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LOCAL GAUGE AND MAGNETIC TRANSLATION GROUPS

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The magnetic translation group was introduced as a set of operators $T(\boldsymbol{R}) = \exp[-\mathrm{i}\boldsymbol{R}\cdot(p-e\boldsymbol{A}/c)/\hbar]$. However, these operators commute with the Hamiltonian for an electron in a periodic potential and a uniform magnetic field if the vector potential $\boldsymbol{A}$ (the gauge) is chosen in a symmetric way. It is showed that a local gauge field $\boldsymbol{A}_{\boldsymbol{R}}(\boldsymbol{r})$ on a crystal lattice leads to operators, which commute with the Hamiltonian for any (global) gauge field $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r})$. Such choice of the local gauge determines a factor system $\omega(\boldsymbol{R},\boldsymbol{R}') = T(\boldsymbol{R})T(\boldsymbol{R}')T(\boldsymbol{R}+\boldsymbol{R}')^{-1}$, which depends on a global gauge only. Moreover, for any potential $\boldsymbol{A}$ a commutator $T(\boldsymbol{R})T(\boldsymbol{R}')T(\boldsymbol{R})^{-1}T(\boldsymbol{R}')^{-1}$ depends only on the magnetic field and not on the gauge.

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1. Introduction

The behavior of electrons in crystalline (periodic) potentials in the presence of a constant (external) magnetic field has been studied since the thirties in many papers (see, e.g., [1, 2]). In the sixties Brown [3] and Zak [4, 5] independently introduced and investigated the so-called *magnetic translation groups.* Their results have been lately applied to a problem of the quantum Hall effect [6, 7] and relations with the Weyl–Heisenberg group have been also studied [8]. Some interesting results have been lately presented by Geyler and Popov [9].

The Hamiltonian for an electron in a periodic potential $V(\boldsymbol{r})$ and a uniform magnetic field (described by the vector potential $\boldsymbol{A}$) is given as [3, 4]

$$\mathcal{H} = \pi^2/2m + V(r), \quad \text{where} \quad \pi = p + eA/c \tag{1}$$

is the (vector) operator of the kinetic momentum. Brown introduced a projective representation of the translation group in the following form:

$$T(\boldsymbol{R}) = \exp[-\mathrm{i}(p - e\boldsymbol{A}/c)\cdot \boldsymbol{R}/\hbar]. \tag{2}$$

These operators commute with Hamiltonian (1) if the vector potential $\boldsymbol{A}$ fulfills the following condition (see [3, 4]):

$$\partial A_j/\partial x_k + \partial A_k/\partial x_j = 0 \text{ for } j,k = 1,2,3. \tag{3}$$

This relation holds, for example, for the gauge $\boldsymbol{A}(\boldsymbol{r}) = (\boldsymbol{H}\times\boldsymbol{r})/2$, which was used by Brown and Zak [3, 4]. On the other hand, this condition is not satisfied by the Landau gauge $\boldsymbol{A}(\boldsymbol{r}) = [-x_2H_3, 0, 0]$ (for $\boldsymbol{H} = [0, 0, H_3]$), which is used in many papers (e.g. [10]).

The aim of this paper is to find such a gauge $\boldsymbol{A}'$ that: (i) $\nabla \times \boldsymbol{A}' = -\boldsymbol{H}$; (ii) operators $T'(\boldsymbol{R}) = \exp[-\mathrm{i}(\boldsymbol{p} + \boldsymbol{A}'e/c)\cdot\boldsymbol{R}/\hbar]$ commute with Hamiltonian (1); (iii) a factor system $\omega(\boldsymbol{R}, \boldsymbol{R}') = T(\boldsymbol{R})T(\boldsymbol{R}')T(\boldsymbol{R}+\boldsymbol{R}')^{-1}$ depends only on a global gauge $\boldsymbol{A}$, which defines the magnetic field (and the generalized momentum π in (1)). It should be underlined that only the constant magnetic field $\boldsymbol{H}$ is considered. It occurs that these conditions are satisfied by a *local* gauge, i.e. an actual form of $\boldsymbol{A}'(\boldsymbol{r})$ depends on a lattice vector $\boldsymbol{R}$.

2. Solution

For the constant magnetic field $\boldsymbol{H} = [H_1, H_2, H_3]$ the vector potential (gauge) $\boldsymbol{A} = [A_1, A_2, A_3]$ can be chosen as a linear function of $\boldsymbol{r} = [x_1, x_2, x_3]$ and can be written as

$$A_j = \sum_{k=1}^{3} a_{jk}x_k, \quad \text{with} \quad a_{jk} \in \mathcal{R}, \quad a_{jj} = 0. \tag{4}$$

Introducing a matrix $\mathcal{A} = (a_{jk})$ it can be written as $\boldsymbol{A} = \mathcal{A}\boldsymbol{r}$. Therefore, the magnetic field $\boldsymbol{H}$ is expressed by the matrix elements a_{jk} as follows:

$$H_j = -\sum_{k,l=1}^{3} \varepsilon_{jkl}a_{kl}, \tag{5}$$

which means that $\boldsymbol{H}$ is related to antisymmetrized matrix $\mathcal{A}$.

The definition (2) of operators $T(\boldsymbol{R})$ can be rewritten as

$$T(\boldsymbol{R}) = \exp\left(-\mathrm{i}\pi'\cdot\boldsymbol{R}/\hbar\right), \tag{6}$$

where

$$\pi' = \boldsymbol{p} + e\boldsymbol{A}'/c \quad \text{and} \quad \boldsymbol{A}'(\boldsymbol{r}) = -\boldsymbol{A}(\boldsymbol{r}),$$

therefore $\nabla\times\boldsymbol{A}' = -\nabla\times\boldsymbol{A} = -\boldsymbol{H}$. Let us consider $\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{H}\times\boldsymbol{r} = \mathcal{A}^{\mathrm{T}}\boldsymbol{r}$, i.e.

$$A'_j = A_j - \sum_{k,l=1}^{3} \varepsilon_{jkl}H_kx_l = \sum_{k=1}^{3} a_{kj}x_k. \tag{7}$$

It is easy to note that $\nabla\times\boldsymbol{A}' = -\boldsymbol{H}$. For example, assuming the Landau gauge (for $\boldsymbol{H} = [0, 0, H_3]$) to be given as $\boldsymbol{A} = [-x_2H_3, 0, 0]$ one obtains $\boldsymbol{A}' = [0, -x_1H_3, 0]$, whereas the symmetric gauge $(\boldsymbol{H}\times\boldsymbol{r})/2$ yields $\boldsymbol{A}' = -(\boldsymbol{H}\times\boldsymbol{r})/2$. We have to check whether the operators $T(\boldsymbol{R})$ determined by the gauge (7) commute with the Hamiltonian. It suffices to calculate commutators $[\pi_j, \pi'_k]$ for $j, k = 1, 2, 3$, for which one obtains

$$[\pi_j, \pi'_k] = [-\mathrm{i}\hbar\partial_j + eA_j/c, -\mathrm{i}\hbar\partial_k + eA'_k/c] = -\mathrm{i}e\hbar([\partial_j, A'_k] + [A_j, \partial_k])/c = 0.$$

To find a factor system of the above determined (projective) representations one has to calculate commutators $[X_j\pi'_j, X'_k\pi'_k]$ ($\boldsymbol{R} = [X_1, X_2, X_3]$):

$$\begin{aligned}[X_j\pi_j, X'_k\pi'_k] &= X_jX'_k[-\mathrm{i}\hbar\partial_j + eA'_j/c, -\mathrm{i}\hbar\partial_k + eA'_k/c] \\ &= -\mathrm{i}X_jX'_ke\hbar/c(\partial_j\ A'_k - \partial_k\ A'_j) = -\mathrm{i}X_jX'_ke\hbar/c(a_{jk} - a_{kj}).\end{aligned}$$

On the other hand, we have

$$(\boldsymbol{R} \times \boldsymbol{R}') \cdot \boldsymbol{H} = -\sum_{l=1}^{3} \left(\sum_{j,k=1}^{3} \varepsilon_{ljk} X_j X'_k \right) \left(\sum_{p,q=1}^{3} \varepsilon_{lpq} a_{pq} \right)$$
$$= -\sum_{j,k=1}^{3} X_j X'_k (a_{jk} - a_{kj})$$

and, therefore

$$T(\boldsymbol{R})T(\boldsymbol{R}') = T(\boldsymbol{R} + \boldsymbol{R}') \exp[-\mathrm{i}(e/\hbar c)(\boldsymbol{R} \times \boldsymbol{R}') \cdot \boldsymbol{H}/2].$$

It is interesting that this result does not depend on the chosen gauge $\boldsymbol{A}$ (and $\boldsymbol{A}'$).

Let us consider now a *local* gauge determined as $\boldsymbol{A}'_{\boldsymbol{R}}(\boldsymbol{r}) = \mathcal{A}(\boldsymbol{r} + \boldsymbol{R}/2)$, i.e.

$$(A'_{\boldsymbol{R}})_j(\boldsymbol{r}) = \sum_{k=1}^{3} a_{kj}(x_k + X_k/2). \tag{8}$$

Similar, but a bit more tedious, calculations lead to the following results:

1. $\nabla \times \boldsymbol{A}'_{\boldsymbol{R}} = -\boldsymbol{H}$.
2. Operators $T(\boldsymbol{R})$ determined by this gauge commute with Hamiltonian (1).
3. The projective representation $T(\boldsymbol{R})$ is characterized by a factor system

$$\omega(\boldsymbol{R}, \boldsymbol{R}') = T(\boldsymbol{R})T(\boldsymbol{R}')T(\boldsymbol{R} + \boldsymbol{R}')^{-1} = \exp[(-\mathrm{i}e/\hbar c)(\boldsymbol{R} \cdot \boldsymbol{A}(\boldsymbol{R}'))]. \tag{9}$$

Note that a scalar product in the last equation can be also written as a bilinear form

$$\boldsymbol{R} \cdot \boldsymbol{A}(\boldsymbol{R}') = \sum_{j,k=1}^{3} a_{jk} X_j X'_k = \boldsymbol{R} \cdot \mathcal{A}\, \boldsymbol{R}' = \mathcal{A}^{\mathrm{T}} \boldsymbol{R} \cdot \boldsymbol{R}', \tag{10}$$

i.e. it is fully determined by the matrix $\mathcal{A}$. It should be stressed that calculating $\omega(\boldsymbol{R}, \boldsymbol{R}')$ one has to take into account that

$$T(\boldsymbol{R} + \boldsymbol{R}') = \exp[-\mathrm{i}(p + e\boldsymbol{A}'_{\boldsymbol{R}+\boldsymbol{R}'}/c) \cdot (\boldsymbol{R} + \boldsymbol{R}')/\hbar].$$

The obtained factors (9) allow to find the commutator $T(\boldsymbol{R})T(\boldsymbol{R}')T(\boldsymbol{R})^{-1}T(\boldsymbol{R}')^{-1}$ as

$$\omega(\boldsymbol{R}, \boldsymbol{R}')\omega(\boldsymbol{R}', \boldsymbol{R})^{-1} = \exp[(-\mathrm{i}e/\hbar c)\boldsymbol{R} \cdot (\mathcal{A} - \mathcal{A}^{\mathrm{T}})\boldsymbol{R}')].$$

Applying (10) this result can be also written as

$$\exp\left[(-\mathrm{i}e/\hbar c)\left(\sum_{j,k=1}^{3} X_j X'_k (a_{jk} - a_{kj})\right)\right].$$

Hence, we have shown that

$$T(\boldsymbol{R})T(\boldsymbol{R}')T(\boldsymbol{R})^{-1}T(\boldsymbol{R}')^{-1} = \exp[(\mathrm{i}e/\hbar c)\Phi],$$

where $\Phi = (\boldsymbol{R} \times \boldsymbol{R}') \cdot \boldsymbol{H}$ is a magnetic flux through the cell spanned by lattice vectors $\boldsymbol{R}$ and $\boldsymbol{R}'$.

3. Conclusion

It was shown that a projective representation (6) of the translation group determined by the *local* gauge (8) has the following properties:

1. Operators $T(\boldsymbol{R})$ commute with Hamiltonian (1).
2. The factor system $\omega(\boldsymbol{R}, \boldsymbol{R}')$ depends on the *global* gauge $\boldsymbol{A}$, i.e. on the matrix $\mathcal{A}$.
3. The commutator of (magnetic) translations $T(\boldsymbol{R})$ and $T(\boldsymbol{R}')$ depends only on the magnetic field $\boldsymbol{H}$.

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