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BRAIDS FOR TWIST KNOTS

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A standardized writing of the words for braid closures of the twist knots allows one to write braid words of twist knots for general values of crossing number.

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1. Introduction

The braid closures are useful for representing knots [1–3]. Vogel [4], using type-II Reidemeister moves, has given a new proof that each knot can be represented by a braid closure. Only a few tables of the braid words for knots have been published [2]. Jones [2] has given braid words for all knots with up to 10 crossings, or nodes, requiring braid index up to 6. The crossing number C, or crossing index of a knot, is the smallest number of crossings that occur in any projection of the knot. The braid index of a knot, or link, is the minimum number of strings in a braid needed to represent the knot, or link, by the braid closure. In the tabulation of Jones [2], the braid word expressions have not been optimized yet. While no method has been proposed so far [1, 2] to find the most economic presentation of braids for knots, there appears a possibility to standardize systematically words for the simple knots.

The $(2, C)$ torus knots have the crossing number C odd. The next simple knots are the twist knots which have a plectonemic region containing $C-2$ crossings interwoven like in the $(2, C-2)$ torus knot, and are locked by an interlock with two crossings [5–7]. We refer to the classical knot designation and the knot diagrams of Rolfsen [5]. Previously [1, 3], a braid generator b_n was defined so that it yielded a negative crossing. We define the braid generator b_n so that it produces a positive crossing [2, 8–10]. As in Ref [2] we write the braid word by a shorthand notation: in a braid word a factor n is written for the braid generator b_n, and $n^{\pm p}$ for $b_n^{\pm p}$ with a natural p. The braid words, $w(C_i)$, with $i = 1, 2, 3$, for twist knots, can be standardized by the use [2, 3, 8–12] of blocks $A(n)$ and $D(n)$, with natural $n > 1$:

$$A(n) = 23 \cdots (n-2)(n-1)n, \qquad D(n) = n(n-1)(n-2) \cdots 321. \tag{1.1}$$

The Conway notation [13], as indicated in Rolfsen diagrams [5], will follow, after the braid word and a comma, for each knot.

2. Twist knots with odd number of crossings

In the classical knot labeling [1–6], if the crossing number C is odd, the $(2, C)$ torus knot is labeled C_1. The first type twist knot which has in a plectonemic region $C-2$ crossings interwoven like in the $(2, C-2)$ torus knot, and is locked by an interlock containing two crossings [5–7], is labeled C_2, and has all its C crossings of the same sign [5–7]. The second type twist knot which is locked by an interlock of three crossings [5], is labeled C_3.

If $C = 2n+1$, with natural n, the braid word for the $(2, C)$ torus knot with positive crossings is [2, 3]:

$$w(C_1) = w((2n+1)_1) = 1^{2n+1}, \qquad 2n+1, \tag{2.1}$$

and, in the case of negative crossings, is

$$w(C_1-) = w((2n+1)_1-) = 1^{-(2n+1)}, \qquad -(2n+1). \tag{2.2}$$

For the twist knots C_2 with low odd crossing numbers the braid words can be written [2] as

$$\begin{aligned} w(5_2) &= 12^3 12^{-1}, && 32, \\ w(7_2) &= 123^3 213^{-1}2^{-1}, && 52, \\ w(9_2) &= 1234^3 3214^{-1}3^{-1}2^{-1}, && 72, \\ w(11_2) &= 12345^3 43215^{-1}4^{-1}3^{-1}2^{-1}, && 92, \end{aligned} \tag{2.3}$$

and, by induction on the number of crossings, for natural $n > 1$,

$$w(C_2) = w((2n+1)_2) = 1A(n)nD(n)(A(n))^{-1}, \qquad (2n-1)2. \tag{2.4}$$

Since the length of each of the three blocks $A(n), (A(n))^{-1}$, and $D(n)$, depends linearly on the number of crossings, an increase in the crossing number C by two, i.e. an increase in the number n by one, enlarges the length of the braid word by three.

The braids for the second type twist knots C_3 with odd number C of crossings, including the three crossings of the interlock, have braid index three, and the braid words are [2, 3]:

$$\begin{aligned} w(7_3) &= 12^{-1}1^4 2^2, && 43, \\ w(9_3) &= 12^{-1}1^6 2^2, && 63, \\ w(11_3) &= 12^{-1}1^8 2^2, && 83, \end{aligned} \tag{2.5}$$

and, for natural $n > 2$,

$$w(C_3) = w((2n+1)_3) = 12^{-1}1^{2n-2}2^2, \qquad (2n-2)3. \tag{2.6}$$

The braid words for knots 7_3 and 9_3 conform in crossing signs to the respective knot diagrams of Rolfsen [5, 13, 14]. Knots equivalent to braids of index three have been illustrated also by diagrams of Akutsu et al. [3]. Perko [15], in his diagrams of knots with crossing number 11, labeled the twist knots by labels which differ from the standard of Rolfsen notation [5, 13, 14, 16] used here.

3. Twist knots with even number of crossings

If the crossing number C of knot is even, the first type twist knot which has in a plectonemic region $C-2$ crossings interwoven like in a torus knot, and has an interlock containing two crossings of the other sign [5–7], is labeled C_1. The second type twist knot, locked by an interlock with three crossings [5], is labeled C_2, and the next type twist knot, locked by an interlock with four crossings [5], is labeled C_3.

The braid words for the twist knots C_1, with low even crossing numbers, can be written [2] as

$$\begin{aligned}
&w(4_1) = 12^{-1}12^{-1}, && 22,\\
&w(6_1) = 123^{-1}212^{-1}3^{-1}, && 42,\\
&w(8_1) = 1234^{-1}3213^{-1}2^{-1}4^{-1}, && 62,\\
&w(10_1) = 12345^{-1}43214^{-1}3^{-1}2^{-1}5^{-1}, && 82,\\
&w(12_1) = 123456^{-1}543215^{-1}4^{-1}3^{-1}2^{-1}6^{-1}, && 10\,2,
\end{aligned} \tag{3.1}$$

and, by induction on the number of crossings, for natural $n > 2$,

$$w(C_1) = w((2n)_1) = 1A(n-1)n^{-1}D(n-1)(A(n-1))^{-1}n^{-1}, \quad (2n-2)2. \tag{3.2}$$

Since the length of each of the three blocks $A(n-1), (A(n-1))^{-1}$, and $D(n-1)$, depends linearly on the number of crossings, an increase in the crossing number C by two, i.e. an increase in the number n by one, enlarges the length of the braid word by three. Knot 6_1 is called the *stevedore knot* [16].

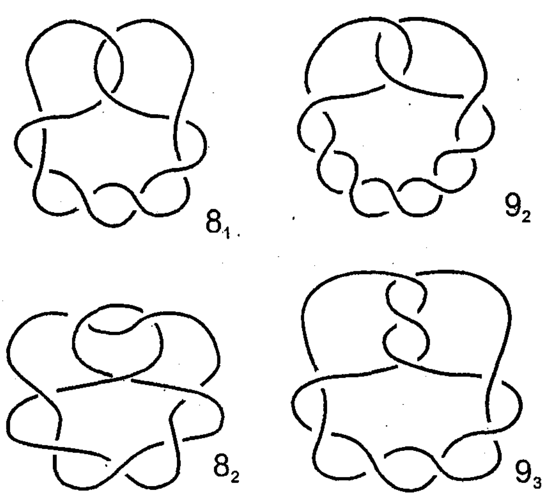

Fig. 1. The interlock in the twist knots 8_1 and 9_2 contains two nodes, in the twist knots 8_2 and 9_3 contains three nodes [5, 13, 14, 16, 17].

The braids for the second type twist knots C_2, with even crossing numbers C, see Fig. 1, have the braid index three, and the braid words are [2, 3]:

$$
\begin{aligned}
&w(6_2) = 12^{-1}1^32^{-1}, && 312,\\
&w(8_2) = 12^{-1}1^52^{-1}, && 512,\\
&w(10_2) = 12^{-1}1^72^{-1}, && 712,\\
&w(12_2) = 12^{-1}1^92^{-1}, && 912,
\end{aligned}
\tag{3.3}
$$

and, for natural $n > 2$,

$$w(C_2) = w((2n)_2) = 12^{-1}1^{2n-3}2^{-1}, \qquad (2n-3)12. \tag{3.4}$$

The braid words for the third type twist knots C_3, with low even crossing numbers $C = 2n$, with $n > 3$, can be written [2] as

$$
\begin{aligned}
&w(8_3) = 1234^{-1}3^{-1}213^{-1}2^{-1}4^{-1}, && 44,\\
&w(10_3) = 12345^{-1}4^{-1}3214^{-1}3^{-1}2^{-1}5^{-1}, && 64,\\
&w(12_3) \doteq 123456^{-1}5^{-1}43215^{-1}4^{-1}3^{-1}2^{-1}6^{-1}, && 84,
\end{aligned}
\tag{3.5}
$$

and, by induction on the number of crossings, for natural $n > 3$,

$$
\begin{aligned}
&w(C_3) = w((2n)_3)\\
&= 1A(n-1)n^{-1}(n-1)^{-1}D(n-2)(A(n-1))^{-1}n^{-1}, \quad (2n-4)4.
\end{aligned}
\tag{3.6}
$$

The amphicheiral [5, 14, 18, 19] knot 6_3 is not a twist knot, it has Conway notation label [13] of a form distinct from that of the amphicheiral knot 8_3, and has a distinct form of braid word [2, 3]:

$$w(6_3) = 12^{-1}1^22^{-2}, \qquad 2112. \tag{3.7}$$

While knot 6_3 has three positive and three negative crossings, knot 8_3 and knots C_3 with even $C > 6$ have $C - 4$ crossings of one sign and four crossings of the other sign [5].

4. Concluding remarks

Except for the 7_3 and 9_3 knots, the braid words are recorded here for the mirror image of Rolfsen [5] knot diagrams: our braid words will conform in crossing signs to Rolfsen diagrams if the sign of power exponents of the braid generators will be reversed, or the braid generators will be interpreted according to Akutsu et al. [3].

Expressions of braid words can be transformed according to the rules known for braid moves [1–4, 8–12]. The standardized expression of braid words for twist knots, $w(C_i)$, with $i = 1, 2, 3$, enables one to see the regular pattern of the words and thus to write the braid words of twist knots for general values of crossing number.

The twist knots 4_1 and 6_2 are synthesized in organic stereochemistry. In laboratory syntheses of long organic molecular chains, in particular of polyether ladders, a "hook and ladder" approach to the synthesis of the twist knot 4_1 is discussed in Ref. [20]. In the processive recombination of a supercoiled DNA ring [17], the successive products appear in a sequence with an increasing number of supercoil crossings [17, 21–24]. In the electron microscope, the supercoils of a DNA ring can be seen. The processive recombination of the DNA rings by a resolvase enzyme [17, 21–24] yields: the two-component link 2_1^2, the twist knot 4_1, the two-component link 5_1^2, and the twist knot 6_2. The braid words for the two-component links, or catenanes, are:

$$\begin{aligned} w(2_1^2) &= 1^2, && 2, \\ w(5_1^2) &= 12^{-1}1^2 2^{-1}, && 212. \end{aligned} \tag{4.1}$$

A review [22] of the mechanism of DNA recombination asserts: "Biologically, the two most important families of knots and catenanes are the torus and twist families", and presents the illustrative examples.

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