Vol. 85 (1994) *ACTA PHYSICA POLONICA A* No. 4

CANONIZATION AND DIAGONALIZATION OF AN INFINITE DIMENSIONAL NONCANONICAL HAMILTONIAN SYSTEM: LINEAR VLASOV THEORY

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Dedicated to Prof. Iwo Białynicki-Birula on the occasion of his 60th birthday

The Vlasov–Poisson equation, which is an infinite dimensional noncanonical Hamiltonian system, is linearized about a stable homogeneous equilibrium. Canonical variables for the resulting linear system are obtained. A coordinate transformation is introduced that brings the system, which possesses a continuous spectrum, into the action-angle form where the linearized energy is diagonal.

PACS numbers: 52.25.Dg, 52.25.Mq, 52.35.Fp

1. Introduction

It is well-known that the Vlasov equation possesses an infinite dimensional Hamiltonian structure [1–4]. An interesting feature of this structure is that the natural variable, the phase space density, constitutes noncanonical coordinates, and as a result the Poisson bracket has a noncanonical form. A feature of this noncanonical form is that the Poisson bracket contains some of the nonlinearity of the theory, a feature that changes the usual procedure for linearization. One must expand both the bracket and the Hamiltonian. The resulting system is an infinite dimensional linear Hamiltonian system — one that still possesses a noncanonical form. Recently [5], motivated by Van Kampen's solution [6], it was shown how to transform this linear system into canonical action-angle coordinates, coordinates in which the Hamiltonian is diagonal. This is an infinite dimensional analogue of the elementary transformation to normal coordinates in finite dimensional oscillator systems. The analogue is not straightforward since the Vlasov–Poisson system has a continuous spectrum.

The purpose of the present paper is to present a significantly simplified method for performing the calculations of Ref. [5]. In Sec. 2 we briefly review the noncanonical Hamiltonian structure of the Vlasov–Poisson system and its linearization about a stable homogeneous equilibrium. In Sec. 3 a family of integral transforms is presented and essential identities are proved. The results of this section allow the simplifications in the calculation. In Sec. 4 we canonize and diagonalize. We conclude with Sec. 5.

2. Noncanonical Hamiltonian structure of Vlasov–Poisson theory

The Vlasov–Poisson equation for a phase space distribution function $f(\boldsymbol{r}, \boldsymbol{v}; t)$ for electrons is

$$\frac{\partial}{\partial t} f + \boldsymbol{v} \cdot \nabla f + \frac{e}{m} \boldsymbol{E}[f] \cdot \nabla_v f = 0. \tag{1}$$

By the notation $\boldsymbol{E}[f]$ we mean the solution of

$$\nabla \cdot \boldsymbol{E} = 4\pi e \left(\int \mathrm{d}^3 v f - N \right), \tag{2}$$

where N is a fixed ion background density and the square brackets emphasize the fact that, through (2), $\boldsymbol{E}$ is a functional of f.

This is an infinite dimensional Hamiltonian system or field theory, but because the distribution function does not constitute canonically conjugate variables, the Poisson bracket is of the following noncanonical form [2]:

$$\{F, G\} = \int \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \, f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right], \tag{3}$$

where F and G are arbitrary functionals, $[\,,\,]$ is the ordinary Poisson bracket

$$[a, b] = \frac{\partial a}{\partial \boldsymbol{r}} \cdot \frac{\partial b}{\partial \boldsymbol{p}} - \frac{\partial b}{\partial \boldsymbol{r}} \cdot \frac{\partial a}{\partial \boldsymbol{p}}, \tag{4}$$

and $\delta F / \delta f$ is the functional derivative. In terms of (3) the Vlasov equation is compactly written as

$$\frac{\partial f}{\partial t} = \{f, H\}, \tag{5}$$

where the Hamiltonian H is the total energy functional,

$$H = \int \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \, \frac{1}{2} m v^2 f + \frac{1}{8\pi} \int \mathrm{d}^3 \boldsymbol{r} E^2. \tag{6}$$

Two features of the bracket, (3), warrant mention: first, the form is obviously not canonical (note e.g. it is an explicit function of f) and second, the bracket is degenerate in the sense that

$$\{C, F\} = 0, \tag{7}$$

for all functionals F, where C, the so-called Casimir invariants, are given by

$$C[f] = \int \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \, \mathcal{C}(f), \tag{8}$$

where $\mathcal{C}$ is any function. Because of this degeneracy the bracket can only generate dynamics in constraint "surfaces" (sometimes called symplectic leaves) determined by the constants C. For details we refer the reader to Refs. [7–11].

We now linearize (1) about an equilibrium $f^{(0)}$ that only depends on v and take for our spatial domain a periodic box of volume V. We are interested in electrostatic perturbations and thus only consider perturbations that have spatial variations in a single direction, x say. The neutralizing background results in no equilibrium field so the linearized Vlasov equation becomes

$$\frac{\partial}{\partial t}f^{(1)} + v\frac{\partial}{\partial x}f^{(1)} + \frac{e}{m}E^{(1)}f^{(0)\prime} = 0, \tag{9}$$

where $E^{(1)}$ is determined by the linearized Poisson equation

$$\frac{\partial}{\partial x}E^{(1)} = 4\pi e\int \mathrm{d}^3\boldsymbol{v} f^{(1)}. \tag{10}$$

An expression for the energy of the linear perturbation was obtained by Kruskal and Oberman [12],

$$\delta^2 F = -\frac{m}{2}\int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\, v\frac{f^{(1)2}}{f^{(0)\prime}} + \frac{1}{8}\pi\int \mathrm{d}^3\boldsymbol{r} E^{(1)2}. \tag{11}$$

This result is the *exact* energy for the linearized equations. The Hamiltonian description of the linearized dynamics of interest here is obtained by expanding both the above noncanonical Poisson bracket and the Hamiltonian. Assuming $f = f^{(0)}(v) + f^{(1)}$ and expanding to first order yields the linearized bracket

$$\{F,G\}_{\mathrm{L}} = \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\, f^{(0)}\left[\frac{\delta F}{\delta f^{(1)}},\frac{\delta G}{\delta f^{(1)}}\right], \tag{12}$$

in terms of which the linearized Vlasov–Poisson equation can be concisely written as

$$\frac{\partial f^{(1)}}{\partial t} = \{f^{(1)},\delta^2 F\}_{\mathrm{L}}. \tag{13}$$

It is a simple matter to show that (12) satisfies the Jacobi identity.

3. A family of integral transforms

Below we discuss some properties of Hilbert transforms, which we then use to define the transformation to action-angle variables.

3.1. Review of Hilbert transforms

The theory of Hilbert transforms relies heavily on the notion of the Hölder continuity. A function ϕ is said to satisfy the Hölder condition of order α if

$$|\phi(x) - \phi(y)| \le A\,|x-y|^{\alpha} \qquad \forall\, x,y \in R, \tag{14}$$

where $A > 0$ and $0 < \alpha < 1$. If in addition to satisfying the Hölder condition, ϕ has a limit, $\phi(\infty)$, as $|x| \to \infty$ and

$$|\phi(x) - \phi(\infty)| \le \frac{A'}{x^{\mu}} \qquad x \to \infty, \tag{15}$$

where $A' > 0$ and $\mu > 0$, then the Hilbert transform of ϕ is guaranteed to exist and satisfies (14) and (15) with the same values of α and μ [13]. The Hilbert transform is defined by

$$\overline{\phi}(x) \equiv \frac{1}{\pi}\mathrm{P}\!\int \mathrm{d}y\frac{\phi(y)}{y-x}, \tag{16}$$

where P denotes the Cauchy principal value. We will state, without proof, several properties of the Hilbert transform that will be used in the following calculations [14, 15].

- The inverse transform exists and is given by

$$\phi = -\overline{\overline{\phi}}. \tag{17}$$

- The Hilbert transform has a convolution theorem

$$\overline{\phi\psi} = \overline{\phi}\psi + \phi\overline{\psi} + \overline{\overline{\phi}\,\overline{\psi}}. \tag{18}$$

- A generalization of Parseval's formula exists

$$\int \mathrm{d}x \phi\psi = \int \mathrm{d}x \overline{\phi}\,\overline{\psi}; \tag{19}$$

$$\int \mathrm{d}x \phi\overline{\psi} = -\int \mathrm{d}x \overline{\phi}\psi. \tag{20}$$

  The existence of the integrals in (19) and (20) is not guaranteed by (14) and (15) and so must be checked separately.

- Given a function α, which has a Hilbert transform, there exists a function, $F(z)$, analytic in the upper half plane, which has the limit $\overline{\alpha} + \mathrm{i}\alpha$ as z approaches the real axis from above. This function is unique up to an additive constant, which can be taken to be the value of the function at infinity.

This last point is of particular importance in that it allows us to compute some otherwise difficult Hilbert transforms. For example, consider functions α and β that satisfy (14) and (15), and are related by

$$\beta = \overline{\alpha} + \beta^{\infty}, \tag{21}$$

where

$$\beta^{\infty} = \lim_{|x|\to\infty} \beta(x). \tag{22}$$

Then $\beta + \mathrm{i}\alpha$ is the limiting value, as z approaches the real axis from above, of the function

$$F(z) = \beta(z) + \mathrm{i}\alpha(z), \tag{23}$$

analytic in the upper half plane. Let

$$\chi + \mathrm{i}\zeta = \frac{1}{F(z)}. \tag{24}$$

Then

$$\chi = \frac{\beta}{\alpha^2 + \beta^2}, \tag{25}$$

and

$$\zeta = -\frac{\alpha}{\alpha^2 + \beta^2}. \tag{26}$$

In view of the above,

$$\chi = \overline{\zeta} + \chi^{\infty}, \tag{27}$$

on the real axis.

3.2. Van Kampen modes and integral transforms

Now we consider a family of integral transformations that are inspired by the Van Kampen mode solution of the linearized Vlasov–Poisson equation [6]. This will ultimately be used to transform the Vlasov–Poisson bracket into canonical action-angle form. Define

$$\psi(x) = \mathcal{G}[\phi](x) \equiv \int \mathrm{d}y\, \mathcal{G}(y,x)\phi(y) \ := \alpha\overline{\phi} + \beta\phi, \tag{28}$$

where α and β are related by (21) and

$$\mathcal{G}(y,x) = \alpha(x)\frac{1}{\pi}\mathrm{P}\frac{1}{y-x} + \beta(x)\delta(y-x). \tag{29}$$

Taking $\alpha = \varepsilon_{\mathrm{I}}$ and $\beta = \varepsilon_{\mathrm{R}}$, we see that $\mathcal{G}$ is a Van Kampen mode and the condition (21) is clearly satisfied. We can think of the transform being parameterized by α, as each function α yields a different transform. We will say more about the structure of $\mathcal{G}$ below.

Using the convolution theorem, we can rewrite (28) as

$$\psi = \overline{\alpha\phi} - \overline{\overline{\alpha}\overline{\phi}} + \beta^{\infty}\phi. \tag{30}$$

It is easily seen that

$$\overline{\psi} = -\alpha\phi + \overline{\alpha}\overline{\phi} + \beta^{\infty}\overline{\phi} = \beta\overline{\phi} - \alpha\phi, \tag{31}$$

and

$$\beta\psi - \alpha\overline{\psi} = \beta^2\phi + \alpha^2\phi.$$

Thus provided $\alpha^2 + \beta^2 \neq 0$, we can solve for ϕ:

$$\phi = \frac{\beta\psi - \alpha\overline{\psi}}{\beta^2 + \alpha^2}; \tag{32}$$

therefore the transformation $\mathcal{G}$ has an inverse, $\widetilde{\mathcal{G}}$, given by

$$\widetilde{\mathcal{G}}[\psi] = \zeta\overline{\psi} + \chi\psi. \tag{33}$$

Here χ and ζ have the same definitions as above. The inverse exists provided that the condition $\alpha^2 + \beta^2 \neq 0$ applies in the *upper half plane*, in addition to on the real axis. Observe that $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ belong to the same family of transforms.

Now consider the chain rule relating functional derivatives with respect to ϕ to those with respect to ψ, where $\psi = \mathcal{G}[\phi]$. Let F be a functional of ϕ and consider its variation

$$\delta F = \int \mathrm{d}x \frac{\delta F}{\delta \phi}\delta\phi = \int \mathrm{d}x \frac{\delta F}{\delta\psi}\delta\psi. \tag{34}$$

Since the relationship between ϕ and ψ is linear,

$$\delta\psi = \mathcal{G}[\delta\phi], \tag{35}$$

which can be used in the expression for δF to obtain

$$\delta F = \int \mathrm{d}x \mathcal{G}^{\dagger}\left[\frac{\delta F}{\delta\phi}\right]\delta\psi, \tag{36}$$

where $\mathcal{G}^{\dagger}$ is defined by

$$\int \mathrm{d}x \xi\mathcal{G}[\lambda] = \int \mathrm{d}x \lambda\mathcal{G}^{\dagger}[\xi]. \tag{37}$$

Comparing (34) with (36) gives

$$\frac{\delta F}{\delta \psi} = \mathcal{G}^{\dagger}\left[\frac{\delta F}{\delta \phi}\right]. \tag{38}$$

Similarly

$$\frac{\delta F}{\delta \phi} = \widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \psi}\right]. \tag{39}$$

Equations (38) and (39) imply $\widetilde{\mathcal{G}}^{\dagger} = \widetilde{\mathcal{G}^{\dagger}}$.

The operator $\widetilde{\mathcal{G}}$ can be determined from the definition (37), namely

$$\int \mathrm{d}x \xi \mathcal{G}[\lambda] = \int \mathrm{d}x \left(\xi \beta \lambda + \xi \alpha \overline{\lambda}\right) = \int \mathrm{d}x \left(\xi \beta \lambda - \overline{\xi \alpha} \lambda\right)$$

$$= \int \mathrm{d}x \lambda \left(\beta \xi - \overline{\alpha \xi}\right) \equiv \int \mathrm{d}x \lambda \mathcal{G}^{\dagger}[\xi]. \tag{40}$$

Since (40) must hold for all ξ and λ, we conclude that

$$\mathcal{G}^{\dagger}[\xi] = -\overline{\alpha \xi} + \beta \xi, \tag{41}$$

and similarly,

$$\widetilde{\mathcal{G}}^{\dagger}[\xi] = -\overline{\zeta \xi} + \chi \xi. \tag{42}$$

There are many identities involving $\mathcal{G}$ and $\mathcal{G}^{\dagger}$ that can be proved. Below we give three identities that will be required subsequently for the transformation to diagonal form

$$\int \mathrm{d}x \alpha \widetilde{\mathcal{G}}^{\dagger}[\phi] \widetilde{\mathcal{G}}^{\dagger}[\psi] = -\int \mathrm{d}x \zeta \phi \psi; \tag{43}$$

$$\int \mathrm{d}x \zeta \mathcal{G}^{\dagger}[\phi] \mathcal{G}^{\dagger}[\psi] = -\int \mathrm{d}x \alpha \phi \psi; \tag{44}$$

$$\int \mathrm{d}x \frac{x}{\alpha} \mathcal{G}[\phi] \mathcal{G}[\psi] = -\int \mathrm{d}x \frac{x}{\zeta} \phi \psi - \frac{\beta^{\infty}}{\pi} \int \mathrm{d}x \phi \int \mathrm{d}x \psi. \tag{45}$$

One can see that the second identity follows from the first through the substitution

$$\phi \to \mathcal{G}^{\dagger}[\phi], \qquad \psi \to \mathcal{G}^{\dagger}[\psi]. \tag{46}$$

The first identity is proved upon substitution of the explicit expressions for $\widetilde{\mathcal{G}}^{\dagger}[\phi]$ and $\widetilde{\mathcal{G}}^{\dagger}[\psi]$, viz.

$$\int \mathrm{d}x \alpha \widetilde{\mathcal{G}}^{\dagger}[\phi] \widetilde{\mathcal{G}}^{\dagger}[\psi] = \int \mathrm{d}x \alpha \left(\chi \phi - \overline{\zeta \phi}\right)\left(\chi \psi - \overline{\zeta \psi}\right). \tag{47}$$

First consider the term that contains two Hilbert transforms,

$$\int \mathrm{d}x \alpha \overline{\zeta \phi}\, \overline{\zeta \psi} = -\int \mathrm{d}x \overline{\alpha \overline{\zeta \psi}} \zeta \phi = \int \mathrm{d}x \zeta \phi \left(\alpha \zeta \psi - \overline{\alpha} \overline{\zeta \psi} + \overline{\overline{\alpha} \zeta \psi}\right)$$

$$= \int \mathrm{d}x \alpha \zeta^2 \phi \psi - \int \mathrm{d}x \overline{\alpha} \left(\zeta \phi \overline{\zeta \psi} + \zeta \psi \overline{\zeta \phi}\right). \tag{48}$$

Using (48), the right hand side of (47) becomes

$$\int \mathrm{d}x\alpha\left(\zeta^2+\chi^2\right)\phi\psi - \int \mathrm{d}x\left(\phi\overline{\zeta\psi}+\psi\overline{\zeta\phi}\right)(\overline{\alpha}\zeta+\alpha\chi), \tag{49}$$

and from the definitions of χ and ζ we see that $\alpha\chi = -\beta\zeta$; whence, the second integral in (49) becomes

$$\int \mathrm{d}x\left(\phi\overline{\zeta\psi}+\psi\overline{\zeta\phi}\right)\zeta(\overline{\alpha}-\beta) = -\beta^{\infty}\int \mathrm{d}x\left(\phi\overline{\zeta\psi}+\psi\overline{\zeta\phi}\right)=0. \tag{50}$$

Therefore

$$\int \mathrm{d}x\alpha\tilde{\mathcal{G}}^{\dagger}[\phi]\tilde{\mathcal{G}}^{\dagger}[\psi] = \int \mathrm{d}x\alpha\left(\chi^2+\zeta^2\right)\phi\psi = -\int \mathrm{d}x\zeta\phi\psi. \tag{51}$$

The proof of the third identity proceeds in a similar manner but requires an additional property of the Hilbert transform, namely

$$\overline{x\phi} = x\overline{\phi}+\frac{1}{\pi}\int \mathrm{d}x\phi.$$

3.3. Group property of $\mathcal{G}$

The family of transforms $\mathcal{G}$ is in fact a family of infinite dimensional linear coordinate changes on function space. Here we describe its group composition law. We adopt a slightly different notation here and denote $\mathcal{G}$ by

$$\mathcal{G}[\phi;\beta,\alpha] = \alpha\overline{\phi}+\beta\phi. \tag{52}$$

The composition of two such transforms can be written out explicitly as follows:

$$\begin{aligned}\mathcal{G}[\mathcal{G}[\phi;\beta_1,\alpha_2];\beta_2,\alpha_2] &= (\alpha_1\beta_2+\beta_2\alpha_1)\overline{\phi}+(\beta_1\beta_2-\alpha_1\alpha_2)\phi\\ &= \mathcal{G}[\phi;\beta_3,\alpha_3],\end{aligned} \tag{53}$$

where

$$\beta_3+\mathrm{i}\alpha_3 = (\beta_1+\mathrm{i}\alpha_1)(\beta_2+\mathrm{i}\alpha_2). \tag{54}$$

This is the group parameter composition rule, which makes the expression for $\tilde{\mathcal{G}}$ clear, since

$$\tilde{\mathcal{G}}[\mathcal{G}[\phi;\alpha,\beta];\beta,\alpha] = \mathcal{G}[\mathcal{G}[\phi;\beta,\alpha];\chi,\zeta] = \mathcal{G}[\phi;1,0]=\phi. \tag{55}$$

The restriction $\beta+\mathrm{i}\alpha\neq 0$ for all x guarantees the existence of the inverse group element. Below we will see that the elements of this group are in essence linear infinite dimensional canonical transformations.

4. Canonization and diagonalization

The first step in making the transformation to action-angle variables is to decompose $f^{(1)}$ into its Fourier modes, i.e.

$$f^{(1)}(x,v,t) = \frac{1}{2}\sum_{k=-\infty}^{\infty} f_k(v,t)\mathrm{e}^{\mathrm{i}kx}, \tag{56}$$

where

$$f_k = \frac{2}{V}\int \mathrm{d}x e^{-ikx} f^{(1)}. \tag{57}$$

Any functional, F, of $f^{(1)}$ can be thought of as a functional of the Fourier amplitudes and the chain rule can be used to relate functional derivatives with respect to $f^{(1)}$ to those with respect to f_k,

$$\frac{\delta F}{\delta f^{(1)}} = \sum_{k=-\infty}^{\infty} \frac{\delta F}{\delta f_k}\frac{\delta f_k}{\delta f^{(1)}}. \tag{58}$$

From (57) we have

$$\frac{\delta f_k}{\delta f^{(1)}} = \frac{2}{V} e^{-ikx}.$$

which gives

$$\frac{\delta F}{\delta f^{(1)}} = \frac{2}{V}\sum_{k=-\infty}^{\infty} \frac{\delta F}{\delta f_k} e^{-ikx}. \tag{59}$$

Using (59) in the expression for the linearized bracket yields

$$\{F,G\}_{\mathrm{L}} = \frac{4i}{mV}\sum_{k=1}^{\infty} k \int \mathrm{d}v f^{(0)\prime}\left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k}\frac{\delta F}{\delta f_{-k}}\right). \tag{60}$$

The above bracket is quite close to canonical form. A simple scaling produces this end; in particular, upon letting $q_k \equiv (mV/4ikf^{(0)\prime})f_k$ and $p_k \equiv f_{-k}$ we obtain

$$\{F,G\}_{\mathrm{L}} = \sum_{k=1}^{\infty} \int \mathrm{d}v \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k}\right). \tag{61}$$

Note, having assumed that the equilibrium is stable there is no problem dividing by $f^{(0)\prime}$.

Now we use the linear transformation $\mathcal{G}$ to change variables

$$f_k = \frac{ik}{4\pi e}\mathcal{G}[\xi_k(u,t)]. \tag{62}$$

For the moment α and β are left unspecified, but we allow for the possibility that they depend on k. From (39)

$$\frac{\delta F}{\delta f_k} = \frac{4\pi e}{ik}\tilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \xi_k}\right], \tag{63}$$

which can be used to write the linearized bracket, (60), in terms of ξ_k:

$$\{F,G\}_{\mathrm{L}} = -\frac{16i}{V}\sum_{k=1}^{\infty} k \int \mathrm{d}v \frac{4\pi e^2}{mk^2}\pi f^{(0)\prime}$$
$$\times\left(\tilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \xi_k}\right]\tilde{\mathcal{G}}^{\dagger}\left[\frac{\delta G}{\delta \xi_{-k}}\right] - \tilde{\mathcal{G}}^{\dagger}\left[\frac{\delta G}{\delta \xi_k}\right]\tilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \xi_{-k}}\right]\right). \tag{64}$$

Clearly (64) is not in canonical form, however, with appropriate choices for α and β it can be made so. The first of the three identities, (43) of the previous section motivates the choices: if $\alpha \propto f^{(0)\prime}$, then

$$\int \mathrm{d}v f^{(0)\prime}\tilde{\mathcal{G}}^{\dagger}\left[\frac{\delta G}{\delta \xi_k}\right]\tilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \xi_{-k}}\right] \propto -\int \mathrm{d}u \zeta \frac{\delta G}{\delta \xi_k}\frac{\delta F}{\delta \xi_{-k}}, \tag{65}$$

which is precisely what is needed. It is convenient to set

$$\alpha(u) = -\frac{4\pi e^2}{mk^2}\pi f^{(0)\prime}(u) \equiv \varepsilon_{\rm I}(k,u), \tag{66}$$

where $\varepsilon_{\rm I}$ is the imaginary part of the plasma dielectric function. Choosing $\beta^\infty = 1$ gives

$$\beta(u) = 1 + \overline{\varepsilon_{\rm I}} \equiv \varepsilon_{\rm R}(k,u), \tag{67}$$

where $\varepsilon_{\rm R}$ is the real part of the plasma dielectric function. With these choices for α and β

$$\zeta(u) = -\frac{\varepsilon_{\rm I}(k,u)}{|\varepsilon(k,u)|^2}, \tag{68}$$

and the expression for the bracket becomes

$$\{F,G\}_{\rm L} = \frac{16{\rm i}}{V}\sum_{k=1}^{\infty} k\int {\rm d}u\frac{\varepsilon_{\rm I}}{|\varepsilon|^2}\left(\frac{\delta F}{\delta\xi_k}\frac{\delta G}{\delta\xi_{-k}} - \frac{\delta G}{\delta\xi_k}\frac{\delta F}{\delta\xi_{-k}}\right). \tag{69}$$

The condition on α and β for the existence of $\widetilde{\mathcal{G}}$ restricts us to considering only stable plasmas that do not support neutral modes, which is consistent with our choice (in this paper) of strictly monotonic equilibria.

Using (45), it is a straightforward matter to write the energy in term of the variables ξ_k. Doing so gives the following form:

$$\delta^2 F = \frac{V}{16\pi}\sum_{k=1}^{\infty}\int {\rm d}u\pi u\frac{|\varepsilon(k,u)|^2}{\varepsilon_{\rm I}(k,u)}|\xi_k|^2, \tag{70}$$

which is seen to be diagonal.

The time dependence of ξ_k is determined by the bracket, (69), and the energy, (70):

$$\frac{\partial}{\partial t}\xi_k = \left\{\xi_k, \delta^2 F\right\}_{\rm L} = -{\rm i}ku\xi_k. \tag{71}$$

Thus

$$\xi_k(u,t) = \xi_k(u){\rm e}^{-{\rm i}kut}. \tag{72}$$

Note that this is the same time dependence that was *assumed* by Van Kampen, but here follows from the diagonalization.

To obtain the physical interpretation of ξ_k, we consider Poisson's equation

$$E^{(1)}(k,t) = -4\pi\frac{{\rm i}e}{k}\int {\rm d}v f^{(1)}$$
$$= \int {\rm d}u\left(\varepsilon_{\rm I}\overline{\xi}_k + \varepsilon_{\rm R}\xi_k\right) = \int {\rm d}u\xi_k(u,t) = \int {\rm d}u{\rm e}^{-{\rm i}kut}\xi_k(u). \tag{73}$$

Thus the $\xi_k(u)$ are the Fourier amplitudes of the electric field corresponding to frequency $\omega = ku$, where k and u are *independent*.

Note that $\delta^2 F$ is not equal to the well-known expression for energy stored in a dielectric [16–18],

$$\mathcal{E}_D = \frac{V}{16\pi}\frac{\partial(\omega\varepsilon_{\rm R})}{\partial\omega}|E(k,\omega)|^2, \tag{74}$$

where ω and k are related through the dispersion relation $\varepsilon(k,\omega/k) = 0$. The expressions differ because plasmas, unlike dielectrics, possess resonant particles (see Ref. [5] for further details).

We can now transform to action-angle variables J_μ and θ_μ, by setting

$$\xi_{-k}(u) = \sqrt{\frac{16|\varepsilon_{\rm I}|}{kV|\varepsilon|^2}J_\mu}\mathrm{e}^{\mathrm{i}\theta_\mu}, \qquad \xi_k(u) = \sqrt{\frac{16|\varepsilon_{\rm I}|}{kV|\varepsilon|^2}J_\mu}\mathrm{e}^{-\mathrm{i}\theta_\mu}. \tag{75}$$

Here $\omega_\mu = ku$ and $\mu = (k,u)$. Thus the energy becomes

$$\delta^2 F = \sum_{k=1}^{\infty}\int \mathrm{d}u\omega_\mu J_\mu. \tag{76}$$

which is the expected action-angle form for an infinite dimensional Hamiltonian system with a continuous spectrum.

Through the chain rule the Poisson bracket obtains the canonical form

$$\{F,G\}_{\rm L} = \sum_{k=1}^{\infty}\int \mathrm{d}u\left(\frac{\delta F}{\delta\theta_\mu}\frac{\delta G}{\delta J_\mu} - \frac{\delta G}{\delta\theta_\mu}\frac{\delta F}{\delta J_\mu}\right). \tag{77}$$

5. Conclusion

Above, we began with the noncanonical Hamiltonian form for the Vlasov–Poisson system, which was linearized about a stable equilibrium state. The noncanonical linear system was then scaled to obtain canonical form, however in these coordinates the Hamiltonian was not diagonal. Then, an infinite dimensional canonical transformation was effected to bring the system into action-angle coordinates, where the Hamiltonian is diagonal.

It is evident that the techniques of this paper are quite general and can apply to a variety of systems. In particular for transverse waves about an homogeneous Vlasov–Maxwell equilibrium, we [19] have obtained the following result:

$$\mathcal{E} = \frac{V}{32}\sum_{k=1}^{\infty}\int \mathrm{d}v_\|v_\|\frac{\left|\varepsilon_{\rm T} - c^2/v_\|\right|^2}{\mathrm{Im}\varepsilon_{\rm T}}|\boldsymbol{E}_k|^2, \tag{78}$$

with obvious definitions of symbols. This result is analogous to that of (70). The details of this calculation will be presented elsewhere. Since quantum mechanics in the Weyl–Wigner formalism [20] possesses a bracket of a form similar to that for the Vlasov–Poisson theory, it is likely that the techniques of this paper are applicable.

Acknowledgments

This work was supported by the U.S. Department of Energy contract No. DE-FG05-80ET-53088.

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