MAC LANE METHOD FOR DETERMINATION OF EXTENSIONS OF FINITE GROUPS. II. AN EXAMPLE FOR THE DIHEDRAL GROUP D_2

T. LULEK

Institute of Physics, A. Mickiewicz University, Matejki 48/49, 60-769 Poznań, Poland

AND R. CHATTERJEE

Department of Physics and Astronomy, University of Calgary
Calgary, Alberta T2N 1N4, Canada

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The Mac Lane method of classification and construction of all extensions of a group Q by an Abelian group T is demonstrated on the case $Q = D_2$, $T = C_2$. Constructions involving free groups and operator homomorphisms are performed in detail, and the complete list of resulting extensions is given. It is shown that there are 8 classes of gauge equivalency, and they fall into 4 classes of isomorphism. The role of gauge transformations is pointed out. Physical contexts of various constructions are reviewed. A comparison with the direct cohomology definitions is performed.

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1. Introduction

In part I [1] of this review the general Mac Lane method of construction of equivalency classes of extensions of a finite group Q by an Abelian group T under a given action $\Delta : Q \to \mathrm{Aut}\ T$ has been presented in detail, and its relevance in physics has been pointed out within the context of crystallography and gauge fields. Here we intend to demonstrate this general method on a specific example of the simplest dihedral group

$$Q = D_2 = \{E, u_x, u_y, u_z\}, \tag{1}$$

where E denotes the unit element, and u_x, u_y, u_z are twofold axes coinciding with a Cartesia coordinate system. In the first stage we assume that T is an arbitrary finite Abelian group, and next we specify it as the two-element group

$$T = C_2 = \{E, \bar{E}\} \tag{2}$$

with the non-trivial element denoted by $\bar{E}$. In particular, we are going to classify all non-equivalent extensions of D_2 by C_2, i.e. to determine the second cohomology group $H^2(D_2, C_2)$ (in this case the action Δ is trivial). This case is simple enough to demonstrate in detail the most of the notions of the Mac Lane method presented in [1], and to provide some physical interpretation to various algebraic and combinatorial constructions. This work is motivated by our belief that such an approach can shed some new light into the fascinating problem of combination ways of geometric and gauge symmetries, and related phenomena like quantum Hall effect [2], Berry phases [3–5], topological invariants [6], flux quantization [7], etc., by exposing the possibilities of construction of extensions of some groups already known in crystallography.

Clearly, the group D_2 has a crystallographic interpretation as a point group, consisting of three mutually perpendicular twofold axes, coinciding with axes of a Cartesian coordinate system in an Euclidean three-dimensional space. In quantum field theories [10] it arises as the group of discrete space-time inversion symmetry (e.g. $u_x = P$, $u_y = T$, $u_z = PT$ with P and T being respectively parity and time reversal); in particular, it labels the four disjoint pieces of the Lorentz group. It is also isomorphic with the automorphism group for some cyclic groups C_N (e.g. $N = 8$, or $N = 12$), and thus serves as the group of hidden symmetry of the recipe of Weyl [11], applied to a closed linear chain of N atoms with an interpretation of fractal symmetry ([12, 13]).

In the following we use the notation introduced in [1].

2. Free groups and their alphabets

The Mac Lane method of construction of all extensions G of the group Q by the group T uses the following exact sequence

$$0 \to R \xrightarrow{i} F \xrightarrow{M} Q \to 1 \tag{3}$$

as a covering prototype of the exact sequence

$$0 \to T \xrightarrow{\kappa} G \xrightarrow{\omega} Q \to 1, \tag{4}$$

which defines the extension G. Here i and κ are injection monomorphisms, M and ω are epimorphisms, F is the free group, generated freely from a set A of generators of the group Q, and R is the kernel of the monomorphism M (cf. [1] for more detail).

In the case $Q = D_2$ we choose the set

$$A = \{u_x, u_y\} \tag{5}$$

as generators of D_2, and thus we write down the alphabet X of the free group F as

$$X = \{x_1, x_2\}, \tag{6}$$

with

$$M(x_1) = u_x, \quad M(x_2) = u_y. \tag{7}$$

The decomposition of the group F into right cosets with respect to the subgroup R, i.e.

$$F = R \cup Rx_1 \cup Rx_2 \cup Rx_1x_2, \tag{8}$$

corresponds to the choice of the Schreier set S as

$$S = \{e_F, x_1, x_2, x_1x_2\}. \tag{9}$$

The Schreier set S determines the factor system $\rho : Q \times Q \to R$ for the covering exact sequence (3) by means of the formula

$$f_{q_1} f_{q_2} = \rho(q_1, q_2) f_{q_1 q_2}, \quad (q_1, q_2) \in Q^2 \tag{10}$$

with the right coset representatives f_q belonging to S, $q \in Q$. The results are listed in Table I. Clearly, entries $\rho(q_1, q_2)$ of the factor system ρ belong to the subgroup $R \lhd F$, but they are expressed in Table I in terms of the alphabet X of the group F.

TABLE I

The factor system $\rho : Q \times Q \to R$, expressed in terms of the alphabet X of the group F. e_F is the unit element of the group F. Observe that $\{x_1, x_2\} = X \subset S = \{e_F, x_1, x_2, x_1x_2\}$, and thus all non-trivial entries of the second and third column constitute the alphabet Y of the subgroup $R \lhd F$.

	e_F	x_1	x_2	x_1x_2
e_F	e_F	e_F	e_F	e_F
x_1	e_F	x_1^2	e_F	x_1^2
x_2	e_F	$x_2x_1x_2^{-1}x_1^{-1}$	x_2^2	$x_2x_1x_2x_1^{-1}$
x_1x_2	e_F	$x_1x_2x_1x_2^{-1}$	$x_1x_2^2x_1^{-1}$	$x_1x_2x_1x_2$

We observe that the alphabet X is a subset of the Schreier set S. This observation allows us to identify the alphabet Y of the subgroup R with the set of all non-trivial elements of the second and third columns of Table I since

$$Y = \{\rho(q_1, q_2) \mid M(q_1) \in S,\ M(q_2) \in X\} \tag{11}$$

(cf. Eqs. (58) and (79) of [1]). Thus we have

$$Y = \{y_1 = x_1^2,\ y_2 = x_2^2,\ y_3 = x_2x_1x_2^{-1}x_1^{-1},$$
$$y_4 = x_1x_2x_1x_2^{-1},\ y_5 = x_1x_2^2x_1^{-1}\}. \tag{12}$$

The number of all letters in the alphabet Y satisfies Eq. (61) of [1], which reads

$$5 = |Y| = 1 + (|X| - 1)|Q| = 1 + (2-1) \cdot 4. \tag{13}$$

Using Eq. (78) of [1], one can readily express the factor system ρ in terms of the alphabet Y. The results are given in Table II. In fact, Eq. (12) determines the monomorphism $i : R \to F$ of the exact sequence (3).

The free group F acts on its subgroup R by inner automorphisms. This action, denoted in [1] by $\Xi : F \to \text{Aut } R$, is defined using both alphabets, X and Y, as

$$\Xi(x) = \begin{pmatrix} y \\ xyx^{-1} \end{pmatrix}, \quad x \in X,\ y \in Y. \tag{14}$$

TABLE II
The factor system $\rho : Q \times Q \to R$, expressed in terms of the alphabet Y of the subgroup $R \triangleleft F$. Trivial entries $\rho(q, E) = \rho(E, q) = e_F$, $q \in D_2$, are omitted.

	u_x	u_y	u_z
u_x	y_1	e_F	y_1
u_y	y_3	y_2	y_3y_5
u_z	y_4	y_5	y_4y_2

TABLE III
The action $\Xi : F \to \mathrm{Aut}\ R$ (cf. Eq. (14)). Entries of the table are xyx^{-1} written in the alphabet Y.

	y_1	y_2	y_3	y_4	y_5
x_1	y_1	y_5	$y_4y_1^{-1}$	y_1y_3	$y_1y_2y_1^{-1}$
x_2	y_3y_4	y_2	$y_2y_5^{-1}y_3^{-1}$	$y_3y_5y_1y_2^{-1}$	$y_3y_5y_3^{-1}$

Table III describes this action by expressing each xyx^{-1} in terms of the alphabet Y.

All the above results, i.e. the alphabet Y of the group R of all relations of the group Q, the factor system $\rho : Q \times Q \to R$, and the action $\Xi : F \to \mathrm{Aut}\ R$, are determined by the active group Q of the exact sequence (4). Further steps of the Mac Lane method involve also the passive group T. They will be considered in the next chapters.

3. Operator homomorphisms and two-cocycles

Construction of the group $\mathrm{Hom}_F(R, T)$ of all operator homomorphisms from the kernel R of the epimorphism $M : F \to Q$ to the group T (which is the kernel of the epimorphism $\omega : G \to Q$) starts from the manifold

$$T^Y = \{\varphi : Y \to T\} \cong \mathrm{Hom}(R, T) \tag{15}$$

of all mappings from the alphabet Y of the group R into the group T. Clearly, each such mapping generates a single homomorphism, therefore, in particular,

$$|\mathrm{Hom}(R, C_2)| = 2^5 = 32. \tag{16}$$

In general, the set (15) is the main subject of combinatorial enumeration theory [14], and its members can be arranged into orbits of some permutation groups, acting on T and Y [15, 16]. Here, the size of the manifold C_2^Y is so small that we do not specify any classification of its elements.

We have to select from the manifold T^Y the submanifold of all operator homomorphisms, i.e. such mappings which intertwine the actions $\Xi : F \to \mathrm{Aut}\ R$ and $\Delta : Q \to \mathrm{Aut}\ T$. They have to satisfy the conditions

$$\varphi(xyx^{-1}) = M(x)\varphi(y), \quad x \in X, \ y \in Y \tag{17}$$

TABLE IV

Conditions (17) for operator homomorphisms $\varphi : R \to T$. Alphabets X and Y classify respectively the columns and the rows of the table. The left-hand side of each entry is $\varphi(xyx^{-1})$ (in additive notation for the group T), whereas the right-hand side is $M(x)\varphi(y)$ (for the trivial action Δ of D_2 in T).

	x_1	x_2
y_1	$\varphi(y_1) = \varphi(y_1)$	$\varphi(y_3) + \varphi(y_4) = \varphi(y_1)$
y_2	$\varphi(y_5) = \varphi(y_2)$	$\varphi(y_2) = \varphi(y_2)$
y_3	$\varphi(y_4) - \varphi(y_1) = \varphi(y_3)$	$\varphi(y_2) - \varphi(y_5) - \varphi(y_3) = \varphi(y_3)$
y_4	$\varphi(y_1) + \varphi(y_3) = \varphi(y_4)$	$\varphi(y_1) - \varphi(y_2) + \varphi(y_3) + \varphi(y_5) = \varphi(y_4)$
y_5	$\varphi(y_2) = \varphi(y_5)$	$\varphi(y_5) = \varphi(y_5)$

(cf. Table III). We rewrite these conditions in Table IV, where we assume the trivial action of Q on T. This table contains $|X| \cdot |Y| = 2 \cdot 5 = 10$ conditions, labelled by pairs (y, x), $y \in Y$, $x \in X$. Clearly, not all these conditions are independent. Three of them, namely (y_1, x_1), (y_2, x_2), and (y_5, x_2), are identities. It is convenient to specify the group T before evaluating further conditions. We assume here the additive notation for the group $T = C_2$, i.e. we put 0 and 1 for E and $\bar{E}$, respectively, with addition modulo 2. Then the conditions (y_2, x_1), (y_5, x_1), and (y_3, x_2) yield

$$\varphi(y_2) = \varphi(y_5), \tag{18}$$

(y_3, x_1), (y_4, x_1) and (y_1, x_2) yield

$$\varphi(y_1) + \varphi(y_3) + \varphi(y_4) = 0, \tag{19}$$

and the last condition (y_4, x_2) depends arithmetically on (18) and (19). Thus the submanifold $\mathrm{Hom}_F(R, T)$ of T^Y is generated by three independent variables, e.g. y_1, y_2, and y_3, so that

$$|\mathrm{Hom}_F(R, C_2)| = 2^3 = 8. \tag{20}$$

The list of all operator homomorphisms is given in Table V.

Each operator homomorphism $\varphi \in \mathrm{Hom}_F(R, C_2)$ yields a two-cocycle $m : Q \times Q \to C$, given by

$$m(q_1, q_2) = \varphi(\rho(q_1, q_2)), \quad (q_1, q_2) \in Q^2, \tag{21}$$

where $\rho : Q \times Q \to R$ is given by Table II. These two-cocycles are listed in Table VI.

TABLE V

The group $\mathrm{Hom}_F(R, C_2)$ of operator homomorphisms. In this case, the group is isomorphic with the second cohomology group $H^2(D_2, C_2)$.

	y_1	y_2	y_3	y_4	y_5
φ_1	0	0	0	0	0
φ_2	0	0	1	1	0
φ_3	0	1	0	0	1
φ_4	0	1	1	1	1
φ_5	1	0	0	1	0
φ_6	1	0	1	0	0
φ_7	1	1	0	1	1
φ_8	1	1	1	0	1

TABLE VI

Factor systems $m : Q \times Q \to C_2$, corresponding to operator homomorphisms of Table V (in additive notation). Each system m is presented as the matrix with the element $m(q_1, q_2)$ in the q_1-th row and q_2-th column. Trivial factors $m(E, q) = m(q, E) = 0$, $q \in D_2$, are omitted. Rows and columns of matrices of m's are labelled consecutively by u_x, u_y, u_z. The second cohomology group $H^2(D_2, C_2)$ coincides with the set $\{m_1, \ldots, m_8\}$ of factor systems of this table with the pointwise matrix addition modulo 2 as the group multiplication.

$$m_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad m_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$m_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad m_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$m_5 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad m_6 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$m_7 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad m_8 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Each cocycle m defines an extension G through the Seitz formula. In order to determine equivalency classes, we proceed to evaluate two-coboundaries, which are associated with crossed homomorphisms.

4. Crossed homomorphisms and the second cohomology group

The group

$$Z^1_{\Delta\circ M}(F,T) \cong T^X = \{\gamma : X \to T\} \tag{22}$$

of all crossed homomorphisms from F to T can be identified with the manifold of all mappings from the alphabet X to T. For the case of $T = C_2$, we obtain four crossed homomorphisms $\gamma_1, \ldots, \gamma_4$, listed in Table VII. Using Eq. (12), it is easy

TABLE VII
The group $Z^1_{\Delta\circ M}(F, C_2)$ of all crossed homomorphisms (in additive notation for C_2).

	x_1	x_2
γ_1	0	0
γ_2	0	1
γ_3	1	0
γ_4	1	1

to evaluate that all restrictions $\gamma|_R = i \circ \gamma$ of these mappings to the subgroup $R \lhd F$ vanish identically, i.e. that

$$Z^1_{\Delta\circ M}(F, C_2)|_R \equiv Z^1_M(F, C_2)|_R = \{0\}\,. \tag{23}$$

Thus the group of restrictions of crossed homomorphisms is in our case trivial (the action Δ in this case is trivial).

By virtue of Mac Lane theorem, the second cohomology group is equal to that of operator homomorphisms, i.e.

$$H^2(D_2, C_2) = \mathrm{Hom}_F(R, C_2) = \{m_1, \ldots, m_8\}\,. \tag{24}$$

Elements of the second cohomology group can be identified with factor systems of Table VI, with the group multiplication defined as addition of corresponding matrices modulo 2. It is an elementary Abelian group, with the unit element m_1, and all other elements of order 2.

5. Classes of extensions of D_2 by C_2

Each element $m \in H^2(D_2, C_2)$ yields the corresponding extension G of D_2 by C_2, determined by the Seitz formula

$$\langle t_1, q_1\rangle\langle t_2, q_2\rangle = \langle t_1 + q_1 t_2 + m(q_1, q_2), q_1 q_2\rangle \tag{25}$$

for multiplication in G. There are thus 8 classes of non-equivalent extensions, which are listed in Table VIII. The trivial factor m_1 yields the elementary Abelian group

TABLE VIII

Extension of D_2 by C_2.

Factor system	Order of element			Isomorphic group
	u_x	u_y	u_z	
m_1	2	2	2	$C_2 \times C_2 \times C_2 \cong D_{2h}$
m_2	2	2	4	D_4^z
m_3	2	4	4	$C_2^x \times C_4$
m_4	2	4	2	D_4^y
m_5	4	2	4	$C_2^y \times C_4$
m_6	4	2	2	D_4^x
m_7	4	4	2	$C_2^z \times C_4$
m_8	4	4	4	D_2'

of order 8, i.e.

$$G_1 = C_2 \times C_2 \times C_2 \cong D_{2h} \tag{26}$$

with all non-trivial elements of order 2. It is isomorphic with the crystallographic point group D_{2h}. It also arises in quantum field theory in the context of CPT theorem [10], as the simplest combination of discrete space and time inversions with an internal symmetry of charge conjugation. We like to mention here that, despite of the simple structure (26) of the group G_1, the appropriate Wigner–Racah calculus suffers from some puzzles concerning proper conventions of phases (cf. Chatterjee and Buckmaster [17] and references therein).

Factor systems m_3, m_5 and m_7 yield extensions isomorphic with the Abelian group $C_2 \times C_4$. They are mutually isomorphic, but not equivalent extensions. The order of an element of the form $\langle t, u_\alpha \rangle$, $t \in C_2$, is either 2 for a single $\alpha = x, y, z$, or 4 for the two other α's. These extensions differ mutually by the distinguished element $u_\alpha \in D_2$ associated with the twofold element of G. It is denoted in Table VIII by the superscript α at the symbol of the isomorphic group.

Factor systems m_6, m_4, and m_2 correspond to dihedral groups D_4 with the fourfold axis associated respectively with x, y, and z. These extensions are thus non-Abelian groups, each with two fourfold elements $\langle t, u_\alpha \rangle$, $t \in C_2$, α fixed.

The factor system m_8 yields the double dihedral group D_2' with the three twofold axes u_x, u_y, and u_z transformed into pairs $\{\langle t, u_\alpha \rangle \mid t \in C_2\}$ of fourfold elements, written usually in a form $\{u_\alpha, \bar{u}_\alpha\}$, $\alpha = x, y, z$, with

$$\bar{u}_\alpha = \bar{E} u_\alpha, \tag{27}$$

where

$$\bar{E} = \langle 1, E \rangle \tag{28}$$

is interpreted as the rotation of the angle 2π (around any axis) for spinors [18]. The double dihedral group D_2' can be readily identified with the group of quaternions, playing an important role in description of rotations of solid bodies (cf.

Altmann [19] for a fascinating history of puzzles accompanying the use of quaternions).

It is worthwhile to observe that various extensions of D_2 by C_2 are non--isomorphic. There are four isomorphic classes of extensions. Two of them, namely G_1 and $\{G_3, G_5, G_7\}$, are Abelian, and the two other, G_8 and $\{G_6, G_4, G_2\}$ are non-Abelian. These isomorphic classes can be nicely reflected in the isomorphism between the second cohomology group $H^2(D_2, C_2)$, and the point group D_{2h}, as given in Table IX. The table shows that extensions within an isomorphic class can be labelled by indices $\alpha = x, y, z$ of the Cartesian coordinate system.

TABLE IX

The isomorphism between the point group D_{2h} and the second cohomology group $H^2(D_2, C_2)$. Vertical lines separate isomorphic classes of extensions of D_2 by C_2.

D_{2h}	E	$\sigma_x\ \sigma_y\ \sigma_z$	I	$u_x\ u_y\ u_z$
$H^2(D_2, C_2)$	m_1	$m_6\ m_4\ m_2$	m_8	$m_3\ m_5\ m_7$
Isomorphic class	$C_2 \times C_2 \times C_2$	$C_2 \times C_4$	D_2'	D_4

Isomorphic classes differ by the number of fourfold elements: 0, 4, 6, and 2 for G_1, $\{G_3, G_5, G_7\}$, G_8, and $\{G_6, G_4, G_2\}$, respectively. Such a doubling, or, more generally, multiplication of order of the element $q \in Q$ in the coset Tg_q of the extension G has a known crystallographic interpretation in terms of screw axes or glide planes with the associated fractional translations.

6. A comparison of Mac Lane method with an immediate application of cohomology

Now we have demonstrated the application of Mac Lane method of classification and construction of all extensions of the group D_2 by C_2. In this section we make some comparison of this method with an immediate adaptation of cohomology to this case.

The cohomology theory starts with the group

$$C^2(D_2, C_2) = \{f : D_2 \times D_2 \to C_2\} \tag{29}$$

of all two-cochains. The order of this group is

$$|C^2(D_2, C_2)| = 2^{4\cdot 4} = 65536. \tag{30}$$

Each two-cochain $f \in C^2(D_2, C_2)$ should satisfy

$$|D_2|^3 = 4^3 = 64 \tag{31}$$

associativity conditions

$$(\delta^2 f)(q_1, q_2, q_3) = 0, \quad (q_1, q_2, q_3) \in D_2^3, \tag{32}$$

where δ^2 is the two-coboundary operator. We observe that the associated numerical problem is rather large even in such a simple case. It increases exponentially with the increase either the active group Q or the passive group T.

After performing such a calculation one arrives at the group

$$Z^2(D_2, C_2) = \{f \in C^2(D_2, C_2) \mid \delta^2 f = 0\} \tag{33}$$

of all two-cocycles. The group classifies all extensions since each $f \in Z^2(D_2, C_2)$ serves as a distinct factor system. All these extensions are different, but some of them can be gauge equivalent. We thus need to find the group

$$B^2(D_2, C_2) = \{\delta^1 c \mid c \in C^1(D_2, C_2)\} = \operatorname{Im} \delta^1 \tag{34}$$

of all two-coboundaries, which is a normal subgroup in $Z^2(D_2, C_2)$. By the definition (34), it can be performed by an application of the one-coboundary operator $\delta^1 : C^1(D_2, C_2) \to C^2(D_2, C_2)$ to the group

$$C^1(D_2, C_2) = \{c : D_2 \to C_2\} \tag{35}$$

of all one-cochains c. $C^1(D_2, C_2)$ is the group of gauge transformations. Each gauge transformation c yields an equivalent extension.

TABLE X

The gauge group $C^1(D_2, C_2)$. Each entry is a gauge $c : D_2 \to C_2$. Each column is a coset of the gauge group $C^1(D_2, C_2)$ with respect to the kernel Ker $\delta^1 = Z^1(D_2, C_2)$ (the group of one-cocycles).

E	u_x	u_y	u_z	E	u_x	u_y	u_z	E	u_x	u_y	u_z	E	u_x	u_y	u_z
0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	1
0	0	1	1	0	0	1	0	1	0	1	1	1	0	1	0
0	1	0	1	0	1	0	0	1	1	0	1	1	1	0	0
0	1	1	0	0	1	1	1	1	1	1	0	1	1	1	1

The order of the gauge group is

$$|C^1(D_2, C_2)| = 2^4 = 16 \tag{36}$$

(cf. Table X). The two-coboundary $\delta^1 c$ associated with a gauge $c \in C^1(D_2, C_2)$ is given by

$$(\delta^1 c)(q_1, q_2) = c(q_1) + c(q_2) - c(q_1 q_2) \tag{37}$$

(since the action of D_2 in C_2 is trivial). The kernel

$$Z^1(D_2, C_2) = \operatorname{Ker} \delta^1 = \{c \in C^1(D_2, C_2) \mid \delta^1 c = 0\} \tag{38}$$

is a subgroup of $C^1(D_2, C_2)$, given in the first column of Table X. It consists of all such gauges which do not change any factor system. In our case, when the action of D_2 on C_2 is trivial, each such a gauge defines a representation of the active group D_2, valued in the passive group C_2.

TABLE XI

The group $B^2(D_2, C_2)$ of all two-coboundaries. Each two-coboundary defines a trivial factor system. Rows and columns of each matrix are labelled consecutively by $q = E, u_x, u_y, u_z$.

$$b_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad b_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$b_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad b_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Equation (37) and Table X yield the group $B^2(D_2, C_2)$ of all two-coboundaries (one has to apply Eq. (37) to a single gauge c for each column of Table X). This group is given in Table XI. We observe that the two-coboundaries b_1 and b_2 satisfy the defining condition

$$m(E, q) = m(q, E) = 0, q \in D_2 \tag{39}$$

for a normalized factor system, whereas b_3 and b_4 are unnormalized.

Thus the group $B^2(D_2, C_2)$ can be easily evaluated merely from the cohomological definition. It is not the case for the group $Z^2(D_2, C_2)$ of all two-cocycles, where much more numerical effort is needed. Using the fact that the order of the second cohomoly group is 8, we obtain

$$|Z^2(D_2, C_2)| = |B^2(D_2, C_2)||H^2(D_2, C_2)| = 4 \cdot 8 = 32. \tag{40}$$

Table XI implies that the factor systems of Table VI are not unique, but they are given only modulo the group $B^2(D_2, C_2)$ of all two-coboundaries. Even if we restrict ourselves to normalized factor systems (39), still each m of Table VI can be substituted by a gauge-equivalent factor system

$$m' = m + b_2. \tag{41}$$

Thus, e.g. the factor system m_8 of Table VI

$$m_8 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} . \tag{42}$$

is gauge-equivalent to

$$m_8' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} . \tag{43}$$

In terms of the extension G_8, i.e. the double group D_2', it means that the relations

$$u_x u_y = u_z \tag{44}$$

and

$$u_x u_y = \bar{u}_z \equiv \bar{E} u_z \tag{45}$$

for the double group D_2' are gauge-equivalent (cf. [20]). This arbitrariness is nicely reflected in the fibre structure of the extension G_8, where $\{u_z, \bar{u}_z\} \in D_2'$ forms the fiber over $u_z \in D_2$, and each element of this fiber is equally good as the candidate for $\langle 0, u_x \rangle \langle 0, u_y \rangle$.

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