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MULTICOMPONENT NUMBER SYSTEMS

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We introduce three types of the four-component number systems which are constructed by joining the complex, binary and dual two-component numbers. We study their algebraic properties and rewrite the Euler and Moivre formulas for them. The most general multicomponent number system joining the complex, binary dual numbers is the eight-component number system, for which we determine the algebraic properties and the generalized Euler and Moivre formulas. Some applications of the multicomponent number systems in differential and integral calculus, which are of physical relevance, are also presented.

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1. Introduction

In the previous paper [1] we have presented the relations between the different two-component number systems and the corresponding space–time symmetry group. A two-component number Z can be written in the form

$$Z = a + \epsilon b,$$

where ϵ is an "imaginary unit" [1]. There are different types of two-component number systems which form an algebraic ring. The algebraic structure of a ring demands that the product Z_3 of any two numbers, Z_1, Z_2, from the ring

$$Z_1 Z_2 = (a + \epsilon b)(c + \epsilon d) = Z_3$$

belonged to this ring as well. Therefore, $\epsilon^2 = \beta + \epsilon\gamma; \beta, \gamma \in \mathrm{R}$. An important mathematical theorem states that any possible two-component number can be reduced to one of the following three types [3]. The "imaginary" unit will be denoted by i, λ and μ, respectively, each of the specific cases listed below:

(i) the complex numbers $Z_\mathrm{c} = a + \epsilon b$, where $\epsilon^2 = \mathrm{i}^2 = -1$,

(ii) the binary numbers $Z_\mathrm{b} = a + \epsilon b$, where $\epsilon^2 = \lambda^2 = 1$ and

(iii) the dual numbers $Z_\mathrm{d} = a + \epsilon b$, where $\epsilon^2 = \mu^2 = 0$.

The criterion for the membership to one of these two-component systems is given by the sign of the expression $Q = (\beta + \gamma^2/4)$. If Q is negative, positive or zero, then we get the complex, binary or dual numbers, respectively.

The usual operations done for the complex numbers, such as absolute value, goniometric form and Euler formula, have corresponding operations in other two-component number systems. For a number, $Z = a+\epsilon b$, from any of these three number systems its conjugate is given as $\overline{Z} = a-\epsilon b$ and its absolute value is defined as $|Z| = \sqrt{Z\overline{Z}}$. All Z can be expressed in the goniometric form $Z = |Z|\exp(\epsilon\phi)$, where $\phi = F(a/b)$, $\phi = \arctan(a/b)$ for the complex, $\phi = \arctan(a/b)$ for the binary and $\phi = (a/b)$ for the dual numbers, respectively.

Geometrically, the complex numbers are related to rotations and dilatations in the plane, whereas the binary numbers are related to the rotations and dilatations of Minkowski space–time [4]. Euler and Moivre formulas for the complex, binary and dual numbers are

$$\exp(\mathrm{i}\phi) = \cos\phi + \mathrm{i}\sin\phi, \quad \exp(n\mathrm{i}\phi) = (\cos\phi + \mathrm{i}\sin\phi)^n, \tag{1a}$$

$$\exp(\lambda\phi) = \cosh\phi + \lambda\sinh\phi, \quad \exp(n\lambda\phi) = (\cosh\phi + \lambda\cosh\phi)^n, \tag{1b}$$

$$\exp(\mu\phi) = 1 + \mu\phi, \quad \exp(n\mu\phi) = 1 + n\mu\phi, \tag{1c}$$

respectively. By means of Eqs. (1a), (1b) and (1c) we get the identities

$$\sin x = [\exp(\mathrm{i}x) - \exp(-\mathrm{i}x)]/2\mathrm{i}, \quad \cos x = [\exp(\mathrm{i}x) + \exp(-\mathrm{i}x)]/2, \tag{2a}$$

$$\sinh x = [\exp(\lambda x) - \exp(-\lambda x)]/2\lambda, \quad \cosh x = [\exp(\lambda x) + \exp(-\lambda x)]/2, \tag{2b}$$

$$x = [\exp(\mu x) - \exp(-\mu x)]/2\mu, \quad 1 = [\exp(\mu x) + \exp(-\mu x)]/2. \tag{2c}$$

Equation (2c) expresses also the variable x and real number 1 in the form containing only the combinations of exponential function. Therefore, a special function of the type $F(x) = P_1(x)P_2(\exp(x))$, where P_1 and P_2 stand for the polynomials, can be expressed as sum of exponential functions

$$F(x) = E_1(x) + E_2(x) + \ldots + E_n(x), \tag{3}$$

the general form of which is

$$E_i = [\exp(a + b\mu)]/(c + \mu d), \qquad i = 1, 2, 3, \ldots \qquad a, b, c, d \in \mathrm{R}.$$

The decomposition (3) makes it possible to perform the integration of the functions $F(x)$ simply as a sum of exponential functions. We have presented an elementary example of this procedure in [2]. When we want to apply the considered procedure for a larger class of functions of the type

$$F(x) = P_1(x)P_2(\sin x, \cos x)P_3(\sinh x, \cosh x)P_4(\exp(x)),$$

where P_1, P_2, P_3 and P_4 stand for the polynomials, we need an extended multicomponent system, whose introduction and analysis we shall next deal.

In what follows we present some elementary algebraic properties as well as the generalized Euler and Moivre formulas for each of these systems and show its possible use in the differential and integral calculus (DIC). It is obvious that in this short article one can only sketch the theory and application of the multicomponent numbers.

2. Four-component number systems

There are three four-component number systems which can be constructed by joining the complex, binary and dual two-component numbers:

(i) The four-component number system joining the complex and binary two-component numbers. The set of these numbers we will call as the complex–binary system S_{cb}. An element of S_{cb} can be written in the form

$$Z = e_1 a + e_2 b + e_3 c + e_4 d.$$

The basis elements of S_{cb} are: $e_1 = 1$, $e_2 = \mathrm{i}$, $e_3 = \lambda$ and $e_4 = p = \mathrm{i}\lambda$. If we demand that the multiplication of the basis units e_1, e_2, e_3 and e_4 should be commutative and associative, i.e. $e_i e_j = e_j e_i$ and $(e_i e_j)e_k = e_i(e_j e_k)$, $i, j, k = 1, 2, 3, 4$, then the multiplication scheme for the basis elements of S_{cb}

	1	i	λ	p
1	1	i	λ	p
i	i	-1	p	$-\lambda$
λ	λ	p	1	i
p	p	$-\lambda$	i	-1

The sum and difference of two numbers of S_{cb}, Z and Z', is

$$Z \pm Z' = (a \pm a') + \mathrm{i}(b \pm b') + \lambda(c \pm c') + p(d \pm d'),$$

respectively. Their product is again a complex–binary number Z'':

$$Z'' = ZZ' = (a + \mathrm{i}b + \lambda c + pd)(a' + \mathrm{i}b' + \lambda c' + pd')$$

$$= aa' - bb' + cc' - dd' + \mathrm{i}(ab' + ba' + cd' - dc')$$

$$+\lambda(ac' + ca' - bd' - db') + p(ad' + da' + cb' + bc').$$

The quotient R of two numbers, Z_1 and Z_2, $R = Z_1/Z_2$, can be rewritten in the form, in which the divisor of R is a real number

$$R = (Z_1\overline{Z}_2)Z_3/(Z_2\overline{Z}_2)Z_3,$$

where $Z_3 = a_2 - \mathrm{i}b_2 - \lambda c_2 + pd_2$ and $\overline{Z}_2 = a_2 - \mathrm{i}b_2 - \lambda c_2 - pd_2$.

The generalized Moivre formula for the complex–binary numbers turns out to be

$$\exp(n\mathrm{i} + m\lambda)x = \exp(n\mathrm{i}x)\exp(m\lambda x) = (\cos nx + \mathrm{i}\sin nx)$$

$$\times(\cosh mx + \lambda\sinh mx) = (\cos x + \mathrm{i}\sin x)^n(\cosh x + \lambda\sinh x)^m. \tag{4}$$

Equation (4) yields a series of interesting identities for the trigonometric and hyperbolic functions of multiple angles.

As an example for the use of the complex–binary numbers in DIC let us consider the derivative of the function

$$F(x) = \exp(\mathrm{i} + \lambda)x = \cos x\cosh x + \mathrm{i}\sin x\cos x + \lambda\cos x\sinh x$$

$$+p\sin x\sinh x = F_1 + \mathrm{i}F_2 + \lambda F_3 + pF_4. \tag{4a}$$

We get this derivative in a simple form

$$\mathrm{d}F/\mathrm{d}x = (\mathrm{i} + \lambda)\exp(\mathrm{i} + \lambda)x = (-\sin x\cosh x + \cos x\cosh x) + \mathrm{i}(\cos x\cosh x$$

$$+\sin x \sinh x) + \lambda(\cos x \cosh x - \sin x \cosh x) + p(\cos x \sinh x + \sin x \cosh x)$$

$$= \mathrm{d}F_1/\mathrm{d}x + \mathrm{i}\mathrm{d}F_2/\mathrm{d}x + \lambda \mathrm{d}F_3/\mathrm{d}x + p\mathrm{d}F_4/\mathrm{d}x. \tag{5}$$

Comparing each part on both sides of Eqs. (4a) and (5), we see that the derivatives of the component functions F_1, F_2, F_3 and F_4 represent the component functions of $\mathrm{d}F/\mathrm{d}x$. It can be easily verified that it holds

$$\int F(x)\mathrm{d}x = (\lambda + \mathrm{i})^{-1}F(x) = \int F_1\mathrm{d}x + \mathrm{i}\int F_2\mathrm{d}x + \lambda\int F_3\mathrm{d}x + p\int F_4\mathrm{d}x.$$

We see that the derivatives and integrals of the component functions can be obtained by simple multiplication of $F(x)$ by factor $(\mathrm{i}+\lambda)$ or $(\mathrm{i}+\lambda)^{-1}$, respectively.

(ii) The complex–dual number system S_{cd} joining the complex and dual numbers. An element of complex–dual numbers has the form

$$Z_{\mathrm{cd}} = a + \mathrm{i}b + \mu c + qd.$$

The basis elements of system S_{bs} obey the multiplication scheme

	1	i	μ	q
1	1	i	μ	q
i	i	-1	q	$-\mu$
μ	μ	q	0	0
q	q	μ	0	0

The product of two numbers Z and Z' is

$$ZZ' = (a + \mathrm{i}b + \mu c + qd)(a' + \mathrm{i}b' + \mu c' + qd')$$

$$= aa' - bb' + \mathrm{i}(ab' + a'b) + \mu(ac' + a'c - b'd - bd') + q(ad' + c'b + cb' + a'd).$$

The quotient Z_3 of Z_1 and Z_2, $Z_3 = Z_1/Z_2$, can be written in the form with real divisor

$$Z_3 = (Z_1\overline{Z}_2)Z_4/(Z_2\overline{Z}_2)Z_4,$$

where $Z_4 = A - qB$ and $A = a_2^2 + b_2^2, B = 2c_2b_2$.

The generalized Moivre formula for the complex–dual numbers is

$$\exp(n\mu x + m\mathrm{i}x) = (1 + \mu x)^n(\cos x + \mathrm{i}\sin x)^m$$

$$= (1 + n\mu x)(\cos mx + \mathrm{i}\sin mz).$$

As an elementary example for the application of the complex–dual numbers in DIC we consider the derivative of the function

$$F(x) = \exp(\mu x)\exp(\mathrm{i}x) = F_1 + \mathrm{i}F_2 + \mu F_3 + qF_4, \tag{6}$$

where $F_1 = \cos x$, $F_2 = \sin x$, $F_3 = x\cos x$ and $F_4 = x\sin x$. Its derivative is

$$\mathrm{d}F/\mathrm{d}x = (\mu + \mathrm{i})F(x) = \mu\cos x + q\sin x + \mathrm{i}\cos x - \sin x + qx\cos x - \mu x\sin x$$

$$-\sin x + \mathrm{i}\cos x + \mu(\cos x - x\sin x) + q(\sin x + x\cos x)$$

$$= \mathrm{d}F_1/\mathrm{d}x + \mathrm{i}\mathrm{d}F_2/\mathrm{d}x + \mu \mathrm{d}F_3/\mathrm{d}x + q\mathrm{d}F_4/\mathrm{d}x.$$

We get the higher derivatives of the functions F_1, F_2, F_3 and F_4 in a similar fashion

$$\mathrm{d}F^n/\mathrm{d}x^n = (\mu + \mathrm{i})^nF(x).$$

Accordingly, we have for the integration

$$\int F(x)\mathrm{d}x = (\mu + \mathrm{i})^{-1}F(x).$$

(iii) The four-component number system S_{bd} joining the binary and dual numbers. Its basis elements are $(1, \lambda, \mu, u)$ with the multiplication scheme

	1	λ	μ	u
1	1	λ	μ	u
λ	λ	1	u	μ
μ	μ	u	0	0
u	u	μ	0	0

The product of two binary–dual numbers, Z, Z' is

$$Z_1 = ZZ' = (a + \lambda b + \mu c + ud)(a' + \lambda b' + \mu c' + ud') = aa' + bb'$$

$$+\lambda(ab' + ba') + u(ad' + cb + a'd + bc') + \mu(ac' + bd' + a'c + b'd).$$

The quotient Z_4 of Z_1 and Z_2 with the real divisor is

$$Z_4 = (Z_1\overline{Z}_2)Z_3/(Z_2\overline{Z}_2)Z_3,$$

where $\overline{Z}_2 = a_2 - \lambda b - \mu c - ud$ and $Z_3 = (a^2 - b^2) + \mu(2bd) + (2cb)u$.

The generalized Moivre formula for the binary–dual number has the form

$$\exp(n\lambda x + m\mu x) = (1 + \mu x)^m(\cosh x + \lambda \cosh x)^n$$

$$= (1 + m\mu x)(\cosh nx + \lambda \sinh nx).$$

As an elementary example of the application of the binary–dual numbers in DIC we consider the derivative of the function

$$F(x) = \exp(\mu + \lambda)x = (1 + \mu x)(\cosh x + \lambda \sinh x) = F_1 + \lambda F_2 + \mu F_3 + uF_4,$$

where the component functions are $F_1 = \cosh x$, $F_2 = \sinh x$, $F_3 = x \cosh x$ and $F_4 = x \sinh x$. It is easily to verify that

$$\mathrm{d}F/\mathrm{d}x = \mathrm{d}F_1/\mathrm{d}x + \lambda \mathrm{d}F_2/\mathrm{d}x + \mu \mathrm{d}F_3/\mathrm{d}x + u\mathrm{d}F_4/\mathrm{d}x.$$

In a similar procedure we find that

$$\mathrm{d}^n F/\mathrm{d}x^n = (\mu + \lambda)^{-n}F(x).$$

3. Eight-component number system

In this section we will deal with the most general eight-component number system joining the complex, binary and dual numbers S_{cbd}. Its element can be written in the form

$$Z = a + \mathrm{i}b + \lambda c + \mu d + ep + fq + gu + hv.$$

The complex–binary, complex–dual and binary–dual numbers represent the special cases of this number system. The basis units of an eight-component number are:

$1, \mathrm{i}, \lambda, \mu, p, q, u$, and v, and their multiplication scheme is

	1	i	λ	μ	p	q	u	v
1	1	i	λ	μ	p	q	u	v
i	i	-1	p	q	$-\lambda$	$-\mu$	v	$-u$
λ	λ	p	1	u	i	v	μ	q
μ	μ	q	u	0	v	0	0	0
p	p	$-\lambda$	i	v	-1	$-u$	0	0
q	$-q$	$-\mu$	v	0	$-u$	0	0	0
u	u	v	μ	0	q	0	0	0
v	v	$-u$	q	0	$-\mu$	0	0	0

The product of two numbers from S_{cbd}, Z and Z', is

$$ZZ' =$$

$$(a + \mathrm{i}b + \lambda c + \mu d + pe + gu + hv)(a' + \mathrm{i}b' + \lambda c' + \mu d' + pe' + qf' + ug' + vh')$$

$$= aa' - bb' + cc' - ee' + \mathrm{i}(ab' + ba' + ce' + ec') + \lambda(ac' - be' + ca' - eb')$$

$$+\mu(ad' - bf' + cg' + da' - fb' + gc') + p(ae' + bc + cb + ea')$$

$$+q(af + bd' + ch' + db' + fa' + hc') + u(ag' - bh' + cd' + dc' + ga' - hb')$$

$$+v(ah' + bg + cf + de' + ed' + fd' + gb' + ha').$$

For the quotient Z_3 of two numbers Z_1 and Z_2, with real divisor, we find

$$Z_3 = (Z_1 Z_2^*)Z_2^{**}/(A^2 + E^2)$$

with $Z_2^* = a_2 - \mathrm{i}b_2 - \lambda c_2 - \mu d_2 + pe_2 + qf_2 + ug_2 + vh_2$ and $Z_2^{**} = A - pE - qF - uG - vH$, where $A = a_2^2 + b_2^2 - c_2^2 + e_2^2$, $E = 2e_2a_2 - 2b_2c_2$, $F = 2a_2f_2 - 2b_2d_2$, $G = 2a_2g_2 - 2c_2d_2$ and $H = 2a_2h_2$.

The generalized Moivre formula for the eight-component number system has the form

$$\exp(n\mathrm{i}x + m\lambda x + l\mu x) = (\cos x + \mathrm{i}\sin x)^n(\cosh x + \lambda\sinh x)^m(1 + \mu x)^l$$

$$= (\cos nx + \mathrm{i}\sin nx)(\cosh mx + \lambda\sinh mx)(1 + l\mu x).$$

As an elementary example for the use of eight-component system in DIC let us consider the derivative of the function

$$F(x) = \exp(\mathrm{i} + \lambda + \mu)x = F_1 + \mathrm{i}F_2 + \lambda F_3 + \mu F_4 + pF_5 + qF_6 + uF_7 + vF_8,$$

where $F_1 = \cos x\cosh x$, $F_2 = \sin x\cosh x$, $F_2 = \cos x\sinh x$, $F_4 = \sin x\sinh x$, $F_5 = x\cos x\cosh x$, $F_6 = x\sin x\cosh x$, $F_7 = x\cos x\sinh x$ and $F_8 = x\sin x\sinh x$.

This derivative has the form

$$\mathrm{d}F/\mathrm{d}x = (\mathrm{i} + \lambda + \mu)\exp(\mathrm{i} + \lambda + \mu)x = [-\sin x\cosh x + \cos x\sinh x]$$

$$+\mathrm{i}(\cos x\cosh x + \sin x\sinh x) + \lambda[-\sin x\sinh x + \cos x\cosh x]$$

$$+\mu[\cos x\sinh x + \sin x\sinh x] + p[x\cos x\sinh x + \cos x\cosh x - x\cos x\cosh x]$$

$$+q[\sin x \cosh x + x \cos x \cosh x + x \sin x \sinh x]$$

$$+u[\cos x \cosh x - x \sin x \sinh x + x \cos x \cosh x]$$

$$+v[\sin x \sinh x + x \cos x \sinh x + x \sin x \cosh x].$$

We see that

$$\mathrm{d}F/\mathrm{d}x = \mathrm{d}F_1/\mathrm{d}x + \mathrm{i}\mathrm{d}F_2/\mathrm{d}x + \lambda \mathrm{d}F_3/\mathrm{d}x + \mu \mathrm{d}F_4/\mathrm{d}x + p\mathrm{d}F_5/\mathrm{d}x$$

$$+q\mathrm{d}F_6/\mathrm{d}x + u\mathrm{d}F_7/\mathrm{d}x + v\mathrm{d}F_8/\mathrm{d}x.$$

In a similar fashion we obtain

$$\int F(x)\mathrm{d}x = (\mathrm{i} + \lambda + \mu)^{-1} F(x).$$

4. Some applications of multicomponent number system in physics

From what has been said so far it follows that the developed formalism of multicomponent number systems can be used in mathematical physics in the following areas:

(i) In trigonometry, where the Euler and Moivre formulas can be also extended to multicomponent numbers. The most important case is the extension of these formulas to the complex–binary number system which leads to many interesting identities between the sines, cosines, hyperbolic sines and hyperbolic cosines of multiple angles.

(ii) In DIC, where generally the function of type

$$F(x) = P_1(x) P_2(\sin x, \cos x) P_3(\sinh x, \cosh x) P_4(\exp(x)),$$

where P_1, P_2, P_3 and P_4 are polynomials, can be, using Euler formula, expressed as a sum of the exponential functions

$$F(x) = E_1(x) + E_2(x) + \ldots + E_n(x),$$

the general form of which is

$$E_i = [\exp(a + bi + c\lambda + d\mu)]/(a' + b'i + c'\lambda + d'\mu), \qquad i = 1, 2, \ldots, n.$$

The derivatives and integrals of these exponential functions are simple and can be written in the form

$$\mathrm{d}E_i(x)/\mathrm{d}x = Z^q E_i(x),$$

where $Z = a + ib + \lambda c + \mu d$ and $q = n$ for n-th derivative and $q = -n$ for n-th integral of E_i. By means of the algebraic rules developed above we can write again the results of the derivation and integration in the form of a sum of exponential functions.

(iii) In the theory of differential equations, especially those of higher orders. A class of linear differential equations with constant coefficients has its solution in the ring of four-component numbers. As an example we take the differential equation occurring in the theory of coupled dual fields characterized by the potentials ψ and ψ'. The field equation for these potentials are [5, 6]:

$$\mathrm{d}^2\psi/\mathrm{d}r^2 + |k_0^2|\psi' = 0, \qquad \mathrm{d}^2\psi'/\mathrm{d}r^2 - |k_0^2|\psi = 0. \tag{7}$$

Eliminating ψ from Eq. (7) we get the differential equation

$$\mathrm{d}^4\psi/\mathrm{d}r^4 + k_0^4\psi = 0,$$

the solution of this equation is

$$\psi = \exp(k_0/\sqrt{2})(\mathrm{i}+\lambda)r = F_1 + \mathrm{i}F_2 + \lambda F_3 + pF_4,$$

where F_1, F_2, F_3 and F_4 are the component functions from Eq. (4a) with the argument $(k_0/\sqrt{2})r$. It is easily to verify that each of the functions F_1, F_2, F_3 and F_4 represents a particular solution of Eqs. (7). Its general solution is

$$\psi = c_1F_1 + c_2F_2 + c_3F_3 + c_4F_4,$$

where c_1, c_2, c_3 and c_4 are the integration constants. There is a lot of other differential equations of the physical relevance, the solutions of which represent the function of four- or eight-component numbers. Their detailed description would, however, exceed the scope of this article.

Finally, let us mention the question how unique are the examples of Secs. 2 and 3. To answer this question we suppose that the relations (1a), (1b) and (1c) are unique and that the usual operations of DIC can be applied also for the functions of multicomponent numbers. The products of the basic elements of the multicomponent number systems are commutative and associative (see the above multiplication schemes) so that the products of functions (1) do not depend on their order. Due to this fact we can calculate the derivatives and integrals of these product functions by the different ways. If we get equal results by these ways we may assume that these operations are unique. Consider as an example the derivative of the function $F(x) = \exp(\mathrm{i}+\lambda)x$. We can derive $F(x)$ in a direct way

$$\mathrm{d}F(x)/\mathrm{d}x = (\mathrm{i}+\lambda)\exp(\mathrm{i}+\lambda)x$$

or as a product of two functions $F_1(x) = \exp(\mathrm{i}x)$ and $F_2(x) = \exp(\lambda x)$

$$\mathrm{d}F(x)/\mathrm{d}x = [\mathrm{d}F_1(x)/\mathrm{d}x]F_2(x) + F_1(x)[\mathrm{d}F_2(x)/\mathrm{d}x] = (\mathrm{i}+\lambda)\exp(\mathrm{i}x+\lambda)x.$$

In both ways we obtain the same result. We can proceed in similar fashion with other examples getting always the equal results. Although it is not a rigorous proof of the uniqueness of the examples presented it represents a strong hint that they are unique.

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