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Nonexponential Decay Law of the $2P$–$1S$ Transition of the H Atom

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We evaluate numerically the survival probability $P(t)$ for the unstable $2P$ excited state of the hydrogen atom that decays into the ground-state $1S$ by emitting one photon ($\tau \sim 1.595$ ns), thus extending the analytic study of Facchi and Pascazio, *Phys. Lett. A* **241**, 139 (1998). To this end, we first determine the analytic expression of the spectral function of the unstable state, which allows for an accurate evaluation of $P(t)$. As expected, for short and long time scales, $P(t)$ shows deviations from the exponential law: a 'Zeno' region occurs at extremely short times (up to ~ 0.3 attosec), followed by a longer 'anti-Zeno' domain (up to ~ 50 attosec); at long times above 125τ the decay law scales as t^{-4}.

topics: nonexponential decay, H-atom, (anti-)Zeno domains

1. Introduction

The fact that the decay law of a given unstable state, described by the survival probability $P(t)$, is not simply an exponential function of the type $P(t) = \mathrm{e}^{-t/\tau} = \mathrm{e}^{-\Gamma t}$, is well understood theoretically, e.g. [1–3]. In particular, deviations are expected at short and long times. At very short times, the quadratic decay law $P(t) = 1 - t^2/\tau_Z^2 + \ldots$ (where the coefficient τ_Z is the so-called Zeno time) is realized, which implies a larger survival probability than $\mathrm{e}^{-t/\tau}$ and in turn renders the quantum Zeno effect possible [4], i.e., a slowing down of the decay if very frequent measurements are performed. Shortly after this Zeno region, an anti-Zeno domain that corresponds to a faster decay than $\mathrm{e}^{-t/\tau}$ usually (but not necessarily) takes place; here a sequence of measurements at an appropriate time interval generates an increased decay rate. At very late times the decay follows a power law, $P(t) \sim t^{-\beta}$, where the exponent $\beta > 0$ depends on the specific system. Quite interestingly, a very similar phenomenology applies also to relativistic decays, which need a quantum field theoretical (QFT) treatment, see e.g. [5–7] (the particular case of strong decays, in which large deviations are expected, is discussed in [8]).

In general, these deviations occur at very short and very long times, which makes them very difficult to observe in natural systems. At the experimental level, deviations from the exponential decay at short times (including both Zeno and anti-Zeno domains and related effects) were confirmed in [9, 10] using engineered tunneling of Na atoms through an optical potential. Deviations at long times were seen in fluorescence decays of chemical compounds in [11]. Short- and late-time deviations were also confirmed by the analogous system of photons propagating in waveguide arrays [12]. A type of 'hidden evidence' of the non-exponential decay law is reported for nuclear beryllium decays in [13]. Indeed, the previous examples show that up to date deviations could be observed only in specific systems.

In this work, we intend to study a natural and very basic decay, namely the $2P$–$1S$ transition of the hydrogen atom. When the electron is located in the $2P$ orbital, it quickly ($\tau \sim 1.595$ ns) 'jumps down' to the ground state by emitting one photon. This decay represents then an optimal test to check what does it mean 'short' and 'long' times when deviations from the exponential decays are considered. In [14] this system was studied by using analytic approximations. Here, we are able to determine numerically $P(t)$ to a very good level of accuracy. This is possible because the spectral function of the unstable state is determined analytically.

Our numerical results confirm in general the outcomes of [14], but they also show some novel aspects: (i) the numerical value of the Zeno-time τ_Z agrees very well with the outcome of [14], but the quadratic approximation is only valid for much shorter times (showing that the coefficient τ_Z is an upper limit); (ii) there is an anti-Zeno region that is much longer than the Zeno one; (iii) the late-time deviations start even later than the estimated value in [14].

The article is organized as follows: in Sect. 2 we recall some general features of the theoretical approach and show the spectral function of the $2P$ state; in Sect. 3 we present the main results of this work — $P(t)$ deviates form the exponential function at short and long times; finally, in Sect. 4 we discuss conclusions and outlooks.

2. The model and the $2P$ spectral function

First, we briefly recall some main properties concerning the decay law, see details in [1]. The results can be also obtained in the specific case of the Lee (or Lee–Friedrichs) Hamiltonian [2, 15, 16]. This is a versatile model that implements the continuum of states and can be also extended to relativistic QFT cases [5–8, 17].

Let us consider a system controlled by the Hamiltonian H describing an unstable system/particle $|S\rangle$ formed at the time $t=0$. The survival probability $P(t)$ that the state has not decayed yet up to time $t>0$ is given by (see e.g. [1])

$$P(t) = |A(t)|^2 \quad \text{with} \quad A(t) = \langle S|\mathrm{e}^{-\mathrm{i}Ht}|S\rangle, \tag{1}$$

where $A(t)$ is denoted as the survival probability amplitude. In turn, $A(t)$ can be expressed as the Fourier transformation of the spectral function (also called energy distribution) $d_S(E)$ of the unstable state,

$$A(t) = \int_{E_{\rm th}}^{\infty} \mathrm{d}E\; d_S(E)\, \mathrm{e}^{-\mathrm{i}Et}, \tag{2}$$

where $E_{\rm th}$ is the lowest admissible 'threshold' energy for the decay. The spectral function $d_S(E)$ emerges as the imaginary part of the propagator $G_S(E) = [E-M+\Pi(E)]^{-1}$, where M is the energy/mass of the unstable state and $\Pi(E)$ the so-called self-energy (intuitively, describing processes of the type $S \to$ decay product $\to S$). It takes the explicit form

$$d_S(E) = -\frac{1}{\pi}\Im[G_S(E)] = \frac{1}{\pi}\frac{\Im[\Pi(E)]}{\left(E-M+\Re[\Pi(E)]\right)^2 + \left(\Im[\Pi(E)]\right)^2}, \tag{3}$$

and is correctly normalized to unity, $\int_{E_{\rm th}}^{\infty} \mathrm{d}E\; d_S(E) = 1$.

Indeed, when the imaginary part $\Im[\Pi(E)]$ is known (or modelled in some way), the real part can be obtained via the dispersion relation,

$$\Re[\Pi(E)] = \frac{1}{\pi} P\int_{-\infty}^{\infty} \mathrm{d}E'\; \frac{\Im[\Pi(E')]}{E'-E}. \tag{4}$$

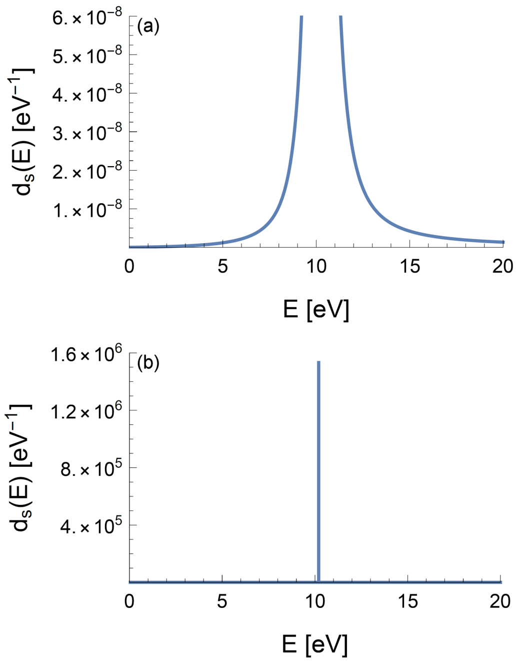

Fig. 1. Plot of the spectral function of the unstable state $S \equiv 2P$ for two different ranges of the vertical axis. As expected, the function is extremely narrow and peaked.

The decay width function $\Gamma(E) = 2\Im[\Pi(E)]$ takes the on-shell value $\Gamma(M) = \Gamma = 2\Im[\Pi(M)] = \tau^{-1}$, with τ being the lifetime of the unstable state.

We can then move to the specific case of the $2P$–$1S$ transition. It is important to recall the explicit formula for the imaginary part of self-energy function as presented in [14] (see also the original calculations in [18, 19])

$$\Im[\Pi(E)] = \pi\,\Lambda\,\chi\,\frac{\frac{E-E_{\rm th}}{\Lambda}}{\left[1+\left(\frac{E-E_{\rm th}}{\Lambda}\right)^2\right]^4}\,\vartheta(E-E_{\rm th}), \tag{5}$$

where

$$\chi = \frac{2}{\pi}\left(\frac{2}{3}\right)^9 \alpha^3 \simeq 6.43509\times10^{-9},$$
$$\Lambda = \frac{3}{2}\alpha m_e \simeq 5593.41 \text{ eV}. \tag{6}$$

Without loss of generality, the threshold energy $E_{\rm th}$ can be set to zero, $E_{\rm th}=0$. Then, the energy of the state $2P$ (neglecting hyperfine splittings) reads

$$M = \frac{3}{8}\alpha^2 m_e \simeq 10.2043 \text{ eV}. \tag{7}$$

The on-shell physical decay width can be written down analytically as

$$\Gamma = \frac{1}{\tau} = \frac{\frac{3}{2}\left(\frac{2}{3}\right)^9 m_e\alpha^5}{\left[1+\left(\frac{\alpha}{4}\right)^2\right]^4} = 4.12582\times10^{-7} \text{ eV}, \tag{8}$$

out of which

$$\tau \simeq 2.42376\times10^{6} \text{ eV}^{-1} = 1.59535\times10^{-9} \text{ s}. \tag{9}$$

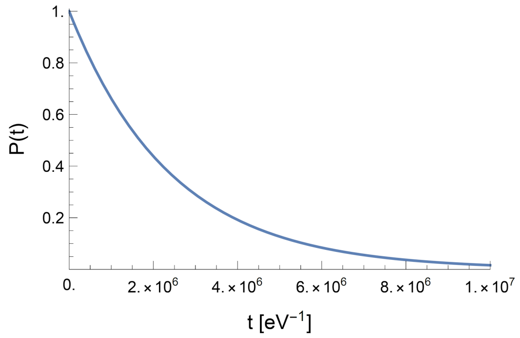

Fig. 2. Survival probability at intermediate times (of the order of $\tau \sim 2.42\times10^6$ eV^{-1}). In this domain, the function is basically indistinguishable from the exponential decay.

The real part of the loop $\Re[\Pi(E)]$ can be determined analytically as

$$\frac{\Re[\Pi(E)]}{\chi\Lambda} - C = -\frac{2\frac{E-E_{\rm th}}{\Lambda}\ln\left(\frac{E-E_{\rm th}}{\Lambda}\right)+\pi\left(\frac{E-E_{\rm th}}{\Lambda}\right)^2}{2\left[1+\left(\frac{E-E_{\rm th}}{\Lambda}\right)^2\right]^4} - \frac{2\frac{E-E_{\rm th}}{\Lambda}+\pi\left(\frac{E-E_{\rm th}}{\Lambda}\right)^2}{4\left[1+\left(\frac{E-E_{\rm th}}{\Lambda}\right)^2\right]^3} - \frac{4\frac{E-E_{\rm th}}{\Lambda}+3\pi\left(\frac{E-E_{\rm th}}{\Lambda}\right)^2}{16\left[1+\left(\frac{E-E_{\rm th}}{\Lambda}\right)^2\right]^2} + \frac{15\pi-16\frac{E-E_{\rm th}}{\Lambda}}{96\left[1+\left(\frac{E-E_{\rm th}}{\Lambda}\right)^2\right]}, \tag{10}$$

where the subtraction constant C is chosen such that $\Re[\Pi(M)] = 0$. The spectral function $d_S(E)$ can be determined by inserting the results from equations (5) and (10) into (3); its form is shown in Fig. 1. As expected, $d_S(E)$ is an extremely narrow function peaked at the energy M. We have numerically verified that it is normalized to 1.

Once the shape of the spectral function is fixed, the decay law can be determined numerically.

3. Results for $P(t)$

At intermediate times (of the order of the lifetime τ), the exponential decay law $P(t) = \mathrm{e}^{-t/\tau}$ provides a very good approximation, see Fig. 2.

At short times, by a direct numerical evaluation of (2), we obtain the results shown in Fig. 3. For times of the order of 0.01 eV^{-1} $\sim 10^{-18}$ s, deviations from the exponential law are visible, but this region is rather of the anti-Zeno type, i.e., an increased decay rate is realized (see the following discussion).

In order to make the quadratic Zeno-region visible, one needs to move to even shorter times of the order of 0.001 eV^{-1} $\sim 0.6\times10^{-18}$ s. In such a time domain, the decay law can be expressed by

$$P(t) \simeq 1 - \frac{1}{2}\left.\frac{\mathrm{d}^2P(t)}{\mathrm{d}t^2}\right|_{t=0} t^2 + \dots = 1 - \frac{t^2}{\tau_Z^2} + \dots . \tag{11}$$

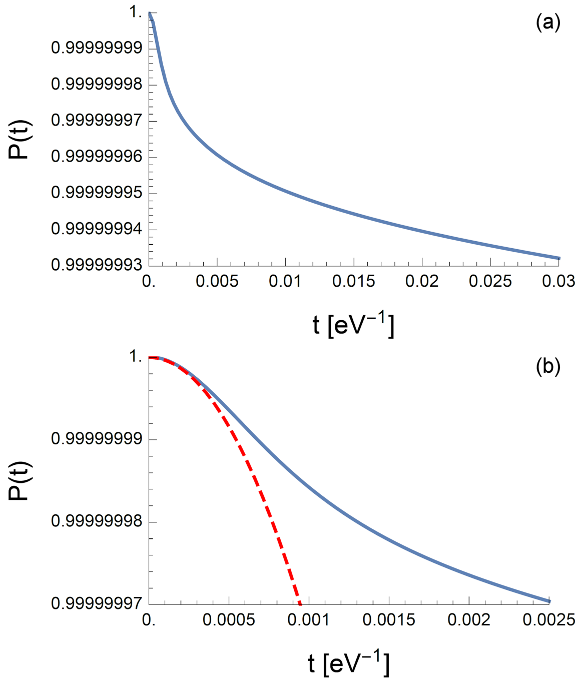

Fig. 3. Survival probability $P(t)$ for short times. In panel (a) we observe an anti-Zeno domain (enhanced decay rate). In panel (b) the Zeno domain is visible, as a direct comparison with (11) shows.

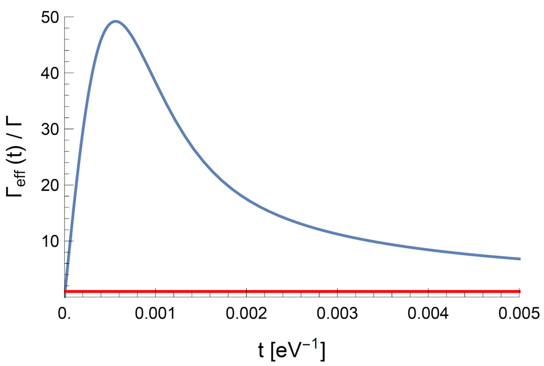

Fig. 4. The ratio of the effective decay width over the exponential one, $\Gamma_{\rm eff}(t)/\Gamma$. The red curve corresponds to unity.

Namely, upon expanding the amplitude as

$$A(t) = 1 - \mathrm{i}t\langle E\rangle - \frac{t^2}{2}\langle E^2\rangle + \dots , \tag{12}$$

one may easily see that $P'(0) = 0$ if $\langle E\rangle$ is finite. Moreover, the Zeno coefficient τ_Z reads

$$\tau_Z = \sqrt{\frac{1}{\langle E^2\rangle - \langle E\rangle^2}} = \frac{1}{\sigma_E} \simeq 5.45911\ \mathrm{eV}^{-1} = 3.59325\times10^{-15}\ \mathrm{s}. \tag{13}$$

It is important to stress that in the present case, the Zeno time τ_Z is actually much longer than the non-exponential region in general and the quadratic

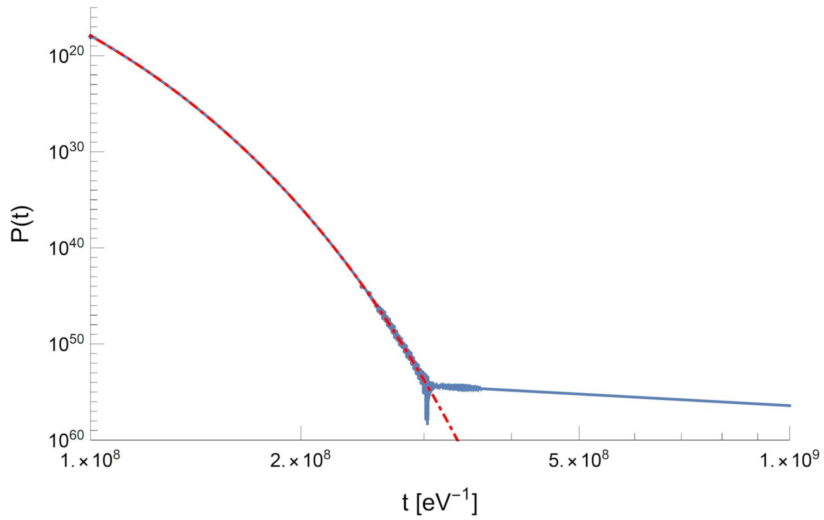

Fig. 5. Survival probability at long times in log–log form. The red curve corresponds to purely exponential decay. An interesting feature is given by the fast oscillations close to the turn-over time.

TABLE I

Selected numerical values of the effective decay width in the anti-Zeno domain together with the corresponding times.

$\Gamma_{\text{eff}}(t)/\Gamma$	Time [eV^{-1}]	Time [s]
2	0.02130	1.40183×10^{-17}
1.1	0.06242	4.10857×10^{-17}
1.01	0.08234	5.41941×10^{-17}

region in particular. Strictly speaking, the Zeno time defined in (11), being the quadratic coefficient of the Taylor expansion of $P(t)$, is not necessarily a good estimate of the non-exponential domain (but rather an upper limit of it). To further investigate the decay rate at short times, we introduce an effective decay width defined as [20, 21]

$$\Gamma_{\text{eff}}(t) = -\frac{\mathrm{d}P(t)}{\mathrm{d}t}\frac{1}{P(t)}. \tag{14}$$

Namely, for a purely exponential decay, $\Gamma_{\text{eff}}(t) = \Gamma$ for each t. In turn, $\Gamma_{\text{eff}}(t) > \Gamma$ signalizes an anti-Zeno domain, while $\Gamma_{\text{eff}}(t) < \Gamma$ — a Zeno one. The function $\Gamma_{\text{eff}}(t)/\Gamma$ is presented in Fig. 4. It is clear that for short times an anti-Zeno region is present, with $\Gamma_{\text{eff}}(t)/\Gamma$ having a maximum at

$$t \simeq 0.00056\ \text{eV}^{-1} = 3.69232 \times 10^{-19}\ \text{s}, \tag{15}$$

at which the decay rate is about 50 times larger than the one in the exponential domain. The Zeno region takes place at such short times that it is barely visible in Fig. 4. It should be noted that quite similar, although non-identical, results for the decay width of electric-dipole transitions were obtained in [22].

We summarize in Table I the relevant times concerning the anti-Zeno domain, which show that the anti-Zeno domain reduces to less than 1% for times above 54 attosec.

Finally, we briefly describe the long-time domain. The pole position for the unstable $2P$ state is given by

$$z_{\text{pole}} = M - \mathrm{i}\frac{\Gamma}{2} = M - \frac{\mathrm{i}}{2\tau}, \tag{16}$$

out of which the survival probability amplitude can be obtained by changing the contour of the integral in (2) in the complex plane, namely it is closed between $(E_{\text{th}} = 0, \infty)$ in the right-lower quadrant, the pole is picked up, and the vertical axis contribution must be subtracted, which leads to

$$A(t) = -\frac{2\mathrm{i}\Im[\Pi(z_{\text{pole}})]\,\mathrm{e}^{-\mathrm{i}\,z_{\text{pole}}t}}{z_{\text{pole}} - M + \Re[\Pi(z_{\text{pole}})] - \mathrm{i}\Im[\Pi(z_{pole})]} - \mathrm{i}\int_0^\infty \mathrm{d}y\ d_S(-\mathrm{i}y)\,\mathrm{e}^{-yt}. \tag{17}$$

While for intermediate times the second term in (17) can be neglected — thus the decay is basically exponential, this is not true at long times where the non-exponential part dominates. Numerical calculations are typically difficult, so the optimal way to treat this problem is to analytically approximate the integral present in (17). At long times, only the linear term of the Taylor expansion of $d_S(-\mathrm{i}z)$ matters, hence with good precision the survival amplitude can be approximated by the formula

$$A(t) = -\frac{2\mathrm{i}\Im[\Pi(z_{\text{pole}})]\,\mathrm{e}^{-\mathrm{i}\,z_{\text{pole}}t}}{z_{\text{pole}} - M + \Re[\Pi(z_{\text{pole}})] - \mathrm{i}\Im[\Pi(z_{\text{pole}})]} - \frac{\chi}{M^2}\,t^{-2}. \tag{18}$$

The corresponding plot can be found in the log–log plot in Fig. 5, where the transition from exponential to power law is clearly visible.

It is also possible to estimate the transition time at which the exponential law breaks down. Upon choosing the turn-over time as the one at which the absolute values of both terms present in expression (18) are equal, one gets

$$t_{\text{turn-over}} \simeq 3.03297 \times 10^{8}\ \text{eV}^{-1} = 1.99634 \times 10^{-7}\ \text{s} \simeq 125.1\tau, \tag{19}$$

basically implying that a detection of such long-time deviations is currently 'de facto' impossible for the $2P$–$1S$ transition.

4. Conclusions

In this work, we have studied the non-exponential decay of a quite natural electromagnetic transition — decay of an electron in the $2P$ state of the H-atom into the $1S$ level via the emission of a photon. While the quadratic Zeno region occurs at very short times (~ 0.3 attosec $\simeq 1.88 \times 10^{-10}\,\tau$), the long-time deviations occur at very long times (0.2 μs $\simeq 125\,\tau$); both of them seem far from any experimental reach.

After a short Zeno region, there is a somewhat longer anti-Zeno domain (up to 50 attosec $\simeq 3.1 \times 10^{-8}\,\tau$). This is in agreement with the general discussion of [3], according to which the anti-Zeno domain (and effect) are in general easier to obtain. Indeed, in [23] the anti-Zeno effect was proposed as an explanation of the neutron-decay anomaly [24] due to the anti-Zeno region (and frequent measurements in the bottle-type experiments).

One can speculate that the detection of the anti-Zeno effect (even if very difficult) might be possible by a continuous measurement [25] of the ground state, in a set-up that would be analogous tho the optical experiment of [26]. Another interesting extension is the study of more decay channels, such as in [7, 27], as well as QFT relativistic systems.

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