the antisymmetric potential we used $A_1 = 0.8021$, $A_3 = -0.1018$, $A_5 = -0.001320$, $A_7 = -0.0003319$, and $A_9 = -0.3811\times 10^{-5}$.

On the basis of the potential function (1) the equations of motion for a single-particle of mass m and charge q may be easily derived. The simplest way to investigate the ion-excitation phenomenon is to consider ion motions along the z axis only. The equation of motion along the z axis can be expressed in dimensionless coordinates as

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2} + [a_z - 2q_z\cos(2\tau)]\,\zeta = q_z v_1 \cos(2\alpha\tau) P(\zeta), \tag{5}$$

where $\zeta = z/z_0$, and $2z_0$ denote the inner distance between end caps [4]. The trapping conditions are given by $q_z = -2qV/\left(mz_0^2\Omega^2\right)$ and $a_z = -2q_zU/V$, where the voltages U and V stand for the dc and ac parts of the trapping voltage, respectively. The level of the ion-excitation is set by the ratio $v_1 = V_1/V$. The time t and the driving frequency ω are scaled to the trap driving frequency Ω by $\tau = \Omega t/2$ and $\alpha = \omega/\Omega$, respectively. Obviously, the polynomial function contains only even terms in ζ, i.e., $P(\zeta) = A_1 + A_3\zeta^2 + A_5\zeta^4 + \ldots$, where the A's coefficients specify the approximation. In the dipole approximation, for instance $P(\zeta) = A_1$.

It is easily seen that an ion released from the trap center with zero kinetic energy moves on the z axis only. Throughout this paper the initial conditions we considered are $\zeta(0) = 0$, $(\mathrm{d}\zeta/\mathrm{d}\tau)(0) = 0$ and zero initial phase, i.e., $\phi = 0$.

The numerical analysis of Eq. (5) showed that, at certain instants, the ion position reaches a local maximum of its departure from the origin. Throughout this paper, the ζ points, where $\mathrm{d}\zeta/\mathrm{d}\tau$ vanishes, are called amplitudes. In the numerical procedure, a set of subsequent increasing amplitudes was sorted out. These data were used to derive the main features of the resonant motion [4, 6].

When the additional voltage is tuned in the proximity of the secular frequency ω_z, the amplitude of the ion motion increases until a maximum amplitude $|\zeta_{\max}|$ value is reached. A graph of $|\zeta_{\max}|$ versus the corresponding frequency ω gives the line-shape of the ion-excitation phenomenon. At low values of v_1, the line-shape may be approximated as

$$|\zeta_{\max}| \cong A(q_z, a_z)v_1/|w - w_c|, \tag{6}$$

where $w = \omega/\omega_z$ and $w_c = \omega_c/\omega_z$ (the observed line center ω_c generally differs from the secular frequency ω_z). The parameter $A(q_z, a_z)$ may be interpreted as a linewidth.

When the driving frequency ω is tuned at the center of the above spectral line ω_c, the amplitude of the ion oscillation $|\zeta_c|$ increases roughly linearly with time as

$$|\zeta_c| \cong R(q_z, a_z)v_1 N, \tag{7}$$

where $n = \Omega t/2\pi$ counts the period number elapsed from the initial moment. Furthermore, the numerical analysis shows that the two parameters $A(q_z, a_z)$ and $R(q_z, a_z)$ in (6) and (7) are related by

$$R(q_z, a_z) \cong A(q_z, a_z)\pi(\omega_z/\Omega). \tag{8}$$

Therefore, the knowledge of one of these parameters will suffice for applications.

The parameter $R(q_z, a_z)$ may be related with the mean-power (averaged over one period) $P = \langle m\dot{z}\ddot{z}\rangle$ absorbed by the ion. Indeed, assuming that the ion moves according to $z \cong (z_0 R v_1 \Omega/2\pi)t\sin(\omega_c t)$ we found that

$$P \cong (m\omega_c/4\pi)(z_0 R(q_z, a_z)v_1\Omega)^2. \tag{9}$$

Moreover, the ion maximum lifetime in such a resonant phenomenon is given by

$$\tau[\mu\text{s}] = \{R(q_z, a_z)v_1 f[\text{MHz}]\}^{-1}, \tag{10}$$

where f is the driving frequency $\Omega/2\pi$. In the derivation of Eq. (10) we used Eq. (7). Obviously, the scaled amplitude $|\zeta_c|$ cannot be greater than 1, i.e., $|\zeta_c| = |z_c|/z_0 \leq 1$, since the only significant motion of the ion belongs to the interior of the trap. This requirement reads for the z axis as $z < z_0$. Consequently, the maximum duration of such a motion τ is given by $1 \cong R(q_z, a_z)v_1\Omega\tau/2\pi$. Substituting $f = \Omega/2\pi$ in the former relation immediately follows Eq. (10).

Due to its importance in applications we give here an approximation of the parameter R, useful in the range $q_z < 0.4$

$$R(q_z, a_z) \cong K(q_z)/(\omega_z/\Omega), \tag{11}$$

where $K(q) = 0.7378q^3 + 0.071687q^2 + 0.64624q - 0.0016147$.

In the next section we will prove the dependences (6) and (7), and also the relation (8).

3. The resonance line-shape in the dipole approximation

We will show that the particular line-shape (6) is mainly determined by the dipole approximation of the antisymmetric potential. In the derivation we looked for those properties of the formal solution responsible for the specific features of the resonance phenomenon in study. In this approximation the ion motion on the z axis is given by

$$\mathrm{d}^2\zeta/\mathrm{d}\tau^2 + (a_z - 2q_z\cos(2\tau))\zeta = p\cos(2\alpha\tau), \tag{12}$$

where p may be expressed as $p \cong 0.8q_z v_1$, because in the dipole approximation $P(\zeta) \cong A_1 \cong 0.8$. By changing the variable ζ to $u = \zeta/p$ in (12) one obtains $\mathrm{d}^2u/\mathrm{d}\tau^2 + (a_z - 2q_z\cos(2\tau))u = \cos(2\alpha\tau)$. This remarkable scaling property tells us that the amplitudes of the ion motion, whatever they are, increase as p increases, and determine the linear dependence of $|\zeta_{\mathrm{max}}|$ and $|\zeta_{\mathrm{c}}|$ on v_1 in (6) and (7).

In the above-mentioned initial conditions, Eq. (12) allows for the even solution only (i.e., $\zeta(-\tau) = \zeta(t)$). As in the classical analysis of the Mathieu equation, we choose as a solution of (12) the expression

$$\zeta(\tau) = \sum_{k=-\infty}^{\infty} \{a_k \cos[(2k+s)\tau] + b_k \cos[(2k+2\alpha)\tau]\}, \tag{13}$$

where s is a parameter similar to the quantity $2\omega_z/\Omega$ [7]. Obviously, at low level of the driving voltage ($v_1 \to 0$), we expect that $s \to 2\omega_z/\Omega$.

A simple analysis shows that the a's amplitudes satisfy

$$q_z(a_{k-1} + a_{k+1}) + \left[(2k+s)^2 - a_z\right] a_k = 0, \tag{14}$$

where $k = 0, \pm 1, \pm 2, \ldots$ The homogeneous system (14) is well known from the theory of Mathieu equation, and it allows for a non trivial solution only if its determinant vanishes. When the point (q_z, a_z) belongs to the stability domain of the Mathieu equation, the value of s, resulting from the zero determinant condition, defines a unique value $s = s(q_z, a_z)$ related to the secular frequency by $s = 2\omega_z/\Omega$. Consequently, in the dipole approximation there is now frequency shift of the resonance motion from the secular frequency (i.e., in the dipole approximation $w_{\mathrm{c}} = 1$).

For the b's amplitudes we found

$$q(b_{k-1} + b_{k+1}) + \left[(2k+2\alpha)^2 - a\right] b_k = -p\delta_{0k}, \tag{15}$$

where δ_{0k} is the Kronecker delta function. The initial condition $\zeta(0) = 0$ implies

$$\sum_k a_k + \sum_k b_k = 0 \tag{16}$$

meaning a global relation between the a's and b's amplitudes.

It must be emphasized that the scaling properties of (15) imply that all the b's should be proportional to p (i.e., to v_1).

We now consider an equivalent form of the even solution given by

$$\zeta(\tau) = \sum_{k=-\infty}^{\infty} \{A_k[\cos(2k+s)\tau - \cos(2\alpha\tau)] + B_k \cos(2k+2\alpha)\tau\}. \tag{17}$$

It is easily seen that the terms $c_k = A_k[\cos(2k+s)\tau - \cos(2\alpha\tau)]$ and $d_k = B_k \cos(2k+2\alpha)\tau$ satisfy the differential equations

$$\ddot{c}_k + (2k+s)^2 c_k = f_k \cos(2\alpha\tau), \tag{18a}$$

$$\ddot{d}_k + (2k+2\alpha)^2 d_k = 0, \tag{18b}$$

where $f_k = A_k[4\alpha^2 - (2k+s)^2]$. Thus, Eq. (18a) describes a harmonic oscillator driven by a sinusoidal force.

We assume now that, near the resonance, the term c_k dominates the other terms. In such a hypothesis, the initial conditions $\zeta(0) = 0$ and $(\mathrm{d}\zeta/\mathrm{d}\tau)(0) = 0$ become $c_k(0) = 0$ and $\dot{c}_k(0) = 0$, respectively.

The analysis now focuses on the term c_0 describing the resonance phenomenon near the secular frequency ω_z. A simple calculus gives the following expression for this term:

$$|c_0(w,\tau)| = \frac{v_1 h_0}{(\omega_z/\Omega)^2}\frac{|\sin(w+w_c)\omega_z\tau/\Omega|}{|w+w_c|}\frac{|\sin(w-w_c)\omega_z\tau/\Omega|}{|w-w_c|}. \tag{19}$$

We admitted (using the above scaling property) that $f_k \propto v_1$, and hence h_0 is a constant. The solution (19) is the well-known expression describing the beats between two oscillations of quasi-equal frequencies.

The maximum values of the motion described by (19), near the secular frequency, are given by

$$|c_{0\,\max}(x,\tau)| = \frac{h_0}{2(\omega_z/\Omega)^2 w_c}\frac{v_1}{|w-w_c|}, \tag{20}$$

which is similar to Eq. (6). Hence the resonant term c_0 dominates the other terms in (17).

Since we assumed $w = w_c$ in the calculation of R, an analogue reasoning gives

$$|c_{0c}| = \frac{\pi h_0}{2(\omega_z/\Omega)w_c}v_1 N, \tag{21}$$

where we used $N = \tau/\pi$. Note that the value of the ratio between the constant factors in (20) and (21) is $\pi(\omega_z/\Omega)$ as in the approximate Eq. (8). Since Eqs. (20) and (21) explain Eqs. (6) and (7), we have reasons to believe that the above hypothesis holds true, i.e., the resonant term dominates the other terms when the driving frequency ω is tuned close to a resonant frequency ω_c.

4. Conclusions

Our analysis shows that the dipole approximation of the antisymmetric potential determines the most of the features of the ion motion in the resonance phenomenon investigated here. The main exception refers to the observed line shift caused by the higher order terms of $P(\zeta)$. It is of interest that all the features of the resonance are determined by the parameter $R(q_z, a_z)$ only.

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