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AN EXTENDED SUPERSYMMETRIC t-J MODEL

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A new lattice model is presented for correlated electrons on the unrestricted 4^L-dimensional electronic Hilbert space $\oplus_{n=1}^{L} C^4$ (where L is the lattice length). It is an extension of the supersymmetric t-J model. The new model has $gl(2|1)$ supersymmetry and contains one symmetry-preserving free real parameter α which has its origin in the one-parameter family of inequivalent typical 4-dimensional irreps of $gl(2|1)$. When $\alpha = 0$, the model reduces to the supersymmetric t-J model.

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In Ref. [1], we presented a $gl(2|1)$-supersymmetric generalization of the Hubbard model of correlated electrons on the unrestricted 4^L-dimensional electronic Hilbert space $\oplus_{n=1}^{L} C^4$ (throughout the paper, L is the lattice length). This model allows doubly occupied sites and contains one symmetry-preserving free real parameter α which has its origin in the one-parameter family of inequivalent typical 4-dimensional irreps of $gl(2|1)$. It contains, in addition to the Hubbard on-site interaction term, other nearest-neighbor interactions. These interactions appear in a different fashion from the ones in the so-called extended Hubbard model proposed by Essler, Korepin and Schoutens (EKS) [2].

In this paper we propose an extension of the supersymmetric t-J model. The t-J model is a lattice model on the restricted 3^L-dimensional electronic Hilbert space $\oplus_{n=1}^{L} C^3$, where the occurrence of two electrons on the same lattice site is forbidden. With the special choice of parameters: $t = 1$ and $J = 2$, the t-J model becomes supersymmetric with the symmetry algebra being the superalgebra $gl(2|1)$ [3, 4]. Here by relaxing this non-double-occupancy restriction, we present yet another new model. Like the model in Ref. [1] and in Ref. [2], this model allows doubly occupied sites but interaction terms are different. This model still has $gl(2|1)$ supersymmetry and contains one symmetry-preserving free real parameter α. When $\alpha = 0$, this model reduces to the supersymmetric t-J model, where the double occupancy of the sites is implicitly projected out. Thus the model can naturally be regarded as a modified supersymmetric t-J model.

Let us begin by introducing some notation. Electrons on a lattice are described by canonical Fermi operators $c_{i\sigma}$ and $c_{i,\sigma}^{\dagger}$ satisfying the anti-commutation relations given by $\{c_{i,\sigma}^{\dagger}, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau}$, where $i, j = 1, 2, \ldots, L$ and $\sigma, \tau = \uparrow, \downarrow$.

The operator $c_{i,\sigma}$ annihilates an electron of spin σ at site i, which implies that the Fock vacuum $|0\rangle$ satisfies $c_{i,\sigma}|0\rangle = 0$. At a given lattice site i there are four possible electronic states

$$|0\rangle, \qquad |\uparrow\rangle_i = c^\dagger_{i,\uparrow}|0\rangle, \qquad |\downarrow\rangle_i = c^\dagger_{i,\downarrow}|0\rangle, \qquad |\uparrow\downarrow\rangle_i = c^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow}|0\rangle. \tag{1}$$

By $n_{i,\sigma} = c^\dagger_{i,\sigma}c_{i,\sigma}$ we denote the number operator for electrons with spin σ on site i, and we write $n_i = n_{i,\uparrow} + n_{i,\downarrow}$. The spin operators S, $S^\dagger$, S^z (in the following, the global operator $\mathcal{O}$ will be always expressed in terms of the local one $\mathcal{O}_i$ as $\mathcal{O} = \sum_{i=1}^{L} \mathcal{O}_i$ in one dimension)

$$S_i = c^\dagger_{i,\uparrow}c_{i,\downarrow}, \qquad S_i^\dagger = c^\dagger_{i,\downarrow}c_{i,\uparrow}, \qquad S_i^z = \frac{1}{2}(n_{i,\downarrow} - n_{i,\uparrow}), \tag{2}$$

form an $sl(2)$ algebra and they commute with the Hamiltonians that we consider below.

The Hamiltonian for our new model on a general d-dimensional lattice reads

$$\begin{aligned} H(\alpha) \equiv \sum_{\langle i,j\rangle} H_{i,j}(\alpha) = &-(\alpha+1)\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c^\dagger_{i,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{i,\sigma}) \\ &+2\sum_{\langle i,j\rangle} \boldsymbol{S}_i\cdot\boldsymbol{S}_j - \frac{1}{2}\sum_{\langle i,j\rangle}(n_i-1)(n_j-1) \\ &+\left(\alpha+1+\sqrt{\alpha(\alpha+1)}\right)\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c^\dagger_{i,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{i,\sigma})(n_{i,-\sigma}+n_{j,-\sigma}) \\ &-\left(2\alpha+1+2\sqrt{\alpha(\alpha+1)}\right)\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c^\dagger_{i,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{i,\sigma})n_{i,-\sigma}n_{j,-\sigma} \\ &+\frac{2\alpha+1}{2}\sum_{\langle i,j\rangle}(n_i+n_j), \end{aligned} \tag{3}$$

where $\langle i,j\rangle$ denote nearest-neighbor links on the lattice. The local Hamiltonian $H_{i,j}(\alpha)$ does not act as graded permutation of the electron states (1) at sites i and j, in contrast to the Hamiltonian in Ref. [2]. Nevertheless, the global number operators of spin up and down are conservative quantities, as will be seen below. Furthermore the Hamiltonian (3) is invariant under spin-reflection $c_{i,\uparrow} \leftrightarrow c_{i,\downarrow}$. Obviously, when $\alpha = 0$ the Hamiltonian $H(\alpha)$ reduces to that of the supersymmetric t-J model, in which the double occupancy of the sites is implicitly projected out.

For non-zero α, our model (3) is the supersymmetric t-J model H^{t-J} plus the perturbation term $H^{\mathrm{P}}(\alpha)$:

$$H(\alpha) = H^{t-J} + H^{\mathrm{P}}(\alpha),$$

$$\begin{aligned} H^{\mathrm{P}}(\alpha) = &-\alpha\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c^\dagger_{i,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{i,\sigma}) + \alpha\sum_{\langle i,j\rangle}(n_i+n_j) \\ &+\left(\alpha+\sqrt{\alpha(\alpha+1)}\right)\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c^\dagger_{i,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{i,\sigma})(n_{i,-\sigma}-n_{j,-\sigma})^2, \end{aligned} \tag{4}$$

which implies that our model can be viewed as an extended supersymmetric t-J model. It is worth mentioning that the supersymmetric t-J Hamiltonian is a graded permutation operator acting on sites i and j but only on the first three electron states in (1)

$$H_{i,j}^{t-J} = - \sum_{\beta,\gamma=0,\uparrow,\downarrow} (-1)^{[[\beta)]} \left(e_\beta^\gamma\right)_i \left(e_\gamma^\beta\right)_j , \tag{5}$$

e_β^γ being matrices defined by $e_\beta^\gamma|\beta'\rangle = \delta_{\beta\beta'}|\gamma\rangle$ or explicitly $e_\beta^\gamma = |\gamma\rangle\langle\beta|$.

For non-zero α, this model can also be written as the EKS Hamiltonian $H_{i,j}^{\rm EKS}$ [2] plus the Hamiltonian $H_{i,j}^Q(\alpha)$ proposed by us in Ref. [1]. Actually it can be shown that the following relation holds:

$$H_{i,j}(\alpha) = -P_{i,j} + \alpha H_{i,j}^Q(\alpha) - (\alpha+1)^2, \tag{6}$$

where $P_{i,j}$ is the graded permutation operator acting on sites i and j

$$P_{i,j} = \sum_{\beta,\gamma=0,\uparrow,\downarrow,\uparrow\downarrow} (-1)^{[[\beta)]} (e_\beta^\gamma)_i (e_\gamma^\beta)_j , \tag{7}$$

where e_β^γ are still equal to $|\gamma\rangle\langle\beta|$ but $\beta, \gamma = 0, \uparrow, \downarrow, \uparrow\downarrow$. Observe that $-P_{i,j}$ is nothing but the EKS Hamiltonian $H_{i,j}^{\rm EKS}$. This immediately implies that our Hamiltonian is a sum of two parts

$$H(\alpha) = H^{\rm EKS} + \alpha H^Q(\alpha), \tag{8}$$

where the constant term $(\alpha+1)^2$ has been disregarded. Note however that the second term on the right hand side does not vanish when $\alpha \to 0$, but gives the difference between the supersymmetric *t-J* and EKS Hamiltonians.

There are four supersymmetries for $H(\alpha)$: $Q_\uparrow, Q_\uparrow^\dagger$, $Q_\downarrow, Q_\downarrow$, and $Q_\downarrow^\dagger$ and $Q_\downarrow^+$ with the corresponding local operators given by

$$Q_{i,\uparrow} = -\sqrt{\alpha} n_{i,\downarrow} c_{i,\uparrow} + \sqrt{\alpha+1}(1-n_{i,\downarrow}) c_{i,\uparrow},$$

$$Q_{i,\downarrow} = -\sqrt{\alpha} n_{i,\uparrow} c_{i,\downarrow} + \sqrt{\alpha+1}(1-n_{i,\uparrow}), c_{i,\downarrow}, \tag{9}$$

where $0 \le \arg\sqrt{Z} < \pi$, $Z = \alpha$ or $\alpha+1$, and $\alpha \ge 0$ or $\alpha < -1$. These generators, together with S, $S^\dagger$, S^z and two others ($E_2^2 + E_3^3$ and E_3^3, defined below), form the superalgebra $gl(2|1)$. To make manifest the symmetries of the Hamiltonian $H(\alpha)$, we denote the generators of $gl(2|1)$ by E_γ^β, $\beta, \gamma = 1, 2, 3$ with grading $[1] = [2] = 0$, $[3] = 1$. In a typical 4-dimensional representation of $gl(2|1)$, the highest weight itself of the representation depends on the free parameter α, thus giving rise to a one-parameter family of inequivalent irreps [5]. Choose the following basis: $|4\rangle = (0,0,0,1)^T$, $|3\rangle = (0,0,1,0)^T$, $|2\rangle = (0,1,0,0)^T$ and $|1\rangle = (1,0,0,0)^T$ with $|1\rangle$, $|4\rangle$ even (bosonic) and $|2\rangle$, $|3\rangle$ odd (fermionic). Then in this typical 4-dimensional representation, E_γ^β are 4×4 supermatrices of the form

$$E_2^1 = |2\rangle\langle 3|,\ E_1^2 = |3\rangle\langle 2|,\ E_1^1 = -|3\rangle\langle 3| - |4\rangle\langle 4|,\ E_2^2 = -|2\rangle\langle\langle 2| - |4\rangle\langle 4|,$$

$$E_3^2 = \sqrt{\alpha}|1\rangle\langle 2| + \sqrt{\alpha+1}|3\rangle\langle 4|,\ E_2^3 = \sqrt{\alpha}|2\rangle\langle 1| + \sqrt{\alpha+1}|4\rangle\langle 3|,$$

$$E_3^1 = -\sqrt{\alpha}|1\rangle\langle 3| + \sqrt{\alpha+1}|2\rangle\langle 4|,\ E_1^3 = -\sqrt{\alpha}|3\rangle\langle 1| + \sqrt{\alpha+1}|4\rangle\langle 2|,$$

$$E_3^3 = \alpha|1\rangle\langle 1| + (\alpha+1)(|2\rangle\langle 2| + |3\rangle\langle 3|) + (\alpha+2)|4\rangle\langle 4|. \tag{10}$$

For $\alpha > 0$, we have $(E_\gamma^\beta)^\dagger = E_\beta^\gamma$ and we call the representation unitary of type I. For $\alpha < -1$, we have $(E_\gamma^\beta)^\dagger = (-1)^{[\beta]+[\gamma]} E_\beta^\gamma$ and we refer to the representation as unitary of type II. In this paper, we are interested in these unitary representations.

For a description and classification of the two types of unitary representations, see Ref. [6].

Further choosing $|4\rangle \equiv |0\rangle$, $|3\rangle \equiv |\uparrow\rangle$, $2\rangle \equiv |\downarrow\rangle$ and $|1\rangle \equiv |\uparrow\downarrow\rangle$ and with the help of the following identities:

$$|0\rangle\langle 0| + |\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| = 1,$$

$$|\uparrow\downarrow\rangle\langle\uparrow\downarrow| = n_\uparrow n_\downarrow,$$

$$|\uparrow\rangle\langle\uparrow| = n_\uparrow - n_\uparrow n_\downarrow, \quad |\downarrow\rangle\langle\downarrow| = n_\downarrow - n_\uparrow n_\downarrow, \tag{11}$$

one has

$$n_i = (\alpha+2) - (E_3^3)_i, \quad n_{i,\downarrow} = (E_1^1)_i + 1, \quad n_{i,\uparrow} = (E_2^2)_i + 1, \tag{12}$$

$$S_i^\dagger = (E_2^1)_i, \quad S_i = (E_1^2)_i, \quad S_i^z = (E_1^1)_i - (E_2^2)_i,$$

$$Q_{i,\downarrow}^\dagger = (E_2^3)_i^\dagger, \quad Q_{i,\downarrow} = (E_2^3)_i, \quad Q_{i,\uparrow}^\dagger = (E_1^3)_i^\dagger, \quad Q_{i,\uparrow} = (E_1^3)_i. \tag{13}$$

It can be checked that the local Hamiltonian $H_{i,j}(\alpha)$ can be cast in a group-theoretical form,

$$H_{i,j}(\alpha) = -\sum_{\sigma=\uparrow,\downarrow} (Q_{i,\sigma}^\dagger Q_{j,\sigma} + Q_{j,\sigma}^\dagger Q_{i,\sigma}) + 2S_i^z S_j^a + S_i^\dagger S_j + S_i S_j^\dagger$$

$$-\frac{1}{2} n_i n_j + (\alpha+1)(n_i + N_j) - (\alpha+2)^2 = \sum_{\beta,\gamma=1} (-1)^{[\gamma]} (E_\gamma^\beta)_i (E_\beta^\gamma)_j. \tag{14}$$

The last expression immediately makes it clear that $H(\alpha)$ commutes with all nine generators of $gl(2|1)$. Equations (12) make it clear that $H^Q(U)$ commutes with the global number operators of spin up and spin down. Remarkably, the model still contains the parameter α as a free parameter without breaking the $gl(2|1)$ supersymmetry. Clearly one can add to the above Hamiltonian an arbitrary chemical potential (coefficient μ) term $\mu \sum_i n_i$ and an external magnetic field (coefficient h) term $h\sum_i (n_{i,\downarrow} - n_{i,\uparrow})$, which commute with $H(\alpha)$ but break its $gl(2|1)$ supersymmetry.

In summary, we have proposed an extension of the supersymmetric *t-J* model, by allowing double occupancy on one site. It has $gl(2|1)$ supersymmetry and still contains a symmetry-preserving parameter α. It is not clear however that our model is exactly solved in one dimension.

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