AN EXTENDED SUPERSYMMETRIC \( t-J \) MODEL

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A new lattice model is presented for correlated electrons on the unrestricted \( 4^L \)-dimensional electronic Hilbert space \( \otimes_{n=1}^L C^4 \) (where \( L \) is the lattice length). It is an extension of the supersymmetric \( t-J \) model. The new model has \( gl(2|1) \) supersymmetry and contains one symmetry-preserving free real parameter \( \alpha \) which has its origin in the one-parameter family of inequivalent typical 4-dimensional irreps of \( gl(2|1) \). When \( \alpha = 0 \), the model reduces to the supersymmetric \( t-J \) model.

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In Ref. [1], we presented a \( gl(2|1) \)-supersymmetric generalization of the Hubbard model of correlated electrons on the unrestricted \( 4^L \)-dimensional electronic Hilbert space \( \otimes_{n=1}^L C^4 \) (throughout the paper, \( L \) is the lattice length). This model allows doubly occupied sites and contains one symmetry-preserving free real parameter \( \alpha \) which has its origin in the one-parameter family of inequivalent typical 4-dimensional irreps of \( gl(2|1) \). It contains, in addition to the Hubbard on-site interaction term, other nearest-neighbor interactions. These interactions appear in a different fashion from the ones in the so-called extended Hubbard model proposed by Essler, Korepin and Schoutens (EKS) [2].

In this paper we propose an extension of the supersymmetric \( t-J \) model. The \( t-J \) model is a lattice model on the restricted \( 3^L \)-dimensional electronic Hilbert space \( \otimes_{n=1}^L C^3 \), where the occurrence of two electrons on the same lattice site is forbidden. With the special choice of parameters: \( t = 1 \) and \( J = 2 \), the \( t-J \) model becomes supersymmetric with the symmetry algebra being the superalgebra \( gl(2|1) \) [3, 4]. Here by relaxing this non-double-occupancy restriction, we present yet another new model. Like the model in Ref. [1] and in Ref. [2], this model allows doubly occupied sites but interaction terms are different. This model still has \( gl(2|1) \) supersymmetry and contains one symmetry-preserving free real parameter \( \alpha \). When \( \alpha = 0 \), this model reduces to the supersymmetric \( t-J \) model, where the double occupancy of the sites is implicitly projected out. Thus the model can naturally be regarded as a modified supersymmetric \( t-J \) model.

Let us begin by introducing some notation. Electrons on a lattice are described by canonical Fermi operators \( c_{i\sigma} \) and \( c_{i\sigma}^\dagger \) satisfying the anti-commutation relations given by \( \{c_{i\sigma}^\dagger, c_{j\tau}\} = \delta_{ij} \delta_{\sigma\tau} \), where \( i, j = 1, 2, \ldots, L \) and \( \sigma, \tau = \uparrow, \downarrow \).

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The operator \(c_{i,\sigma}\) annihilates an electron of spin \(\sigma\) at site \(i\), which implies that the Fock vacuum \(|0\rangle\) satisfies \(c_{i,\sigma}|0\rangle = 0\). At a given lattice site \(i\) there are four possible electronic states

\[
|0\rangle, \quad |\uparrow\rangle_i = c_{i,\uparrow}^\dagger |0\rangle, \quad |\downarrow\rangle_i = c_{i,\downarrow}^\dagger |0\rangle, \quad |\uparrow\downarrow\rangle_i = c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger |0\rangle.
\]

By \(n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma}\) we denote the number operator for electrons with spin \(\sigma\) on site \(i\), and we write \(n_i = n_{i,\uparrow} + n_{i,\downarrow}\). The spin operators \(S_i, S_i^\dagger, S_i^z\) (in the following, the global operator \(\mathcal{O}\) will be always expressed in terms of the local one \(\mathcal{O}_i\) as \(\mathcal{O} = \sum_{i=1}^L \mathcal{O}_i\) in one dimension)

\[
S_i = c_{i,\uparrow}^\dagger c_{i,\downarrow}, \quad S_i^\dagger = c_{i,\downarrow}^\dagger c_{i,\uparrow}, \quad S_i^z = \frac{1}{2}(n_{i,\downarrow} - n_{i,\uparrow}),
\]

form an \(sl(2)\) algebra and they commute with the Hamiltonians that we consider below.

The Hamiltonian for our new model on a general \(d\)-dimensional lattice reads

\[
H(\alpha) \equiv \sum_{(i,j)} H_{i,j}(\alpha) = -(\alpha + 1) \sum_{(i,j)} \sum_{\sigma = \uparrow, \downarrow} (c_{i,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{i,\sigma})
\]

\[
+ 2 \sum_{(i,j)} S_i \cdot S_j - \frac{1}{2} \sum_{(i,j)} (n_i - 1)(n_j - 1)
\]

\[
+ \left(\alpha + 1 + \sqrt{\alpha(\alpha + 1)}\right) \sum_{(i,j)} \sum_{\sigma = \uparrow, \downarrow} (c_{i,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{i,\sigma})(n_{i,-\sigma} + n_{j,-\sigma})
\]

\[
- \left(2\alpha + 1 + 2\sqrt{\alpha(\alpha + 1)}\right) \sum_{(i,j)} \sum_{\sigma = \uparrow, \downarrow} (c_{i,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{i,\sigma})n_{i,-\sigma}n_{j,-\sigma}
\]

\[
+ \frac{2\alpha + 1}{2} \sum_{(i,j)} (n_i + n_j),
\]

where \((i,j)\) denote nearest-neighbor links on the lattice. The local Hamiltonian \(H_{i,j}(\alpha)\) does not act as graded permutation of the electron states \((1)\) at sites \(i\) and \(j\), in contrast to the Hamiltonian in Ref. [2]. Nevertheless, the global number operators of spin up and down are conservative quantities, as will be seen below. Furthermore the Hamiltonian \((3)\) is invariant under spin-reflection \(c_{i,\uparrow} \leftrightarrow c_{i,\downarrow}\). Obviously, when \(\alpha = 0\) the Hamiltonian \(H(\alpha)\) reduces to that of the supersymmetric \(t-J\) model, in which the double occupancy of the sites is implicitly projected out.

For non-zero \(\alpha\), our model \((3)\) is the supersymmetric \(t-J\) model \(H^{t-J}\) plus the perturbation term \(H^P(\alpha)\):

\[
H(\alpha) = H^{t-J} + H^P(\alpha),
\]

\[
H^P(\alpha) = -\alpha \sum_{(i,j)} \sum_{\sigma = \uparrow, \downarrow} (c_{i,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{i,\sigma}) + \alpha \sum_{(i,j)} (n_i + n_j)
\]

\[
+ \left(\alpha + \sqrt{\alpha(\alpha + 1)}\right) \sum_{(i,j)} \sum_{\sigma = \uparrow, \downarrow} (c_{i,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{i,\sigma})(n_{i,-\sigma} - n_{j,-\sigma})^2,
\]

which implies that our model can be viewed as an extended supersymmetric \(t-J\) model. It is worth mentioning that the supersymmetric \(t-J\) Hamiltonian is a graded permutation operator acting on sites \(i\) and \(j\) but only on the first three electron states in \((1)\)
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For non-zero $\alpha$, this model can also be written as the EKS Hamiltonian $H_{ij}^{EKS}$ [2] plus the Hamiltonian $H^Q_i(\alpha)$ proposed by us in Ref. [1]. Actually it can be shown that the following relation holds:

$$H_{ij}(\alpha) = -P_{i,j} + \alpha H^Q_{ij}(\alpha) - (\alpha + 1)^2,$$

where $P_{i,j}$ is the graded permutation operator acting on sites $i$ and $j$

$$P_{i,j} = \sum_{\beta, \gamma = 0, 1, \downarrow} (-1)^{\|\beta\|} (e^\gamma_\beta)_i (e^\beta_\gamma)_j,$$

where $e^\gamma_\beta$ are still equal to $|\gamma\rangle\langle\beta|$ but $\beta, \gamma = 0, 1, \downarrow$. Observe that $-P_{i,j}$ is nothing but the EKS Hamiltonian $H_{ij}^{EKS}$. This immediately implies that our Hamiltonian is a sum of two parts

$$H(\alpha) = H^{EKS} + \alpha H^Q(\alpha),$$

where the constant term $(\alpha + 1)^2$ has been disregarded. Note however that the second term on the right hand side does not vanish when $\alpha \to 0$, but gives the difference between the supersymmetric $t$-$J$ and EKS Hamiltonians.

There are four supersymmetries for $H(\alpha)$: $Q_T, Q_t, Q_T, Q_t$ with the corresponding local operators given by

$$Q_{i,\downarrow} = -\sqrt{\alpha n_{i,\downarrow}} c_{i,\downarrow} + \sqrt{\alpha + 1}(1 - n_{i,\downarrow}) c_{i,\downarrow},$$

$$Q_{i,\uparrow} = -\sqrt{\alpha n_{i,\uparrow}} c_{i,\uparrow} + \sqrt{\alpha + 1}(1 - n_{i,\uparrow}) c_{i,\uparrow},$$

where $0 \leq \text{arg} \sqrt{Z} < \pi$, $Z = \alpha$ or $\alpha + 1$, and $\alpha \geq 0$ or $\alpha < -1$. These generators, together with $S, S^\dagger, S^z$ and two others ($E_3^+ + E_3^-$ and $E_3^-$, defined below), form the superalgebra $gl(2|1)$. To make manifest the symmetries of the Hamiltonian $H(\alpha)$, we denote the generators of $gl(2|1)$ by $E_\beta^\gamma, \beta, \gamma = 1, 2, 3$ with grading $[1] = [2] = 0, [3] = 1$. In a typical 4-dimensional representation of $gl(2|1)$, the highest weight itself of the representation depends on the free parameter $\alpha$, thus giving rise to a one-parameter family of inequivalent irreps [5]. Choose the following basis: $|4\rangle = (0, 0, 0, 1)^T, |3\rangle = (0, 0, 1, 0)^T, |2\rangle = (0, 1, 0, 0)^T$ and $|1\rangle = (1, 0, 0, 0)^T$ with $|1\rangle, |4\rangle$ even (bosonic) and $|2\rangle, |3\rangle$ odd (fermionic). Then in this typical 4-dimensional representation, $E_\gamma^\beta$ are $4 \times 4$ supermatrices of the form

$$E_2^1 = |2\rangle\langle3|, \quad E_2^3 = |3\rangle\langle2|, \quad E_1^1 = -|3\rangle\langle3| - |4\rangle\langle4|, \quad E_2^2 = -|2\rangle\langle2| - |4\rangle\langle4|,$$

$$E_3^2 = \sqrt{\alpha}|1\rangle\langle2| + \sqrt{\alpha + 1}|3\rangle\langle4|, \quad E_3^3 = \sqrt{\alpha}|2\rangle\langle1| + \sqrt{\alpha + 1}|4\rangle\langle3|,$$

$$E_3^1 = -\sqrt{\alpha}|1\rangle\langle3| + \sqrt{\alpha + 1}|2\rangle\langle4|, \quad E_3^4 = -\sqrt{\alpha}|3\rangle\langle1| + \sqrt{\alpha + 1}|4\rangle\langle2|,$$

$$E_3^3 = (\alpha + 1)^2 |1\rangle\langle1| + (\alpha + 1)(|2\rangle\langle2| + |3\rangle\langle3|) + (\alpha + 2)|4\rangle\langle4|.$$

For $\alpha > 0$, we have $(E_3^\beta)^\dagger = E_3^\beta$ and we call the representation unitary of type I. For $\alpha < -1$, we have $(E_3^\beta)^\dagger = (-1)^{|\beta|+|\gamma|} E_3^\beta$ and we refer to the representation as unitary of type II. In this paper, we are interested in these unitary representations.
For a description and classification of the two types of unitary representations, see Ref. [6].

Further choosing \(|4\rangle \equiv |0\rangle, \ |3\rangle \equiv |\uparrow\rangle, \ |2\rangle \equiv |\downarrow\rangle \) and \(|1\rangle \equiv |\uparrow\downarrow\rangle\) and with the help of the following identities:

\[
\begin{align*}
|0\rangle\langle 0| + |\uparrow\rangle\langle \uparrow| + |\downarrow\rangle\langle \downarrow| &= 1, \\
|\uparrow\downarrow\rangle\langle \uparrow\downarrow| &= n_{\uparrow}n_{\downarrow}, \\
|\uparrow\rangle\langle \uparrow| &= n_{\uparrow} - n_{\downarrow}n_{\downarrow}, \\
|\downarrow\rangle\langle \downarrow| &= n_{\downarrow} - n_{\uparrow}n_{\downarrow},
\end{align*}
\]

one has

\[
\begin{align*}
n_i &= (\alpha + 2) - (E^3_{\uparrow})_i, \quad n_{i,\downarrow} = (E^3_{\downarrow})_i + 1, \quad n_{i,\uparrow} = (E^3_{\uparrow})_i + 1, \\
S^i_1 &= (E^3_{\downarrow})_i, \quad S_i = (E^3_{\uparrow})_i, \quad S^i_2 = (E^3_{\downarrow})_i - (E^3_{\uparrow})_i, \\
Q^i_1 &= (E^3_{\downarrow})_i, \quad Q_{i,\downarrow} = (E^3_{\downarrow})_i, \quad Q^i_2 = (E^3_{\uparrow})_i, \quad Q_{i,\uparrow} = (E^3_{\uparrow})_i.
\end{align*}
\]

It can be checked that the local Hamiltonian \(H_{i,j}(\alpha)\) can be cast in a group-theoretical form,

\[
H_{i,j}(\alpha) = -\sum_{\sigma=\uparrow,\downarrow} (Q^i_{\sigma}Q_j,\sigma + Q^i_\sigma Q_{\sigma,j}) + 2S^i_1S^j_1 + S^i_1S_j + S_jS^j_1.
\]

The last expression immediately makes it clear that \(H(\alpha)\) commutes with all nine generators of \(gl(2|1)\). Equations (12) make it clear that \(H^Q(U)\) commutes with the global number operators of spin up and spin down. Remarkably, the model still contains the parameter \(\alpha\) as a free parameter without breaking the \(gl(2|1)\) supersymmetry. Clearly one can add to the above Hamiltonian an arbitrary chemical potential (coefficient \(\mu\)) term \(\mu\sum n_i\) and an external magnetic field (coefficient \(h\)) term \(h\sum (n_{i,\downarrow} - n_{i,\uparrow})\), which commute with \(H(\alpha)\) but break its \(gl(2|1)\) supersymmetry.

In summary, we have proposed an extension of the supersymmetric \(t-J\) model, by allowing double occupancy on one site. It has \(gl(2|1)\) supersymmetry and still contains a symmetry-preserving parameter \(\alpha\). It is not clear however that our model is exactly solved in one dimension.

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References