NORMAL TERMS OF HAMILTONIAN SYSTEMS

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All the transformation matrices for a Hamiltonian with two and three degrees of freedom are found. They are calculated using algorithms for the linear normalization of Hamiltonian systems near the equilibrium point.

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1. Introduction

There are many possibilities of describing physical systems. Here we use Hamilton's formalism. This formalism is used in all the problems of celestial mechanics, in which the potential forces fulfill some general conditions. Here are some examples of such problems:

- The motion near any libration point in any restricted problem;
- The rotational motion of artificial and natural satellites with the condition that they are rigid bodies;
- The motion of the planets and asteroids.

Let us consider a Hamiltonian system, which has n degrees of freedom. Let

$$q = (q_1, \ldots, q_n), \quad p = (p_1, \ldots, p_n), \quad x^T = (q, p),$$

$$H = H(x)$$

denote canonical variables, and a time independent Hamiltonian. Let us introduce a matrix J, which consists of $2n$ rows and $2n$ columns.

$$J = -J^T = \begin{bmatrix} 0_n & E_n \\ -E_n & 0_n \end{bmatrix},$$

where 0_n and E_n are null and unit matrices, respectively. Hamilton's equations of motion may be written in the following way:

$$\frac{dx}{dt} = J \frac{\partial H}{\partial x}. \tag{1}$$

We assume that this system has an equilibrium position x^r, i.e.

$$\frac{\partial H}{\partial x}(x^r) = 0.$$

One can assume that $x^r = 0$. This can be achieved by using a canonical transformation — the linear translation

$$x = x^* + x^r$$

to move this point to zero. If one assumes the analyticity of the Hamiltonian function at this point one can get, by the expanding the Hamiltonian into MacLaurin's series about the canonical coordinates,

$$H = H_2 + H_3 + \cdots + H_m + \cdots, \tag{2}$$

where H_m is a homogeneous polynomial of order m in the canonical variables.

Equations (1) are, apart from a few particular cases, nonintegrable. Moreover there is no information about qualitative and quantitative solutions of this system. A very useful way to overcome these difficulties is a treatment called the normalization procedure. It is based on simplification of the terms in the Hamiltonian (2), using canonical transformations. After some steps of this procedure one can:

- establish the stability of the equilibrium position,
- find the approximate, analytical solution of equation (1),
- find the periodic and quasiperiodic families of the solutions,
- establish the initial conditions of the asymptotical solutions, which tend to the equilibrium position.

The aim of this paper is to formulate a general algorithm for the first step of the normalization procedure (called the linear normalization), which includes all possible Hamiltonian functions in the case of two and three degrees of freedom.

2. Normal forms

The whole normalization procedure depends very strongly on the H_2 term, so it is very important to simplify it as much as possible. It is the first step of this procedure. In order to do that let us take the Hamiltonian system with the function $H = H_2$ [1]. The H_2 term is of the second order in x, so one can write it in the following way:

$$H_2 = \frac{1}{2}x^T hx, \tag{3}$$

where

$$h = \frac{\partial^2 H}{\partial x^2}$$

is a symmetric $2n \times 2n$ matrix. Hamilton's equations of motion for this system are linear

$$\dot{x} = Jhx. \tag{4}$$

Let us now find the linear transformation

$$x = Ny, \tag{5}$$

which is the canonical one

$$N^T J N = J, \tag{6}$$

and transforms (3) into the normal form

$$H_2^* = \frac{1}{2} y^T h^* y. \tag{7}$$

Using the equations given above one gets the condition for the N matrix

$$N J h^* = J h N. \tag{8}$$

This means that the matrices Jh and Jh^* are similar to each other. Let us notice that the normal form (7) and N depend on the structure of the matrix Jh. The greater the simplification of Jh the more simple will be the resulted form of H_2. The most effective way to do that is to find the transformation, which transforms this matrix into Jordan's form (more general theory about Jordan's forms [2]). Generally this transformation does not have to be the canonical one, so we look for the matrix N in the following form:

$$N = AB,$$

where the matrix A transforms the matrix Jh into Jordan's form, and B completes A in order to fulfill the condition (6). Let us consider now the practical using of the process given, ahead of looking for the normal forms. We look for the matrices, which transform Jh into their simplest form. It is convenient to divide our problem into two parts. In the first part we consider all the Hamiltonian functions H_2, for which Jh can be diagonalized, and in the second part the other ones. This division is very important, because the process of looking for the normal forms in these two cases is different.

2.1. Case of simple elementary divisors

In this part we consider all those Hamiltonians, for which the matrix Jh can be transformed into a diagonal form.

2.1.1. Way of constructing the matrix

If e_k and e_{k+n} are eigenvectors of the matrix Jh:

$$Jhe_k = \lambda_k e_k,$$

$$Jhe_{k+n} = -\lambda_k e_{k+n}, \quad k = 1, \ldots, n,$$

then the matrix A is constructed in the following way:

$$A = [e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}]. \tag{9}$$

2.1.2. Structure of the matrix $A^T J A$

Let us put the term $N = AB$ into the condition (6)

$$B^T A^T J A B = J.$$

Let us consider the structure of the matrix $A^T J A$, which is interesting because of the way of construction of the matrix A. Let us denote

$$F = A^T J A.$$

J is a skew-symmetric matrix and due to this fact the matrix F is also skew-symmetric. Transforming both sides of the following equation:

$$e_k^T J^2 h e_l = e_k^T h J^2 e_l,$$

which is true for any e_j (because $J^2 = -E_{2n}$ and the matrix h is symmetric), one gets

$$e_k^T J^2 h e_l = \ J^T h^T e_k J e_l, \qquad e_k^T J J h e_l = -J h e_k J e_l.$$

e_k are eigenvectors of the matrix Jh, so one gets

$$e_k^T J \lambda_l e_l = -\lambda_k e_k^T J e_l, \qquad \lambda_l e_k^T J e_l = -\lambda_k e_k^T J e_l.$$

These equations may be written in the following way:

$$(\lambda_k + \lambda_l) f_{kl} = 0, \quad f_{kl} = e_k^T J e_l. \tag{10}$$

From this equation $f_{kl} = 0$ in the case when

$$\lambda_k \neq -\lambda_l.$$

Knowing the structure of A and the conclusion given above one can write

$$F = \begin{bmatrix} 0_n & F_n^* \\ -F_n^* & 0_n \end{bmatrix},$$

where F_n^* is a diagonal matrix, with elements $f_{kk}^* = e_k^T J e_{k+n}$. None of these elements is equal to zero, because

$$\det F = \det A^T \det J \det A = (\det A)^2 \neq 0,$$

since the matrix A consists of $2n$ eigenvectors (linearly independent).

2.1.3. Normalization matrix

The construction of the matrix B is the next step in the above algorithm. This matrix, multiplied by A, gives the symplectic transformation. Let us assume that $B = E_{2n}$. We have assumed nothing about the length of the vectors e_k and e_{k+n}, so it will be true that $\alpha_k e_k$ and $\alpha_{k+n} e_{k+n}$ are also eigenvectors of the matrix Jh. Let us write A in the following way:

$$A = [\alpha_1 e_1, \ldots, \alpha_k e_k, \ldots, \alpha_{k+n} e_{k+n}, \ldots, \alpha_{2n} e_{2n}] \tag{11}$$

and let $\alpha_k = \alpha_{k+n}$. Then the condition (6) transforms into the following equation:

$$\alpha_k^2 e_k^T J e_{k+n} = 1, \tag{12}$$

where $k = 1, \ldots, n$ (the left-hand side of this equation is not equal to zero, as shown earlier). Solving Eq. (12) one can find the α_k values. Then multiplying the k-th and the $(k+n)$-th vector by proper α_k values one gets the matrix N, which defines (by the relation $N^T h N = h^*$ following from Eqs. (6), (8)) the normal form of the Hamiltonian

$$H_2^* = \sum_{j=1}^n \lambda_j y_j y_{j+n}. \tag{13}$$

It is easy to notice this is the complex normal form, because generally λ_j is complex (and also x_j is complex for $j = 1, \ldots, 2n$). This form is much simpler and convenient for the next step of the normalization procedure than the real one. However, the Hamiltonian is real and after normalization we would like to get the

real normal form. Such a form may be obtained from Eq. (13) by a standard, linear canonical transformation Q^{-1} (Bruno [3]). This transformation is the reason for another condition, which the matrix N has to fulfill. This condition will be fulfilled if $e_k^T J e_{k+n} > 0$ for real λ_k or $\text{Im}(e_k^T J e_{k+n}) < 0$ (Im means here the imaginary part of the product) for imaginary λ_k. One has to exchange λ_k with λ_{k+n} and the k-th and $(k+n)$-th column of the matrix A, when any of the two products has the opposite sign. By doing that the normal form (13) is the complex notation of the real normal form [1, 4].

2.2. Multidimensional Jordan's boxes

Let us consider now the second part of the problem. We look for the matrix N in the following form: $N = AB$ (as before). Also as before the matrix A transforms Jh into Jordan's form and B completes A, in this way the transformation defining by N is the canonical one.

Let us introduce some notation. Let the symbol e denote the columns of the matrix A and the eigenvectors or adjoint vectors of the matrix Jh. The subscript of e indicates the position of the column of the matrix A. The superscript defines the connection of the adjoint vectors to the eigenvector with the same subscript. It is obvious that the eigenvectors of the Jh matrix have only one symbol — the subscript. Let us denote e_l as a vector connected to the eigenvalue λ_l, and e_{l+n} to the eigenvalue λ_l. In the case of the elementary divisor $\lambda^{2(l+s_l)}$ (there are $1 + 2s_l$ of adjoint vectors) we rewrite the last group of equations in another form. Let us assume that e_{l+n}^l is the $(s_l + 1)$-th adjoint vector, e_{l+n+1}^l is the $(s_l + 2)$-th one and so on. Thus

$$Jh e_{l+n} = e_{l+s_l}^l,$$

$$Jh e_{l+n+1}^l = e_{l+s_l+1}^l,$$

$$\cdots\cdots\cdots\cdots$$

$$Jh e_{l+n+s_l}^l = e_{l+2s_l}^l.$$

Finally A may be written as follows:

$$A = \left[e_1, \ldots, e_l, e_{l+1}^l, \ldots, e_{l+s_l}^l, \ldots, e_{l+n}, e_{l+n+1}^{l+n}, \ldots, e_{l+n+s_l}^{l+n}, \ldots \right]. \tag{14}$$

Let $B = C^{-1}$, where C transforms the matrix Jh^* into Jordan's form. Thus we can write

$$BJh^* = GB$$

(G is Jordan's form of the matrix Jh^* and Jh). It is easy to find the general form of the matrix B, because one knows all Jordan's forms and the normal forms of Jh. Then one can find the elements of the matrix B by using the condition (6) and the fact that the matrix N has to be real. Let us consider now the problem for three cases, when all the eigenvalues of Jh

1. are equal to zero,

2. are real and different than zero,

3. are imaginary.

These three cases describe all the Hamiltonian functions, which are contained in this section.

2.2.1. Eigenvalues equal to zero

Let us consider now only such Hamiltonian functions, for which the matrix Jh has only vanishing eigenvalues. This problem has been solved by Sokolski [5]. Based upon that paper one has:

Two degrees of freedom

There are the following normal forms, if all the Jh values are equal to zero. These forms depend on the order of the matrix h:

1. $H_2^* = 0$ if $\dim(h) = 0$,
2. $H_2^* = \frac{1}{2}\delta p_1^{*2}$ $\delta = \pm 1$ if $\dim(h) = 1$,
3. $H_2^* = \frac{1}{2}\delta_1 p_1^{*2} + \frac{1}{2}\delta_2 p_2^{*2}$ $\delta_1 = \delta_2 = \pm 1$ if $\dim(h) = 3$,
4. $H_2^* = \frac{1}{2}\delta p_1^{*2} + q_1^* p_2^*$ $\delta = \pm 1$ if $\dim(h) = 4$.

The difference between Sokolski and our fourth relation follows from the fact that the basic material on which this paper is based on is Briuno's paper. In that paper for the pair of elementary divisors λ^l, where l is an odd number, H_2^* is given in the following form:

$$H_2^* = \varepsilon \sum_{j=1}^{l-1} y_j y_{j+n+1}.$$

For the elementary divisor λ^{2l}

$$H_2^* = \varepsilon \sum_{j=1}^{l-1} y_j y_{j+n+1} + \frac{1}{2}\delta y_1^2.$$

One can assume that $\varepsilon = 1$, because ε is any complex number.
At the beginning let us consider the third and the fourth case.

Case 3

If

$$Jhe_1 = 0, \quad Jhe_2 = e_1, \quad Jhe_3 = 0, \quad Jhe_4 = e_3,$$

then

$$F = \begin{bmatrix} 0 & f_{12} & 0 & 0 \\ -f_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{34} \\ 0 & 0 & -f_{34} & 0 \end{bmatrix},$$

where $f_{12} \neq 0$, $f_{34} \neq 0$, because $f_{12}^2 f_{34}^2 = \det(F) = \det(A)^2 \neq 0$, and

$$B = \begin{bmatrix} b_1 & 0 & b_3 & 0 \\ 0 & 0 & \delta_1 b_1 & 0 \\ 0 & b_2 & 0 & b_4 \\ 0 & 0 & 0 & \delta_2 b_2 \end{bmatrix}.$$

From the condition (6) one gets

$$\delta_1 b_1^2 f_{12} = 1, \quad \delta_2 b_2^2 f_{34} = 1.$$

Assuming, for simplicity, $b_3 = b_4 = 0$ one gets

$$N\left[\ b_1 e_1, \quad b_2 e_3, \quad \delta_1 b_1 e_2, \quad \delta_2 b_2 e4\ \right],$$

where

$$b_1 = \left(e_1^T J e_2\right)^{-1/2}, \quad b_2 = \left(e_3^T J e_4\right)^{-1/2}, \quad \delta_1 = \operatorname{sgn}(e_1^T J e_2), \quad \delta_2 = \operatorname{sgn}(e_3^T J e_4).$$

Case 4

Let

$$J h e_1 = 0, \quad J h e_2 = e_1, \quad J h e_3 = e_2, \quad J h e_4 = e_3,$$

then

$$F = \begin{bmatrix} 0 & 0 & 0 & f_{14} \\ 0 & 0 & -f_{14} & 0 \\ 0 & f_{14} & 0 & f_{34} \\ -f_{14} & 0 & -f_{34} & 0 \end{bmatrix}.$$

F is a non-singular matrix so $f_{14} \neq 0$. The B matrix in this case can be written in the following form:

$$B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_2 & 0 & \delta b_1 & -b_3 \\ 0 & 0 & \delta b_2 & -\delta b_1 \\ 0 & 0 & 0 & -\delta b_2 \end{bmatrix},$$

and the condition (6) is fulfilled if

$$-\delta b_2 b_2 f_{14} = 1,$$

$$b_1 b_1 f_{14} - 2\delta b_2 b_3 f_{14} - b_2 b_2 f_{34} = 0.$$

Assuming that $b_1 = 0$, we have

$$N\left[\ b_2 e_2, \quad b_2 e_1, \quad \delta b_2 e_3 + b_3 e_1, \quad -\delta b_2 e_4 - b_3 e_2\ \right]$$

and

$$b_4 = 0, \quad \delta = -\operatorname{sgn}(e_1^T J e_4), \quad b_2 = (e_1 J e_4)^{-1/2}, \quad b_3 = \frac{b_2 e_3^T J e_4}{-2\delta e_1^T J e_4}.$$

The first case is omitted, because it was described in the preceding section. It is useful to notice that if $\dim(h) = 4$ then H_2 is already written in the simplest form. So the process of simplification is not needed in this case. The second case is omitted, because it is a combination of the first and the third one.

Three degrees of freedom

There are eight different normal forms for this number of degrees of freedom.

1. $H_2^* = 0$ if $\dim(h) = 0$,

2. $H_2^* = \frac{1}{2}\delta p_1^{*2}$ if $\dim(h) = 1$, where $\delta = \pm 1$,

3. $H_2^* = \frac{1}{2}\delta_1 p_1^{*2} + \frac{1}{2}\delta_2 p_2^{*2}$ if $\dim(h) = 2$, where $\delta_1 = \delta_2 = \pm 1$,

4. $H_2^* = \frac{1}{2}\delta p_1^{*2} + q_1^* p_2^*$ if $\dim(h) = 3$, where $\delta = \pm 1$, or

5. $H_2^* = \frac{1}{2}\delta_1 p_1^{*2} + \frac{1}{2}\delta_2 p_2^{*2} + \frac{1}{2}\delta_3 p_3^{*2}$, where $\delta_1 = \delta_2 = \delta_3 = \pm 1$,

6. $H_2^* = q_1^* p_2^* + q_2^* p_3^*$ if $\dim(h) = 4$, or

7. $H_2^* = \frac{1}{2}\delta_1 p_1^{*2} + q_2 p_3 + \frac{1}{2}\delta_2 p_2^{*2}$, where $\delta_1 = \delta_2 = \pm 1$,

8. $H_2^* = q_1^* p_2^* + q_2^* p_3^* + \frac{1}{2}\delta p_2^{*2}$ if $\dim(h) = 5$, where $\delta = \pm 1$.

At first let us try to find the normalization matrices only for the 6-th and 8-th cases.

Case 6

For this case we have the following equations:

$$Jhe_1 = 0, \quad Jhe_2 = e_1, \quad Jhe_3 = e_2, \quad Jhe_4 = 0, \quad Jhe_5 = e_4, \quad Jhe_6 = e_5,$$

so

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & f_{16} \\ 0 & 0 & f_{23} & 0 & -f_{16} & f_{26} \\ 0 & -f_{23} & 0 & f_{16} & -f_{26} & f_{36} \\ 0 & 0 & -f_{16} & 0 & 0 & 0 \\ 0 & f_{16} & f_{26} & 0 & 0 & f_{56} \\ -f_{16} & -f_{26} & -f_{36} & 0 & -f_{56} & 0 \end{bmatrix}$$

and $f_{16} \neq 0$. Looking for the matrix B, according to the algorithm one gets

$$B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_2 & b_3 & 0 & 0 & -b_4 & -b_5 \\ b_3 & 0 & 0 & 0 & 0 & b_4 \\ b_{10} & b_8 & b_9 & b_7 & b_{11} & b_{12} \\ b_8 & b_9 & 0 & 0 & -b_7 & -b_{11} \\ b_9 & 0 & 0 & 0 & 0 & b_7 \end{bmatrix}.$$

The normalization conditions can be written as

$$b_7 b_3 f_{16} - b_4 b_9 f_{16} = 1,$$

$$-2b_2 b_9 f_{16} - 2b_3 b_9 f_{26} + 2b_3 b_8 f_{16} - b_3^2 f_{23} - b_9 f_{56} = 0,$$

$$b_2 b_7 f_{16} + b_4 b_3 f_{23} + b_{11} b_3 f_{16} + b_7 b_3 f_{26} + b_4 b_8 f_{16} - b_5 b_9 f_{16} - b_4 b_9 f_{26}$$

$$+ b_7 b_9 f_{56} = 0,$$

$$2b_5 b_7 f_{16} - 2b_5 b_9 f_{26} - 2b_4 b_{11} f_{16} - b_4^2 f_{23} - b_7^2 f_{56} = 0,$$

$$b_7 b_1 f_{16} + b_7 b_2 f_{26} + b_2 b_{11} f_{16} + b_4 b_2 f_{23} + b_3 b_5 f_{23} + b_{12} b_3 f_{16} + b_{11} b_3 f_{26}$$

$$+ b_7 b_3 f_{36} - b_4 b_{10} f_{16} - b_8 b_5 f_{16} + b_4 b_8 f_{26} + b_8 b_7 f_{56} - b_6 b_9 f_{16} + b_9 b_4 f_{26}$$

$$- b_4 b_9 f_{36} + b_{11} b_9 f_{56} = 0.$$

Then by assuming (for algebraic simplicity) $b_6 = b_4 = b_9 = b_{12} = 0$, $b_{10} = b_{11} = 0$ one gets

$$b_3 = 1, \quad b_7 = \frac{1}{f_{16}}, \quad b_2 = \frac{-b_3 f_{26}}{f_{16}}, \quad b_8 = \frac{b_3 f_{23}}{2f_{16}}, \quad b_5 = \frac{b_7 f_{56}}{2f_{16}},$$

$$b_{10} = \frac{b_8 b_5 f_{16} - b_8 b_7 f_{56} - b_2 b_7 f_{26} - b_3 b_5 f_{23} - b_7 b_3 f_{36}}{b_7 f_{16}},$$

$$N^T = \begin{bmatrix} b_1 e_1 + b_2 e_2 + b_3 e_3 + b_8 e_5 \\ b_2 e_1 + b_3 e_2 \\ b_3 e_1 \\ b_7 e_4 \\ b_5 e_1 - b_7 e_5 \\ b_7 e_6 - b_5 e_2 \end{bmatrix}.$$

Case 8

In this case

$$Jhe_1 = 0, \quad Jhe_2 = e_1, \quad Jhe_3 = e_2, \quad Jhe_4 = e_3, \quad Jhe_5 = e_4, \quad Jhe_6 = e_5,$$

so

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & f_{16} \\ 0 & 0 & 0 & 0 & -f_{16} & 0 \\ 0 & 0 & 0 & f_{16} & 0 & f_{36} \\ 0 & 0 & -f_{16} & 0 & -f_{36} & 0 \\ 0 & f_{16} & 0 & f_{36} & 0 & f_{56} \\ -f_{16} & 0 & -f_{36} & 0 & -f_{56} & 0 \end{bmatrix}.$$

The element $f_{16} \neq 0$, because the matrix F is non-singular. The matrix B for this form is the following:

$$B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_2 & b_3 & 0 & \delta b_1 & -b_4 & -b_5 \\ b_3 & 0 & 0 & \delta b_2 & -\delta b_1 & b_4 \\ 0 & 0 & 0 & \delta b_3 & -\delta b_2 & \delta b_1 \\ 0 & 0 & 0 & 0 & -\delta b_3 & \delta b_2 \\ 0 & 0 & 0 & 0 & 0 & \delta b_3 \end{bmatrix}.$$

The condition (6) is fulfilled when

$$\delta b_3 b_3 f_{16} = 1$$

$$2\delta b_3 b_1 f_{16} + \delta b_3 b_3 f_{36} - \delta b_2 b_2 f_{16} = 0,$$

$$b_2 b_2 f_{36} + 2\delta b_2 b_4 f_{16} + 2\delta b_3 b_5 f_{16} - b_3 b_3 f_{56} - 2b_1 b_3 f_{36} - b_1 b_1 f_{16} = 0.$$

Assuming $b_2 = 0$ and $b_4 = 0$ one gets

$$N^T = \begin{bmatrix} b_1 e_1 + b_3 e_3 \\ b_3 e_2 \\ b_3 e_2 \\ \delta b_1 e_2 + \delta b_3 e_4 \\ b_5 e_1 - \delta b_1 e_3 - \delta b_3 e_5 \\ -b_5 e_2 + \delta b_1 e_4 + \delta b_3 e_6, \end{bmatrix},$$

where

$$\delta = \text{sign}(f_{16}), \quad b_3 = (f_{16})^{-1/2}, \quad b_1 = \frac{-f_{36}b_3}{2f_{16}},$$

$$b_5 = \frac{b_1 b_1 f_{16} + 2b_3 b_1 f_{36} + b_3 b_3 f_{56}}{2\delta f_{16} b_3}.$$

We do not consider other cases, because they may be reduced to combinations of the previous cases.

2.2.2. Real eigenvalues

For the pair of elementary divisors $(\lambda^* - \lambda)^l$ and $(\lambda^* + \lambda)^l$, where λ is the real eigenvalue, the normal form given by Briuno [2] may be written in the following way:

$$H_2^* = \lambda \sum_{j=1}^{l} y_j y_{j+n} + \varepsilon \sum_{j=1}^{l-1} y_j y_{j+n+1}. \tag{15}$$

Let us consider only two cases. In the first one

$$Jhe_1 = \lambda_1 e_1, \quad Jhe_2 = \lambda_1 e_2 + e_1, \quad Jhe_3 = -\lambda_1 e_3, \quad Jhe_4 = -\lambda_1 e_4 + e_3,$$

and so

$$F = \begin{bmatrix} 0 & 0 & 0 & f_{14} \\ 0 & 0 & -f_{14} & f_{24} \\ 0 & f_{14} & 0 & 0 \\ -f_{14} & -f_{24} & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} b_1 & b_2 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ 0 & 0 & b_3 & b_4 \\ 0 & 0 & 0 & -b_3 \end{bmatrix}$$

From the condition (6) it follows that

$$-f_{14}b_3 b_2 = 1,$$

$$-b_1 b_3 f_{14} - b_2 b_4 f_{14} - f_{24}b_2 b_3 = 0.$$

Now let $b_1 = 0$, then

$$b_2 = (f_{14})^{-1/2}, \quad \delta = \text{sgn}(f_{14}), \quad b_3 = -\delta (f_{14})^{-1/2}, \quad b_4 = \frac{f_{24}b_3}{-f_{14}}.$$

The matrix N may be written as

$$N = \begin{bmatrix} b_2 e_2, & b_2 e_1, & b_3 e_3, & b_4 e_3 - b_3 e_4 \end{bmatrix}.$$

In the second case

$$Jhe_1 = \lambda_1 e_1, \quad Jhe_2 = \lambda_1 e_2 + e_1, \quad Jhe_3 = \lambda_1 e_3 + e_2,$$

$$Jhe_4 = -\lambda_1 e_4, \quad Jhe_5 = -\lambda_1 e_5 + e_4, \quad Jhe_6 = -\lambda_1 e_6 + e_5,$$

so

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & f_{16} \\ 0 & 0 & 0 & 0 & -f_{16} & f_{26} \\ 0 & 0 & 0 & f_{16} & -f_{26} & f_{36} \\ 0 & 0 & -f_{16} & 0 & 0 & 0 \\ 0 & f_{16} & f_{26} & 0 & 0 & 0 \\ -f_{16} & -f_{26} & -f_{36} & 0 & 0 & 0 \end{bmatrix}$$

and $f_{16} \neq 0$. The matrix B may be written in the following way:

$$B = \begin{bmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ b_2 & b_3 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & 0 & -b_4 & -b_5 \\ 0 & 0 & 0 & 0 & 0 & b_4 \end{bmatrix}.$$

The transformation matrix fulfills the condition (6) if

$$b_3 b_4 f_{16} = 1,$$

$$b_4 b_2 f_{16} + b_3 b_5 f_{16} + b_3 b_4 f_{26} = 0,$$

$$b_4 b_1 f_{16} + b_2 b_5 f_{16} + b_4 b_2 f_{26} + b_6 b_3 f_{16} + b_5 b_3 f_{26} + b_4 b_3 f_{36} = 0.$$

One can assume for algebraic simplicity $b_1 = b_5 = 0$, then

$$b_3 = \frac{1}{f_{16}}, \quad b_4 = 1, \quad b_2 = \frac{-b_3 f_{26}}{f_{16}}, \quad b_6 = \frac{b_4(f_{26}^2/f_{16} - f_{36})}{f_{16}}$$

and

$$N = \begin{bmatrix} b_2 e_2 + b_3 e_3, & b_2 e_1 + b_3 e_2, & b_3 e_1, & b_4 e_4, & -b_4 e_5, & b_4 e_6 + b_6 e_4 \end{bmatrix}$$

2.2.3. Imaginary eigenvalues

In this section we consider the pair of elementary divisors $(\lambda^* - \lambda)^l$ and $(\lambda^* + \lambda)^l$, where λ is the imaginary eigenvalue ($\lambda = i\beta$). The normal form for this case, given by Briuno [2], is the following:

$$H_2^* = \frac{1}{2}\beta \sum_{j=1}^{l} (y_j y_{l+1-j} + y_{j+n} y_{l+n+1-j})$$

$$+ \frac{1}{2}\delta \sum_{j=1}^{l-1} (y_j y_{l-j} + y_{1+j+n} y_{l+n+1-j}), \tag{16}$$

where $\delta = \pm 1$. Let us consider (like for the real eigenvalues) two cases: when l is equal to two and three. The matrices F are the same like in the case of real eigenvalues. It is worth noticing that for $l = 2$ the element f_{14} is the real value, and f_{24} the imaginary one, but if $l = 3$ both f_{16} and f_{36} are imaginary and f_{26}

is real. Let the power of elementary divisor be equal to two, and let the matrix A be constructed according to the recipe described at the beginning of this section. Then

$$
B = \begin{bmatrix}
b_1 & b_2 & -i\delta b_2 & -i\delta b_1 \\
ib_2 & 0 & 0 & \delta b_2 \\
b_9 & b_{10} & i\delta b_{10} & i\delta b_9 \\
-ib_{10} & 0 & 0 & \delta b_{10}
\end{bmatrix} .
$$

The normalization conditions have the form

$$
2\delta b_2 b_{10} f_{14} = 1,
$$

$$
2\delta b_1 b_{10} f_{14} + 2\delta b_2 b_9 f_{14} + i2\delta b_{10} b_2 f_{24} = 0.
$$

So the matrix N is given by

$$
N^T = \begin{bmatrix}
b_1 e_1 + ib_2 e_2 + b_9 e_3 - ib_{10} e_4 \\
b_2 e_1 + b_{10} e_3 \\
-i\delta b_2 e_2 + i\delta e_3 \\
-i\delta b_1 e_1 + \delta b_2 e_2 + i\delta b_9 e_3 + \delta b_{10} e_4
\end{bmatrix} ,
$$

where

$$
\delta = \mathrm{sgn}(f_{14}), \quad b_2 = b_{10} = (2\delta f_{14})^{-1/2}, \quad b_1 = b_9 = -\frac{ib_2 f_{24}}{2f_{14}}
$$

and

$$
e_3 = \overline{e_1}, \quad e_4 = \overline{e_2}.
$$

The straight line over e_k means the coupling of the vector. For $l = 3$

$$
B = \begin{bmatrix}
b_1 & b_2 & b_3 & -i\delta b_3 & -i\delta b_2 & -i\delta b_1 \\
ib_2 & ib_3 & 0 & 0 & \delta b_3 & \delta b_2 \\
-b_3 & 0 & 0 & 0 & 0 & i\delta b_3 \\
b_{19} & b_{20} & b_{21} & i\delta b_{21} & i\delta b_{20} & i\delta b_{19} \\
-ib_{20} & -ib_{21} & 0 & 0 & \delta b_{21} & \delta b_{20} \\
-b_{21} & 0 & 0 & 0 & 0 & -i\delta b_{21}
\end{bmatrix} .
$$

The condition (6) is fulfilled if

$$
-2i\delta b_{21} b_3 f_{16} = 1,
$$

$$
-2i\delta b_{21} b_2 f_{16} - 2i\delta b_{20} b_3 f_{16} + 2\delta b_3 b_{21} f_{26} = 0,
$$

$$
-2i\delta b_{21} b_1 f_{16} - 2i\delta b_{20} b_2 f_{16} - 2i\delta b_{19} b_3 f_{16}
$$
$$
+2\delta b_2 b_{21} f_{26} + 2\delta b_{20} b_3 f_{26} + 2i\delta b_3 b_{21} f_{36} = 0.
$$

Finally

$$
N^T = \begin{bmatrix}
b_1 e_1 - b_3 e_3 + ib_2 e_2 + b_{19} e_4 - ib_{20} e_5 - b_{21} e_6 \\
b_2 e_1 + ib_3 e_2 + b_{20} e_4 - ib_{21} e_5 \\
b_3 e_1 + b_{21} e_4 \\
-i\delta b_3 e_1 + i\delta b_{21} e_4 \\
-i\delta b_2 e_1 + \delta b_3 e_2 + i\delta b_{20} e_4 + \delta b_{21} e_5 \\
-i\delta b_1 e_1 + i\delta b_3 e_3 + \delta b_2 e_2 + i\delta b_{19} e_4 + \delta b_{20} e_5 - i\delta b_{21} e_6
\end{bmatrix} .
$$

and

$$\delta = \text{sgn}(\text{Im} f_{16}), \quad b_3 = b_{21} = (2\text{Im}(f_{16}))^{-1/2},$$

$$b_2 = b_{20} = \frac{b_3 f_{26}}{2\text{i} f_{16}}, \quad b_1 = b_{19} = \frac{\text{i} b_3 f_{36} + 1.5 b_2 f_{26}}{2\text{i} f_{16}},$$

$$e_4 = \overline{e_1}, \quad e_5 = \overline{e_2}, \quad e_6 = \overline{e_3}.$$

Like for the real eigenvalues these two cases are sufficient to get the complete solution of the Hamiltonian function, for which the Jh matrices have degenerate eigenvalues.

3. Conclusion

None of the above ways of finding N matrices depend upon the number of degrees of freedom. One can easily find, using the algorithms described in this paper, the transformation matrices for any Hamiltonian (fulfilling the initial conditions) with any degrees of freedom. It is enough to construct the B and F matrices for such systems of the elementary divisor (according to the properties of the Jh matrices), which we have not considered in the case of two and three degrees of freedom.

Acknowledgments

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