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UNIVERSAL PROPAGATOR FOR GROUP-RELATED COHERENT STATES

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Dedicated to Prof. Iwo Białynicki-Birula on the occasion of his 60th birthday

In earlier work universal propagators were introduced for the Heisenberg –Weyl group, the affine group, and the rotation group. By generalizing these constructions we show here that it is possible to introduce a universal propagator for a rather general unitary Lie group. In the context of coherent-state representations, the universal propagator is a single function independent of any particular choice of fiducial vector, which nonetheless, propagates all coherent state Hilbert space representatives correctly.

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1. Introduction

A universal propagator has been constructed for the canonical coherent states (see [1]), for the affine [or SU(1,1)] coherent states (see [2]), and for the spin [or SU(2)] coherent states (see [3]). As a prelude to discussing the construction of the universal propagator for general group-related coherent states, we first outline its construction for the case of the canonical coherent states. Let P and Q denote an irreducible pair of self-adjoint Heisenberg operators satisfying the CCR, $[Q,P]=\mathrm{i}$, where $\hbar=1$, then

$$\eta^{pq}=e^{-\mathrm{i}qP}e^{\mathrm{i}pQ}\eta$$

defined for all pairs $(p,q)\in \boldsymbol{R}^2$, denotes a family of normalized, overcomplete states for a fixed, normalized fiducial vector η. These states give rise to representation of Hilbert space $\boldsymbol{H}$ by bounded functions,

$$\psi_\eta(p,q)\equiv\langle\eta^{pq},\psi\rangle,$$

defined for all $\psi\in\boldsymbol{H}$, which evidently depends on the choice of η. If $\mathcal{H}$ denotes the self-adjoint Hamiltonian for the quantum system under consideration, then the abstract Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi=\mathcal{H}\psi,$$

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and its formal solution in terms of the evolution operator $U(t) = \exp(-\mathrm{i}t\mathcal{H})$,

$$\psi(t'') = U(t''-t')\psi(t'),$$

are given in this representation by

$$\mathrm{i}\frac{\partial}{\partial t}\psi_\eta(p,q,t) = \langle \eta^{pq}, \mathcal{H}(P,Q)\psi(t)\rangle$$

and

$$\psi_\eta(p'',q'',t'') = \int K_\eta(p'',q'',t'';p',q',t')\psi_\eta(p',q',t')\frac{\mathrm{d}p'\mathrm{d}q'}{2\pi},$$

where

$$K_\eta(p'',q'',t'';p',q',t') = \langle \eta^{p''q''}, U(t''-t')\eta^{p'q'}\rangle.$$

Clearly, K_η depends on the choice of the fiducial vector η. In contrast the *universal propagator* $K(p'',q'',t'';p',q',t')$ is a single function *independent* of any particular fiducial vector, which, nevertheless, propagates ψ_η correctly, i.e.

$$\psi_\eta(p'',q'',t'') = \int K(p'',q'',t'';p',q',t')\psi_\eta(p',q',t')\frac{\mathrm{d}p'\mathrm{d}q'}{2\pi}$$

for any choice of fiducial vector. This function is constructed in two steps, first one observes that

$$\left(-\mathrm{i}\frac{\partial}{\partial q}\right)\psi_\eta(p,q) = \langle \eta^{pq}, P\psi\rangle,$$

$$\left(q+\mathrm{i}\frac{\partial}{\partial p}\right)\psi_\eta(p,q) = \langle \eta^{pq}, Q\psi\rangle$$

hold independently of η. Thus if $\mathcal{H} = \mathcal{H}(P,Q)$ denotes the Hamiltonian it follows that Schrödinger's equation takes the form

$$\mathrm{i}\frac{\partial}{\partial t}\psi_\eta(p,q,t) = \langle \eta^{pq}, \mathcal{H}(P,Q)\psi(t)\rangle = \mathcal{H}\left(-\mathrm{i}\frac{\partial}{\partial q}, q+\mathrm{i}\frac{\partial}{\partial p}\right)\psi_\eta(p,q,t)$$

valid for all η. Since the universal propagator is also a solution to Schrödinger's equation we have that

$$\mathrm{i}\frac{\partial}{\partial t}K(p,q,t;p',q',t') = \mathcal{H}\left(-\mathrm{i}\frac{\partial}{\partial q}, q+\mathrm{i}\frac{\partial}{\partial p}\right)K(p,q,t;p',q',t'). \tag{1}$$

One then interprets this resulting Schrödinger equation (1) as an equation for *two* degrees of freedom. In this interpretation $y_1 = q$ and $y_2 = p$ are viewed as two "coordinates", and one is looking at the irreducible Schrödinger representation of a special class of two-variable Hamiltonians, ones where the classical Hamiltonian is restricted to have the form $H_c(p_1, y_1 - p_2)$.

Based on this interpretation a standard phase-space path integral solution may be given for the universal propagator between sharp Schrödinger states. In particular, it follows after some change of variables that the universal propagator is given by the formal path integral

$$K(p'',q'',t'';p',q',t')$$
$$= \mathcal{M}\int \exp\left(\mathrm{i}\int [q\dot{p} + k\dot{q} - x\dot{p} - \mathcal{H}(k,x)]\,dt\right)\mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x, \tag{2}$$

where "x" and "k" are "momenta" conjugate to the "coordinates" "p" and "q".

Despite the fact that the universal propagator has been constructed by interpreting the appropriate Schrödinger equation (1) as an equation for two degrees of freedom, it is nonetheless true that the classical limit corresponds to a single (canonical) degree of freedom (see [1]). This is a direct consequence of the restricted form chosen for the quantum or classical Hamiltonian.

We will now show that it is possible to introduce an appropriate universal propagator for group-related coherent states by generalizing the constructions for canonical, affine, and spin coherent states.

2. Group-related coherent states and their propagators

2.1. Group-related coherent states

Let us denote by $X_a, a = 1, \ldots, N$, the hermitian generators of an N-dimensional Lie group G. The X_a are elements of the associated Lie-algebra g, whose commutation relations are given by

$$[X_a, X_b] = \mathrm{i} c_{ab}{}^d X_d,$$

wherein the summation convention has been adopted. From the structure constants $c_{ab}{}^d$ we form the $N(N \times N)$ matrices

$$c_a \equiv (-c_{ab}{}^d).$$

These matrices provide the adjoint representation for the Lie-algebra g, (see [4]). In the following we denote by $V[\ell]$ a strongly continuous (faithful) unitary representation of the Lie group G on an appropriate Hilbert space. Each element of the representation is characterized by N-parameters, $\ell^m, m = 1, \ldots, N$. For definiteness we assume that the elements $V[\ell]$ are given in canonical coordinates of the first kind, i.e.,

$$V[\ell] = \exp(-\mathrm{i}\ell^m X_m).$$

With each such representation we associate a left invariant one-form, the so-called Maurer–Cartan form,

$$\mathrm{i}V^\dagger[\ell](V[\ell + \mathrm{d}\ell] - V[\ell]) = \omega_m{}^k(\ell)\mathrm{d}\ell^m X_k. \tag{3}$$

In addition, we introduce a second set of coefficients $U_a{}^b(\ell)$, which are defined by

$$V^\dagger[\ell]X_a V[\ell] = U_a{}^b(\ell)X_b. \tag{4}$$

From these coefficients a matrix $U(\ell) = \left[U_a{}^b(\ell)\right]$ is formed, and an implicit characterization of this matrix is given by (see [5])

$$U(\ell) = \exp(c_k \ell^k).$$

For all ℓ and a fixed, normalized fiducial vector $\eta \in \boldsymbol{H}$ we define the set of normalized coherent states corresponding to a Lie group G as

$$\eta^\ell \equiv V[\ell]\eta,$$

and we assume that these states give rise to a resolution of unity in the form,

$$I_{\boldsymbol{H}} = \int_G \eta^\ell \langle \eta^\ell, \cdot \rangle \mathrm{d}\mu(\ell),$$

where $\mathrm{d}\mu(\ell)$ denotes the left-invariant group measure given by

$$\mathrm{d}\mu(\ell) = \det({\omega_m}^b) \prod_{k=1}^{N} \mathrm{d}\ell^k .$$

Restricting attention to the so-called square integrable representations of G (see [6, p. 39]), the existence of such a resolution of unity is a direct consequence of Schur's Lemma for irreducible representations. In case the representation is reducible, the resolution of unity must be established as one of the defining properties of the coherent states. This may pose an additional restriction on the choice of fiducial vectors allowed in the coherent state definition. For both compact groups and non-compact groups we normalized the measure so that

$$\int |\langle \eta, U[\ell]\eta \rangle|^2 \mathrm{d}\mu(\ell) = 1;$$

in all cases we denote this normalization by $1/|G|$. Hereafter we assume that the group measure has the appropriately normalized form

$$\mathrm{d}\mu(\ell) = \frac{\det({\omega_m}^b)}{|G|} \prod_{k=1}^{N} \mathrm{d}\ell^k .$$

The normalized coherent states give rise to a representation of the Hilbert space $\boldsymbol{H}$, by bounded continuous functions,

$$\psi_\eta(\ell) \equiv \langle \eta^\ell, \psi \rangle,$$

defined for all $\psi \in \boldsymbol{H}$, which evidently depends on the choice of η. An inner product in this representation is introduced in the following way:

$$\langle \phi, \psi \rangle = \int \phi^*_\eta(\ell) \psi_\eta(\ell) \mathrm{d}\mu(\ell),$$

the result of which is independent of the choice of the fiducial vector η. The representation space is given by $L^2_\eta(G, \mathrm{d}\mu(\ell))$; this space is spanned by the $\psi_\eta(\ell)$ for a fixed η and for all $\psi \in \boldsymbol{H}$.

2.2. Propagators

If we denote by $\mathcal{H}(X_1, \ldots, X_N)$ the self-adjoint Hamiltonian of a quantum mechanical system on $\boldsymbol{H}$, then the abstract Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi = \mathcal{H}\psi,$$

and its formal solution in terms of the evolution operator $U(t) = \exp(-\mathrm{i}t\mathcal{H})$

$$\psi(t'') = U(t'' - t')\psi(t')$$

take the following form:

$$\mathrm{i}\frac{\partial}{\partial t}\psi_\eta(\ell, t) = \langle \eta^\ell, \mathcal{H}(X_1, \ldots, X_N)\psi(t) \rangle,$$

and

$$\psi_\eta(\ell'', t) = \int K_\eta(\ell'', t''; \ell', t')\psi_\eta(\ell')\mathrm{d}\mu(\ell').$$

Here,

$$K_\eta(\ell'',t'';\ell',t') = \langle \eta^{\ell''}, U(t''-t')\eta^{\ell'}\rangle$$

denotes the coherent state propagator for G, which clearly depends on the choice of the fiducial vector η as does ψ_η. In contrast the *universal propagator* $K(\ell'',t'';\ell',t')$ is a single function *independent* of any particular fiducial vector, which, nonetheless, propagates the ψ_η correctly, i.e.

$$\psi_\eta(\ell'',t'') = \int K(\ell'',t'';\ell',t')\psi_\eta(\ell',t')\mathrm{d}\mu(\ell') \tag{5}$$

for any choice of fiducial vector. That the functions K_η and K are qualitatively different can be seen from their behavior in the limit $t'' \to t'$. On the one hand we have

$$\lim_{t''\to t'} K_\eta(\ell'',t'';\ell',t') = \langle \eta^{\ell''}, \eta^{\ell'}\rangle, \tag{6}$$

since

$$s-\lim_{t''\to t'} U(t''-t') = I_{\boldsymbol{H}}.$$

This is the usual reproducing kernel for group-related coherent states, which is a *projection operator* in $L^2(G)$ onto the representation space $L^2_\eta(G)$. While on the other hand if (5) is to hold for arbitrary η, we must require that

$$\lim_{t''\to t'} K(\ell'',t'';\ell',t') = \delta(\ell'';\ell'), \tag{7}$$

where $\delta(\ell'';\ell')$ is defined as

$$\delta(\ell'';\ell') \equiv \frac{|G|}{\det(\omega_m{}^b)}\prod_{k=1}^N \delta(\ell''^k - \ell'^k),$$

so that

$$f(\ell') = \int f(\ell)\delta(\ell;\ell')\mathrm{d}\mu(\ell).$$

As a first step in our construction of the universal propagator we express the group generators $X_a, a = 1,\ldots,N$ in terms of differential operators that describe their action on the representation space $L^2_\eta(G)$ of $\boldsymbol{H}$, independently of the chosen fiducial vector η. Using (3) and (4) it is not hard to show that

$$U_k{}^m \omega^{-1}{}_m{}^n \left(-\mathrm{i}\frac{\partial}{\partial \ell^n}\right)\langle \eta^\ell, \psi\rangle = \langle \eta^\ell, X_k\psi\rangle, \qquad k = 1,\ldots,N, \tag{8}$$

hold for arbitrary η. These operators are hermitian with respect to the group invariant measure for G. With $p_{\ell^m} = -\mathrm{i}\frac{\partial}{\partial \ell^m}, m = 1,\ldots,N$ we define the differential operators

$$x_k(p_\ell,\ell) = M^{-1}{}_k{}^m p_{\ell^m}, \qquad k = 1,\ldots,N$$

where $M_k{}^m \equiv \omega_k{}^j U^{-1}{}_j{}^m$. Thus if $\mathcal{H}(X_1,\ldots,X_N)$ denotes the Hamiltonian it follows that Schrödinger's equation takes the form

$$\mathrm{i}\frac{\partial}{\partial t}\psi_\eta(\ell) = \langle \eta^\ell, \mathcal{H}(X_1,\ldots,X_N)\psi(t)\rangle = \mathcal{H}(x_1(p_\ell,\ell),\ldots,x_N(p_\ell,\ell))\psi_\eta(\ell).$$

Since the propagators are also solutions to Schrödinger's equation, then

$$\mathrm{i}\frac{\partial}{\partial t}K_\#(\ell,t;\ell',t') = \mathcal{H}(x_1(p_\ell,\ell),\ldots,x_N(p_\ell,\ell))K_\#(\ell,t;\ell',t'). \tag{9}$$

where $K_\#$ denotes either K_η or K. Note that the *initial conditions* (at $t = t'$), i.e. either (6) or (7), determine which function is under consideration.

Equation (9) admits two qualitatively different interpretations, one suitable for K_η, the other suitable for K. When one considers K_η the operators $x_k(p_\ell, \ell)$ are simply representatives of the N operators X_k acting on the representation space $L^2_\eta(G)$. For K a different interpretation is appropriate.

3. The universal propagator

When the universal propagator K is under consideration one interprets the resulting Schrödinger equation (9) as an equation appropriate to *N separate and independent canonical* degrees of freedom. In this interpretation $q_1 = \ell_1, \ldots, q_N = \ell_N$ are viewed as N "coordinates", and one is looking at the irreducible Schrödinger representation of a special class of N-variable Hamiltonians, the ones where the classical Hamiltonian is restricted to have the form $\mathcal{H}(\tilde{x}_1(p,q), \ldots, \tilde{x}_N(p,q))$, instead of the most general form $\mathcal{H}(p,q) = \mathcal{H}(p_1, \ldots, p_N, q_1, \ldots, q_N)$.

Based on this interpretation a standard phase-space path integral solution may be given for the universal propagator for group-related coherent states between sharp Schrödinger states. In particular, it follows that

$$K(q'', t''; q', t') = \mathcal{M} \int \exp\left(\mathrm{i} \int [p_m \dot{q}^m - \mathcal{H}(\tilde{x}_1(p,q), \ldots, \tilde{x}_N(p,q))]\,\mathrm{d}t\right) \times \prod \mathrm{d}p(t)\mathrm{d}\mu(q(t)), \tag{10}$$

where "p_1", ..., "p_N" denote "momenta" conjugate to the "coordinates" "q_1", ..., "q_N". Note that the Hamiltonian has been used in the special form discussed above and that its arguments are given by the following functions:

$$\tilde{x}_k(p,q) = {M^{-1}}_k{}^m p_m, \qquad k = 1, \ldots, N.$$

Since this is a standard phase-space path integral representation, the number of integrals over the momenta $p_1, \ldots, p_N$ is always one more than the number of integrals over the coordinates $q_1, \ldots, q_N$. The integration over the coordinates is restricted to the group manifold G. If part of the group manifold is compact then the momenta conjugate to the restricted range or periodic "coordinates" of this part of the group manifold are discrete variables. For this class of momenta the notation $\int \prod \mathrm{d}p(t)$ is then properly to be understood as sums rather than integrals.

4. Propagation with the universal propagator

For convenience in the following construction we assume that the representation $V[\ell]$ of G is irreducible. In that case we may choose a group element as a "basic" propagator. Let us set

$$V_T[\beta] \equiv \exp(-\mathrm{i}T\beta^a X_a).$$

Then one finds for the time evolution of an arbitrary element of $L^2_\eta(G)$ under $V_T[\beta]$ the following

$$\psi_\eta(\ell, T) = \langle \eta^\ell, V_T[\beta]\psi\rangle = \langle V_T^\dagger[\beta] V[\ell]\eta, \psi\rangle = \psi_\eta(\ell_T)$$

which holds for any η. Here

$$\ell_T = (T\beta)^{-1} \circ \ell,$$

where $\circ$ denotes group multiplication. In the present case it is clear that there exists a universal propagator such that

$$\psi_\eta(\ell'', T) = \int K_\beta(\ell'', T; \ell', 0)\psi_\eta(\ell')\mathrm{d}\mu(\ell')$$

or stated otherwise,

$$\psi_\eta(\ell''_T) = \int K_\beta(\ell'', T; \ell', 0)\psi_\eta(\ell')\mathrm{d}\mu(\ell').$$

If this last equation is to be valid for arbitrary η and ψ, then we must require that

$$K_\beta(\ell'', T; \ell', 0) = \delta(\ell''_T; \ell').$$

Now since we have chosen a unitary irreducible representation $V[\ell]$ for the Lie group G, we are assured that any bounded operator may be constructed as the (weak) limit of sums of such unitary operators (see [7, p. 45]). In particular, we can represent any time evolution operator $\exp(-\mathrm{i}T\mathcal{H})$ as the weak limit of finite linear combinations of the $V[\beta]$. Let

$$B_n^T = \sum_{j=0}^{n} \gamma_j V[\beta_j],$$

and correspondingly

$$K_n(\ell'', T; \ell', 0) = \sum_{j=0}^{n} \gamma_j K_{\beta_j}(\ell'', 1; \ell', 0).$$

Then for suitable $\{\gamma_j\}$ and $\{\beta_j\}$,

$$\langle\phi, \exp(-\mathrm{i}T\mathcal{H})\psi\rangle = \lim_{n\to\infty} \langle\phi, B_n^T\psi\rangle \qquad \forall\phi, \psi \in \boldsymbol{H},$$

and in particular

$$\langle\eta^{\ell''}, \exp(-\mathrm{i}T\mathcal{H})\psi\rangle = \lim_{n\to\infty} \langle\eta^{\ell''}, B_n^T\psi\rangle \qquad \forall\, \psi \in \boldsymbol{H}, \qquad \ell'' \in G.$$

If $K(\ell'', T; \ell, 0)$ denotes the universal propagator associated with the time evolution operator $\exp(-\mathrm{i}T\mathcal{H})$, then it follows that

$$\int K(\ell'', T; \ell, 0)\psi_\eta(\ell)\mathrm{d}\mu(\ell) = \lim_{n\to\infty} \int K_n(\ell'', T; \ell, 0)\psi_\eta(\ell)\mathrm{d}\mu(\ell)$$

$$\forall\psi_\eta \in L^2_\eta(G).$$

This relation asserts that any desired universal propagator K can be written as the limit of the K_n in the indicated sense. Stated otherwise any universal propagator can be written as the weak* – limit of the set $\{K_n\}_{n\in N}$ (see [8, p. 160]), i.e.,

$$K(\ell'', T; \ell, 0) = \mathrm{w}^* - \lim_{n\to\infty} K_n(\ell'', T; \ell, 0).$$

Although the point is clear from the foregoing it is worth emphasizing that the universal propagator evolves any state in a way that leaves the choice of η invariant. Inasmuch as the choice of η corresponds to the choice of polarization in the sense of geometric quantization it follows that the universal propagator evolves the system while at the same time preserving the polarization. Since this fact holds for a general Hamiltonian we see that the universal propagator for the unitary Lie group G provides an acceptable solution to the longstanding problem posed by geometric quantization.

5. Classical limit

5.1. Classical limit

Even though the universal propagator has been constructed by interpreting the appropriate Schrödinger equation (9) as an equation for N (canonical) degrees of freedom, it should nonetheless be true that the classical limit corresponds to degree(s) of freedom associated with the group G. As we will show this property holds because the equations of motion obtained from the action functional for the universal propagator of the Lie group G imply the equations of motion obtained from the action functional for the group-related coherent state path integral.

Observe, for an arbitrary fiducial vector η, that the classical action appropriate to the group-related coherent state path integral is given by (see [6, p. 64])

$$\begin{aligned} I_{cl} &= \lim_{\hbar\to 0} \int \left[\mathrm{i}\langle \eta^{\ell}, \frac{\mathrm{d}}{\mathrm{d}t}\eta^{\ell}\rangle - \langle \eta^{\ell}, \mathcal{H}(X_1,\ldots,X_N)\eta^{\ell}\rangle \right] \mathrm{d}t \\ &= \int \left[\omega_m{}^k \dot{\ell}^m v_k - \mathcal{H}(U_1{}^b v_b,\ldots,U_N{}^b v_b) \right] \mathrm{d}t, \end{aligned} \tag{11}$$

where $v_k \equiv \langle \eta, X_k \eta\rangle$, $k = 1,\ldots,N$, which are real constants. To achieve this classical limit we must restrict η so that

$$\lim_{\hbar\to 0} \langle \eta, (X_k - v_k)^2 \eta\rangle = 0,$$

i.e., we insist on vanishing dispersion as $\hbar \to 0$.

Extremal variation of this action functional, with respect to the independent labels ℓ^b, holding the end points fixed, yields the equations of motion

$$v_d \left[\frac{\partial}{\partial \ell^c}(\omega_b{}^d) - \frac{\partial}{\partial \ell^b}(\omega_c{}^d) \right] \dot{\ell}^b = \mathcal{H}^a \frac{\partial}{\partial \ell^c}(U_a{}^f) v_f, \tag{12}$$

where $\mathcal{H}^a$ denotes the partial derivative of $\mathcal{H}$ with respect to the a-th argument $a = 1,\ldots,N$.

Observe that the generally nonvanishing values of $v_1,\ldots,v_N$ are the *vestiges of the group-related coherent state representation induced by η that remain even after the limit $\hbar \to 0$.*

5.2. Classical limit of the universal propagator

For the universal propagator the classical action functional is identified as [see Eq. (10)]

$$I_{cl} = \int \left[p_j \dot{\ell}^j - \mathcal{H}(\tilde{x}_1(p,\ell),\ldots,\tilde{x}_N(p,\ell)) \right] \mathrm{d}t$$

$$= \int \left[p_j \dot{\ell}^j - \mathcal{H}\left({M^{-1}}_1{}^j p_j, \ldots, {M^{-1}}_N{}^j p_j\right)\right] \mathrm{d}t. \tag{13}$$

Extremal variation of this action functional holding the end points fixed yields the equations of motion

$$\dot{\ell}^b = \mathcal{H}^a {M^{-1}}_a{}^b, \tag{14}$$

$$\dot{p}_c = -\mathcal{H}^a \frac{\partial}{\partial \ell^c}\left({M^{-1}}_a{}^j\right) p_j. \tag{15}$$

If we substitute $\mathcal{H}^d = M_f{}^d \dot{\ell}^f$ into equation (15) and contract both sides with ${U^{-1}}_h{}^b {M^{-1}}_b{}^c$, we find

$${U^{-1}}_h{}^b {M^{-1}}_b{}^c \dot{p}_c = \dot{\ell}^f {U^{-1}}_h{}^b {M^{-1}}_b{}^c \frac{\partial}{\partial \ell^c}(M_f{}^d) {M^{-1}}_d{}^j p_j, \tag{16}$$

where we have used the fact that

$$M_f{}^d \frac{\partial}{\partial \ell^c}({M^{-1}}_d{}^j) = -\frac{\partial}{\partial \ell^c}(M_f{}^d) {M^{-1}}_d{}^j.$$

We now assume the relation

$$\dot{\ell}^f {U^{-1}}_h{}^b {M^{-1}}_b{}^c \frac{\partial}{\partial \ell^c}(M_f{}^d) {M^{-1}}_d{}^j p_j = -\frac{\partial}{\partial \ell^m}({U^{-1}}_h{}^b {M^{-1}}_b{}^j) \dot{\ell}^m p_j, \tag{17}$$

which we shall prove below. If we insert equation (17) into equation (16) then we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({U^{-1}}_k{}^b {M^{-1}}_b{}^j p_j\right) = 0. \tag{18}$$

Therefore, we can introduce a set of integration constants, $c_1, \ldots, c_N$, such that

$$p_j = \omega_j{}^m c_m, \tag{19}$$

where we have used the identity $\omega_j{}^m = M_j{}^d U_d{}^m$. If we now substitute this form of p_j into Eqs. (14) and (15), we find the following set of $2N$ equations:

$$\dot{\ell}^b = \mathcal{H}^a(U_1{}^d c_d, \ldots, U_N{}^d c_d) {M^{-1}}_a{}^b,$$

$$\frac{\partial}{\partial t}(\omega_c{}^d) c_d = -\mathcal{H}^a(U_1{}^d c_d, \ldots, U_N{}^d c_d) \frac{\partial}{\partial \ell^c}({M^{-1}}_a{}^j) \omega_j{}^m c_m.$$

Carrying out the indicated partial differentiation with respect to time these equations become

$$\dot{\ell}^b = \mathcal{H}^a {M^{-1}}_a{}^b, \tag{20}$$

$$\frac{\partial}{\partial \ell^b}(\omega_c{}^d) \dot{\ell}^b c_d = -\mathcal{H}^a \frac{\partial}{\partial \ell^c}({M^{-1}}_a{}^j) \omega_j{}^m c_m. \tag{21}$$

Next we contract (20) with $\partial/\partial \ell^c \left(\omega_b{}^d\right) c_d$, which yields

$$\frac{\partial}{\partial \ell^c}(\omega_b{}^d) \dot{\ell}^b c_d = \mathcal{H}^a {M^{-1}}_a{}^b \frac{\partial}{\partial \ell^c}(\omega_b{}^d) c_d, \tag{22}$$

$$\frac{\partial}{\partial \ell^b}(\omega_c{}^d) \dot{\ell}^b c_d = -\mathcal{H}^a \frac{\partial}{\partial \ell^c}({M^{-1}}_a{}^j) \omega_j{}^m c_m. \tag{23}$$

If we now subtract equation (23) from equation (22) then we find

$$c_d \left(\frac{\partial}{\partial \ell^c}(\omega_b{}^d) - \frac{\partial}{\partial \ell^b}(\omega_c{}^d)\right) \dot{\ell}^b = \mathcal{H}^a \frac{\partial}{\partial \ell^c}(U_a{}^f) c_f, \tag{24}$$

where we have used the fact that $U_a{}^f = M^{-1}{}_a{}^g \omega_g{}^f$. Among all possible allowed values of $c_1, \ldots, c_N$ are those that coincide with $v_1, \ldots, v_N$ for an arbitrary fiducial vector. Hence, the above equations can be identified with the equation of motion obtained from the action functional for group-related coherent states (see Eq. (12)). Therefore, the set of classical equations of motion for the universal propagator implies the set of classical equations of motion appropriate to the group-related coherent state propagator for an arbitrary fiducial vector. Thus we find that the set of solutions of the universal classical equations of motion appropriate to the universal propagator for the Lie group G includes *every possible solution* of the classical equations of motion appropriate to the group-related coherent state propagator for arbitrary η.

We close this section by giving a proof of Eq. (17). To show Eq. (17) it is sufficient to show that

$$\frac{\partial}{\partial \ell^m}\left(U^{-1}{}_h{}^c M^{-1}{}_c{}^b\right) = -U^{-1}{}_h{}^c M^{-1}{}_c{}^f \frac{\partial}{\partial \ell^f}(M_m{}^d) M^{-1}{}_d{}^b \tag{25}$$

holds. This equation can be simplified as follows

$$M_n{}^j U_j{}^h \frac{\partial}{\partial \ell^m}\left(U^{-1}{}_h{}^c M^{-1}{}_c{}^b\right) = -\frac{\partial}{\partial \ell^n}(M_m{}^d) M^{-1}{}_d{}^b$$

$$U^{-1}{}_h{}^f \frac{\partial}{\partial \ell^m}(M_n{}^j U_j{}^h) = \frac{\partial}{\partial \ell^n}(M_m{}^f).$$

After carrying out the indicated partial differentiation of the product and rearranging the terms we find

$$\frac{\partial}{\partial \ell^m}(M_n{}^f) - \frac{\partial}{\partial \ell^n}(M_m{}^f) = -M_n{}^j \frac{\partial}{\partial \ell^m}(U_j{}^h) U^{-1}{}_h{}^f.$$

Next we use the relation (see [5])

$$\frac{\partial}{\partial \ell^m}(U_j{}^h) = M_m{}^d c_{jd}{}^n U_n{}^h$$

which leads to

$$\frac{\partial}{\partial \ell^m}(M_n{}^f) - \frac{\partial}{\partial \ell^n}(M_m{}^f) = -M_n{}^j M_m{}^d c_{jd}{}^f. \tag{26}$$

Now let (see [5])

$$\overline{X}_a \equiv M_a{}^c X_c = \int_0^1 \exp(-\mathrm{i}s\ell^b X_b) X_a \exp(\mathrm{i}s\ell^c X_c) \mathrm{d}s. \tag{27}$$

Using Eq. (27), Eq. (26) becomes

$$\frac{\partial}{\partial \ell^m}\overline{X}_n - \frac{\partial}{\partial \ell^n}\overline{X}_m = \mathrm{i}[\overline{X}_n, \overline{X}_m], \tag{28}$$

Hence, we have reduced our problem to showing that (28) holds. To show this we make use of the general rule

$$\frac{\partial}{\partial \ell^n}\exp(-\mathrm{i}\xi\ell^a X_a) = -\mathrm{i}\xi \int_0^1 \exp(-\mathrm{i}\xi t\ell^b X_b) X_n \exp(-\mathrm{i}\xi(1-t)\ell^c X_c) \mathrm{d}t.$$

Therefore,

$$\frac{\partial}{\partial \ell^m}(\overline{X}_n) - \frac{\partial}{\partial \ell^n}(\overline{X}_m)$$

$$= \frac{\partial}{\partial \ell^m} \int_0^1 \exp(-\mathrm{i}s\ell^a X_a) X_n \exp(\mathrm{i}s\ell^b X_b) \mathrm{d}s$$
$$- \frac{\partial}{\partial \ell^n} \int_0^1 \exp(-\mathrm{i}s\ell^c X_c) X_m \exp(\mathrm{i}s\ell^d X_d) \mathrm{d}s$$
$$= -\mathrm{i} \int_0^1 \mathrm{d}ss \int_0^1 \exp(-\mathrm{i}st\ell^a X_a) X_m \exp[-\mathrm{i}s(1-t)\ell^b X_b] X_n \exp(\mathrm{i}s\ell^c X_c) \mathrm{d}t$$
$$+\mathrm{i} \int_0^1 \mathrm{d}ss \int_0^1 \exp(-\mathrm{i}s\ell^a X_a) X_n \exp(-\mathrm{i}st\ell^b X_b) X_m \exp[\mathrm{i}s(1-t)\ell^c X_c] \mathrm{d}t$$
$$+\mathrm{i} \int_0^1 \mathrm{d}ss \int_0^1 \exp(-\mathrm{i}st\ell^a X_a) X_n \exp[-\mathrm{i}s(1-t)\ell^b X_b] X_m \exp(\mathrm{i}s\ell^c X_c) \mathrm{d}t$$
$$-\mathrm{i} \int_0^1 \mathrm{d}ss \int_0^1 \exp(-\mathrm{i}s\ell^a X_a) X_m \exp(-\mathrm{i}st\ell^b X_b) X_n \exp[\mathrm{i}s(1-t)\ell^c X_c] \mathrm{d}t.$$

If we now change $t \to t/s$, then we find

$$\frac{\partial}{\partial \ell^m}(\overline{X}_n) - \frac{\partial}{\partial \ell^n}(\overline{X}_m)$$
$$= \mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}t\ell^a X_a) X_n \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_m \exp(\mathrm{i}s\ell^c X_c) \mathrm{d}t$$
$$+\mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}s\ell^a X_a) X_n \exp(-\mathrm{i}t\ell^b X_b) X_m \exp[\mathrm{i}(s-t)\ell^c X_c] \mathrm{d}t$$
$$-\mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}t\ell^a X_a) X_m \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_n \exp(\mathrm{i}s\ell^c X_c) \mathrm{d}t$$
$$-\mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}s\ell^a X_a) X_m \exp(-\mathrm{i}t\ell^b X_b) X_n \exp[\mathrm{i}(s-t)\ell^c X_c] \mathrm{d}t.$$

In the second and the fourth integral we now change $t \to (s-t)$

$$\frac{\partial}{\partial \ell^m}(\overline{X}_n) - \frac{\partial}{\partial \ell^n}(\overline{X}_m)$$
$$= \mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}t\ell^a X_a) X_n \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_m \exp(\mathrm{i}s\ell^c X_c) \mathrm{d}t$$
$$+\mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}s\ell^a X_a) X_n \exp\left[\mathrm{i}(s-t)\ell^b X_b\right] X_m \exp(\mathrm{i}t\ell^c X_c) \mathrm{d}t$$
$$-\mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}t\ell^a X_a) X_m \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_n \exp(\mathrm{i}s\ell^c X_c) \mathrm{d}t$$
$$-\mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}s\ell^a X_a) X_m \exp\left[\mathrm{i}(s-t)\ell^b X_b\right] X_n \exp(\mathrm{i}t\ell^c X_c) \mathrm{d}t.$$

Changing variables one more time by exchanging s and t in the second and fourth integral, we find

$$\frac{\partial}{\partial \ell^m}(\overline{X}_n) - \frac{\partial}{\partial \ell^n}(\overline{X}_m)$$
$$= \mathrm{i} \int_0^1 \mathrm{d}s \int_0^s \exp(-\mathrm{i}t\ell^a X_a) X_n \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_m \exp(\mathrm{i}s\ell^c X_c) \mathrm{d}t$$

$$+\mathrm{i}\int_0^1 \mathrm{d}t\int_0^t \exp(-\mathrm{i}t\ell^a X_a)X_n \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_m \exp(\mathrm{i}s\ell^c X_c)\mathrm{d}s$$
$$-\left\{\mathrm{i}\int_0^1 \mathrm{d}s\int_0^s \exp(-\mathrm{i}t\ell^a X_a)X_m \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_n \exp(\mathrm{i}s\ell^c X_c)\mathrm{d}t\right.$$
$$\left.+\mathrm{i}\int_0^1 \mathrm{d}t\int_0^t \exp(-\mathrm{i}t\ell^a X_a)X_m \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_n \exp(\mathrm{i}s\ell^c X_c)\mathrm{d}s\right\}.$$

Combining terms we obtain our final result

$$\begin{aligned}
&\frac{\partial}{\partial\ell^m}(\overline{X}_n) - \frac{\partial}{\partial\ell^n}(\overline{X}_m)\\
&\quad = \mathrm{i}\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\exp(-\mathrm{i}t\ell^a X_a)X_n \exp\left[-\mathrm{i}(s-t)\ell^b X_b\right] X_m \exp(\mathrm{i}s\ell^c X_c)\\
&\quad -\mathrm{i}\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\exp(-\mathrm{i}t\ell^a X_a)X_m \exp\left[-\mathrm{i}(s-t)\ell^b X_b)\right] X_n \exp(\mathrm{i}s\ell^c X_c)\\
&\quad = \mathrm{i}(\overline{X}_n\overline{X}_m - \overline{X}_m\overline{X}_n) = \mathrm{i}\left[\overline{X}_n, \overline{X}_m\right],
\end{aligned}$$

which establishes equation (17).

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