BASIC SPACE-TIME TRANSFORMATIONS EXPRESSED BY MEANS OF TWO-COMPONENT NUMBER SYSTEMS

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We show that the Lorentz and Galilei transformations can be expressed in the algebraic structures called the rings of two-component binary and dual number systems.

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1. Introduction

It is well known that a number of a two-component number system forming an algebraic ring can be written in the form $z = a + \varepsilon b$, where ε is an "imaginary unit". The algebraic structure of a ring demands that the product of two two-component numbers $Z_1 = a + \varepsilon b$ and $Z_2 = c + \varepsilon d$,

$$Z_1 Z_2 = (a + \varepsilon b)(c + \varepsilon d) = ac + \varepsilon(ad + bc) + \varepsilon^2 bd$$

should belong to the ring as well. Therefore

$$\varepsilon^2 = \beta + \varepsilon\gamma, \qquad \beta, \gamma \in R.$$

For the complex numbers $\beta = -1$ and $\gamma = 0$. In mathematics it was shown that except for the complex numbers there exist other two-component number systems forming commutative rings called binary or dual numbers [1]. As it is well known the rotation of the vector r with the components x and y can be expressed by means of complex numbers as $Z' = AZ$, where $Z = x + \mathrm{i}y$ and $A = \exp(\mathrm{i}\phi)$. We show that the special Lorentz and Galilei transformations can be, in similar fashion, expressed in the rings of binary and dual numbers.

An important mathematical theorem states that any possible two-component number $Z = a + \varepsilon b$, $a, b, \in R$, $\varepsilon^2 = \beta + \varepsilon\gamma$, $\beta, \gamma \in R$ can be reduced to one of the following three types [1]:

(i) the complex numbers, $Z_c = a + \varepsilon b$, $a, b \in R$ with $\varepsilon^2 = \mathrm{i}^2 = -1$;

(ii) the binary [1] (known as perplex [2], double [3] and anormal-complex [4] numbers) $Z_b = a + \varepsilon b,\ a, b \in R$ with $\varepsilon^2 = \lambda^2 = 1$;

(iii) the dual numbers, $Z_d = a + \varepsilon b,\ a, b, \in R$ with $\varepsilon^2 = \mu^2 = 0$.

The criterion for the membership to one of these two-component number systems is given by the sign of the expression $Q = (\beta + \gamma^2/4)$. If Q is negative, positive or zero then we get the complex, binary or dual numbers, respectively.

The usual operations done for the complex numbers, such as absolute value, goniometric form, and Euler formula, have corresponding operations in other two-component number systems. For a number of any of these three number systems $Z = A + \varepsilon b$ its conjugate is $\overline{Z} = a - \varepsilon b$ with i, λ, μ, respectively. The absolute value of Z can be defined as

$$|Z| = \sqrt{|Z\overline{Z}|}.$$

Z can be expressed also in the goniometric form

$$Z = |Z|\left\{\frac{a}{|Z|} + \frac{\varepsilon b}{|Z|}\right\} = |Z|\exp(\varepsilon\Phi), \quad \Phi = F\left(\frac{a}{b}\right).$$

The relevant properties of the individual number systems can be summarized in the following Table:

TABLE

Kind of numbers	Goniometric form	For
complex numbers	$\lvert Z_c\rvert e^{i\phi}, \quad \phi = \arctan(b/a)$	$a > 0$
$Z_c = a + ib$	$-\lvert Z_c\rvert e^{i\phi}, \quad \phi = \arctan(b/a)$	$a < 0$
$\lvert Z_c\rvert = \sqrt{\lvert a^2 + b^2\rvert}$	$\lvert Z_c\rvert e^{i\phi}, \quad \phi = (\operatorname{sgn} b)\pi/2$	$a = 0$
binary numbers	$\lvert Z_b\rvert e^{\lambda\Phi}, \quad \Phi = \operatorname{arctanh}(b/a)$	$a^2 > b^2 \quad a > 0$
$Z_b = a + \lambda b$	$-\lvert Z_b\rvert e^{\lambda\Phi}, \quad \Phi = \operatorname{arctanh}(b/a)$	$a^2 > b^2 \quad a < 0$
$\lvert Z_b\rvert = \sqrt{\lvert a^2 - b^2\rvert}$	Φ is undefined	$a = b$
dual numbers	$\lvert Z_d\rvert e^{\mu\alpha}, \quad \alpha = b/a$	$a > 0$
$Z_d = a + \mu b$	$-\lvert Z_d\rvert e^{\mu\alpha}, \quad \alpha = b/a$	$a < 0$
$Z_d = \sqrt{\lvert a^2\rvert}$	α is undefined	$a = 0$

The role of the complex numbers in the mathematical expression of natural laws is generally well known. Here, the question arises whether other two-component number systems can also be used for the mathematical description of natural laws as it is the case of the complex numbers. In what follows we show that this question can be answered affirmatively.

2. Geometrical transformations in the two-component number systems

The rotation of a vector r with the components x and y can be expressed as $Z' = A_c Z_c$, where $Z_c = x + iy$ and $A_c = \exp(i\phi)$. This transformation leads to the familiar formulas for the rotation of a two-component vector $x' = x(\cos\phi) - y(\sin\phi)$ and $y' = x(\sin\phi) + c(\cos\phi)$ which leaves the expression $Z_c\overline{Z}_c = x^2 + y^2$ unchanged. In the similar way we can define the transformation of the components of the binary number, $Z_b' = A_b Z_b$, where $Z_b = a + \lambda b$, $a, b, \in R$, and $A_b = \exp(\lambda\Phi)$. Taking into account the relation [2, 6]

$$\exp(\lambda\Phi) = \cosh\Phi + \lambda\sinh\Phi, \tag{1}$$

which can be derived by the expansion of the left hand side of Eq. (1) into series and the separation of the real and non-real terms. Inserting (1) into the transformation of the binary numbers we obtain

$$a' = a(\cosh\Phi) + b(\sinh\Phi) \text{ and } b' = a(\sinh\Phi) + b(\cosh\Phi). \tag{2}$$

This transformation leaves the expression $Z_b\overline{Z}_b = a^2 - b^2$ unchanged. A similar transformation in the ring of dual numbers reads

$$Z_d' = A_d Z_d, \tag{3}$$

where $Z = f + \mu g$, $f, g \in R$ and $A_d = \exp(\mu\alpha)$. Keeping in mind that $\mu^n = 0$ for $n > 1$, we have

$$\exp(\mu\alpha) = 1 + \mu\alpha + \frac{\mu^2\alpha^2}{2!} + \ldots = 1 + \mu\alpha. \tag{4}$$

Inserting Eq. (4) into Eq. (3), we obtain the transformation

$$f' = f \text{ and } g' = g + \alpha f, \tag{5}$$

which leaves the expression $Z_d\overline{Z}_d \equiv f^2$ unchanged.

3. Space-time transformations in two-component number systems

All the described transformations can be interpreted geometrically as the special transformations of the components of the two-component number systems. They play, however, an important role in physics. To rewrite them in familiar forms we have to specify physically the components of the actual number system. Let us start with the transformation in the ring of the binary numbers. If we put in the binary number $z = a + \lambda b$, $a = x$ and $b = ct$ as well as the set $\Phi = \operatorname{arctanh}(v/c)$, then the transformation turns out to be the special Lorentz transformation, $Z_b' = \exp(\lambda\Phi)Z_b$, or explicitly

$$x' = x(\cosh\Phi) + ct(\sinh\Phi) \text{ and } ct' = x(\sinh\Phi) + ct(\cosh\Phi), \tag{6}$$

which leaves the expression $x^2 - (ct)^2$ unchanged. It is interesting that the transformation (6) can be also written in the form of the product of two binary numbers $O_b = V_b(|V_b|)^{-1}$ with $V_b = c + \lambda v$, and $Z = x + \lambda(ct)$, where c is the velocity of light and v is the relative velocity of two reference frames:

$$Z_b' = O_b Z_b = \left[\frac{c}{(c^2 - v^2)^{1/2}} + \frac{\lambda v}{(c^2 - v^2)^{1/2}}\right][x + \lambda(ct)]. \tag{7}$$

The separation of the real and non-real terms in Eq. (7) yields the special Lorentz transformation (6). Since the successive application of the operators

$$O_b(1) = \frac{c}{(c^2 - v_1^2)^{1/2}} + \frac{\lambda v_1}{(c^2 - v_1^2)^{1/2}}$$

and

$$O_b(2) = \frac{c}{(c^2 - v_2^2)^{1/2}} + \frac{\lambda v_2}{(c^2 - v_2^2)^{1/2}}$$

is equivalent to the application of the operator

$$O_b(1,2) = \left[\frac{c}{(c^2 - v_{1,2}^2)^{1/2}} + \frac{\lambda v_{1,2}}{(c^2 - v_{1,2}^2)^{1/2}}\right],$$

the following equation is valid:

$$\left[\frac{c}{(c^2 - v_1^2)^{1/2}} + \frac{\lambda v_1}{(c^2 - v_1^2)^{1/2}}\right]\left[\frac{c}{(c^2 - v_2^2)^{1/2}} + \frac{\lambda v_2}{(c^2 - v_2^2)^{1/2}}\right] = \left[\frac{c}{(c^2 - v_{1,2}^2)^{1/2}} + \frac{\lambda v_{1,2}}{(c^2 - v_{1,2}^2)^{1/2}}\right].$$

This equation leads to the addition theorem of the velocities

$$v_{1,2} = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}$$

well-known in special theory of relativity.

If we put $f = ct$, $g = x$ in a dual number $Z = f + \mu g$, $f, g \in R$, and take $A_d = \exp(\mu\alpha)$, where $\alpha = v/c$, then we obtain immediately the Galilei transformation

$$x' = x + vt \quad \text{and} \quad t' = t,$$

which leaves the expression $Z_d\overline{Z}_d = t^2$ unchanged. This transformation can be also written in the form of the product of two dual numbers $O_d = V_d(|V_d|)^{-1}$, where $V_d = c + \mu v$, and $Z_d = ct + \mu x$, $Z'_d = O_d Z_d = [(c + \mu v)/c](ct + \mu x)$.

This successive application of the operator O_d

$$\left(\frac{c + \lambda v_1}{c}\right)\left(\frac{c + \lambda v_2}{c}\right) = \frac{c + \lambda v_{1,2}}{c}$$

leads to the classical addition of the velocities

$$v_{1,2} = v_1 + v_2,$$

which is typical for the Galilei transformation.

From what has been said so far it follows that:

(i) The transition from the Lorentz to Galilei transformation can be performed by means of the subset of all two-component numbers, the elements of which are given as $a + \varepsilon b$, where $\varepsilon^2 = \beta, \beta \in \langle 0.1\rangle$. As the value of β changes from 1 to 0, one can move continuously from special relativity to Newtonian physics;

(ii) It is interesting that a Lorentz-like transformation does not exist in the field of the complex numbers, i.e. a transformation of the type $Z'_c = A_c Z_c$, where $Z_c = x + \mathrm{i}(ct)$ and $A_c = \exp[\mathrm{i}\arctan(v/c)]$. Although such a transformation would have many appealing theoretical properties it is discarded through the physical experiments;

(iii) The invariants of the Galilei transformation are the square of time t^2 and the square of the space distance of two points $d^2 = (x_2 - x_1)^2$;

(iv) The corresponding Cauchy–Rieman equations for each two-component number systems, which are necessary and sufficient conditions of a function $f(Z = x + \varepsilon y) = u(x, y) + \varepsilon v(x, y)$ to be analytic, are

$$u_x = v_y - \gamma v_x, \quad u_y = \beta v_x.$$

Both u and v satisfy the partial differential equation

$$y_{yy} - \gamma u_{xy} - \beta u_{xx} = 0, \quad v_{yy} - \gamma v_{xy} - \beta v_{xx} = 0.$$

If $\gamma = 0, \beta > 0$ and $y = ct$ then this equation represents the classical wave equation with the velocity of the wave propagation $c = 1/\sqrt{\beta}$. If $\gamma = 0$ and $\beta = -1$ then this equation represents the two-dimensional Laplace equation.

For the complete formulation of the physical laws in the four-dimensional space time it is necessary to expand the two-dimensional algebraic numbers to the four-dimensional ones. The four-dimensional numbers are called quaternions. As it is well known the quaternions have been successfully applied for the mathematical formulation of many physical laws (see e.g. [5] and [7]).

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