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Two-Photon Decay of Para-Positronium Within a Composite Approach

M. Piotrowska[a,*] and F. Giacosa[a,b]

[a] *Institute of Physics, Jan Kochanowski University, ul. Uniwersytecka 7, 25-406 Kielce, Poland*
[b] *Institute for Theoretical Physics, J.W. Goethe University, Max-von-Laue-Str. 1, 60438 Frankfurt, Germany*

Doi: 10.12693/APhysPolA.146.699 *e-mail: milena.piotrowska@ujk.edu.pl

The decay of the para-positronium into two photons is studied in the framework of a composite quantum field theoretical approach. This amounts to the evaluation of the electron–positron dressing, the Weinberg compositeness condition for the positronium, and the triangle-shaped diagram with virtual electrons circulating in it, leading to the final two-photon state. An important role is played by the positronium–electron–positron vertex, which is linked to the wave function of the para-positronium. We show how possible choices for the vertex function affect the $\gamma\gamma$ decay rate. Outlooks to other decay channels and other positronia are presented.

topics: positronium, compositeness condition, vertex function

1. Introduction

Positronium is a bound state of an electron–positron pair (e^-–e^+), that arises in the framework of quantum electrodynamics (QED). As such, it plays an important role in fundamental physics, being the lightest 'element' [1, 2], as well as in medical physics, mainly due to its applications in the technique of positron emission tomography (PET) [1, 3–7]. The ground state of the positronium, known under the name para-positronium (p-Ps), corresponds to a spin-singlet state described by the spectroscopic quantum numbers $n\ ^{2S+1}L_J = 1\ ^1S_0$, or equivalently, by using the relativistic notation as $J^{PC} = 0^{-+}$, hence being a pseudoscalar state just like the pion in quantum chromodynamics (QCD).

The para-positronium is an unstable short-living state with a mean lifetime of $\sim$ 0.125 ns. It decays into photons due to the annihilation of its components. Charge-conjugation conservation requires that the decay of p-Ps occurs only into an even number of photons, with the $\gamma\gamma$ mode (p$-$Ps $\to \gamma\gamma$) being by far the dominant one. This decay width can be expressed by (see e.g. [8])

$$\Gamma(\mathrm{Ps} \to n\gamma)=\frac{1}{2J+1}\,|\psi(0)|^2 \lim_{v\to 0}\Big[4v\sigma(e^+e^-\to n\gamma)\Big], \tag{1}$$

where v and σ are the relative velocity and the annihilation cross-section of e^-–e^+, respectively, and where $|\psi(0)|^2 = m_e^2\alpha^3/(8\pi)$ is the spatial wave function at the origin (annihilation part of the amplitude). Namely, the ground-state wave function is $\psi(\boldsymbol{x}) = (\pi a^3)^{-\frac{1}{2}}\,\mathrm{e}^{-r/a}$, where $r = |\boldsymbol{x}|$ and $a = 2a_0$ (twice the Bohr radius of the hydrogen atom). At the lowest order, it becomes

$$\Gamma\left(^1S_0 \to 2\gamma\right) = \frac{1}{2}\frac{e^4|\psi(\boldsymbol{x}{=}0)|^2}{\pi m^4}\int_0^\infty \mathrm{d}|\boldsymbol{k}_1|\ \delta\big(2m{-}2|\boldsymbol{k}_1|\big)\,|\boldsymbol{k}_1|^2 = \frac{e^4|\psi(\boldsymbol{x}{=}0)|^2}{4\pi m^2} = \frac{4\pi\alpha^2}{m^2}|\psi(\boldsymbol{x}{=}0)|^2 = \alpha^5\frac{m_e}{2}, \tag{2}$$

where the last result is the famous Wheeler–Pirenne formula [9, 10] involving solely the fine-structure constant α and the electron mass m_e. In the framework of QED, higher orders can be systematically evaluated, see e.g. [11–13] and Table I [9–16] for a summary.

Here, we intend to study the p-Ps within a composite model that makes use of the compositeness condition, originally proposed to describe the deuteron [17, 18], in a way that resembles the treatment of QCD bound states [19–21]. To this end, we extend the scalar model described in [22]. The triangle diagram in Fig. 1 leading to the $\gamma\gamma$ decay is calculated within this approach. Of course, the aim

TABLE I

Summary of theoretical and experimental results for the decay rate p−Ps → 2γ. The quantity $\Gamma_0 = \alpha^5 m_e/2$ is the lowest-order (LO) result. The constants A, B, C describe further corrections (NLO — next-to-leading-order, NNLO — next-to-next-to-leading-order, etc.) and are reported in [12].

	$\Gamma(\text{p−Ps} \to 2\gamma)$			
	Formula	Result [μs^{-1}]	Comment	Reference
Theory	$\frac{\alpha^5 m_e}{2}$	8032.5028(1)	lowest order (LO)	[9, 10]
	$\Gamma_0\left\{1+\frac{\alpha}{\pi}\left(\frac{\pi^2}{4}-5\right)\right\}$	7985.249	LO+NLO corrections	[11]
	$\Gamma_0\Big\{1+\frac{\alpha}{\pi}\left(\frac{\pi^2}{4}-5\right)-2\alpha^2\ln(\alpha)+B_{2\gamma}\left(\frac{\alpha}{\pi}\right)^2 -\frac{3\alpha^3}{2\pi}\ln^2(\alpha)+C\frac{\alpha^3}{\pi}\ln(\alpha)+D\left(\frac{\alpha}{\pi}\right)^3\Big\}$	7989.6178(2)	LO+NLO+NNLO corrections	[12–15]
Experiment		7990.9(1.7)		[16]

is not to go beyond the precision studies of QED, but to learn how to deal with the nonperturbative vertex linking positronium to its constituents [23]. In the following, a detailed discussion of this topic will be provided.

2. The composite model

The Lagrangian of our model, describing the decay (and interaction) of para-positronium (with mass M_P) into two massless photons, reads

$$\mathcal{L}_{int} = g_P P(x)\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x) - eA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x), \tag{3}$$

where $P(x)$ stands for the pseudoscalar positronium field, $\psi(x)$ is the electron field, A_μ is the photon field, e is the electric charge of the proton, and g_P is the positronium-constituent coupling constant. Note, there is no direct coupling between p-Ps and photons; the process p-Ps→ $\gamma\gamma$ is realized through a triangle-shaped diagram with virtual electrons circulating in it, see Fig. 1.

At this point one needs to specify the kinematics of the two-body decay illustrated in Fig. 1. As for the external momenta, one has $p^\mu = (M_P, \mathbf{0})$, $k_1^\mu = (\omega, 0, 0, \omega)$, $k_2^\mu = (\omega, 0, 0, -\omega)$, with $\omega = \frac{M_P}{2}$, while the internal momenta are $q_1 = \frac{p}{2} + q$, $q_2 = \frac{p}{2} - q$ and finally $q_3 = \frac{p}{2} + q - k_1$.

Let us now introduce one of the most important objects of our model, i.e., the triangle amplitude I related to the diagram of the process depicted in Fig. 1, which reads

$$I = \mathrm{i}\int \frac{\mathrm{d}^4q/(2\pi)^4\ \mathcal{F}(q,p)}{(q_1^2 - m_e^2+\mathrm{i}\varepsilon)\,(q_2^2-m_e^2+\mathrm{i}\varepsilon)\,(q_3^2-m_e^2+\mathrm{i}\varepsilon)}, \tag{4}$$

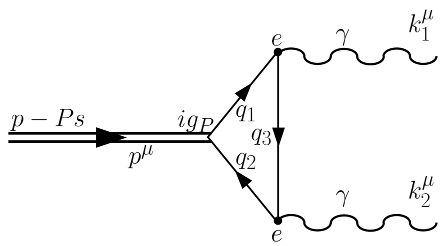

Fig. 1. Triangle-shaped diagram for the process p−Ps → $\gamma\gamma$.

where $\mathcal{F}(q,p)$ is the 'nonlocal' vertex function proportional to the Fourier transform of the wave function of the para-positronium. Note, this function is formally not present in (3), since the latter is local. It could be however easily included by rendering it nonlocal, see details in [19–21, 24].

The evaluation of the integral of (4) has been done in two independent ways. The first one is performed by using the Wick rotation, and the second one by using the residue theorem. The basics of our formalism, including the discussion about the convergence of integral of (4), has been presented in the quantum field theoretical (QFT) scalar toy model in [22].

The triangle amplitude of (4) can be written as

$$I = \mathrm{i}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathcal{F}(q,p)}{D_1\,D_2\,D_3}, \tag{5}$$

where

$$\begin{aligned} D_{1,2} &= (p/2 \pm q)^2 - m_e^2 + \mathrm{i}\varepsilon = \\ &\quad \left(M_P/2 \pm q^0\right)^2 - \boldsymbol{q}^2 - m_e^2 + \mathrm{i}\varepsilon, \\ D_3 &= \left(M_P/2 + q^0 - k_1^0\right)^2 - (\boldsymbol{q} - \boldsymbol{k_1})^2 - m_e^2 + \mathrm{i}\varepsilon, \end{aligned} \tag{6}$$

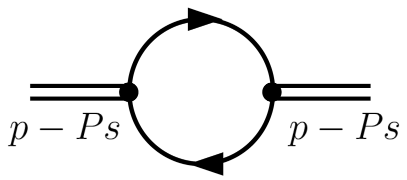

Fig. 2. Loop diagram of the process p−Ps $\to$ $e^- e^+ \to$ p−Ps.

By setting $D_{1,2,3} = 0$, one gets the corresponding poles

$$\text{Poles of} D_1 : \begin{cases} L_1 = -\frac{M_P}{2} - \sqrt{\rho^2+q_z^2+m_e^2} + \mathrm{i}\delta, \\ R_1 = -\frac{M_P}{2} + \sqrt{\rho^2+q_z^2+m_e^2} - \mathrm{i}\delta, \end{cases} \tag{7}$$

$$\text{Poles of} D_2 : \begin{cases} L_2 = \frac{M_P}{2} - \sqrt{\rho^2+q_z^2+m_e^2} + \mathrm{i}\delta, \\ R_2 = \frac{M_P}{2} + \sqrt{\rho^2+q_z^2+m_e^2} - \mathrm{i}\delta, \end{cases} \tag{8}$$

$$\text{Poles of } D_3 : \begin{cases} L_3 = -\sqrt{\rho^2+(q_z-k_z)^2+m_e^2} + \mathrm{i}\delta, \\ R_3 = \sqrt{\rho^2+(q_z-k_z)^2+m_e^2} - \mathrm{i}\delta. \end{cases} \tag{9}$$

The resulting decay width into $\gamma\gamma$ within our approach reads

$$\Gamma_{P\to\gamma\gamma} = \frac{1}{2}\frac{2|\boldsymbol{k}_1|}{8\pi M_P^2}\left|4\pi\, 8 m_e \alpha\, g_P\, I\, \frac{M_P^2}{4}\right|^2, \tag{10}$$

with $|\boldsymbol{k}_1| = \frac{1}{2}M_P$.

The para-positronium is not an elementary particle, but an extended object emerging as a bound state of one electron and one positron. Thus, the positronium-constituent coupling constant g_P entering the Lagrangian in (3) is not a free parameter of our model and it can be obtained by using the Weinberg's compositeness condition [17, 18]

$$g_P = \sqrt{\frac{1}{\Sigma'(s{=}M_p^2)}} \tag{11}$$

with

$$\Sigma(s) = -\mathrm{i}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathcal{F}^2(\boldsymbol{q})^2}{D_1\, D_2}\left(-\frac{p^2}{4}+q^2-m_e^2\right), \tag{12}$$

where $\Sigma(s)$ is the loop function depicted in Fig. 2 as a self-energy loop diagram.

Within our approach the vertex function $\mathcal{F}(q,p)$ turns out to be proportional to the wave function of the positronium in momentum coordinates, $A(\boldsymbol{q})$,

$$\mathcal{F}(q,p) = \mathcal{F}(\boldsymbol{q}^2) \sim A(\boldsymbol{q}) = \left(1 + \frac{4}{\alpha^4+m_e^2}\boldsymbol{q}^2\right)^{-2}. \tag{13}$$

However, simply setting it as equal $\mathcal{F}(q,p) = A(\boldsymbol{q})$ does not work, because for this naive Ansatz the value of the decay rate of the process p−Ps $\to \gamma\gamma$ turns out to be too small by a factor of 2 compared to the experimental data.

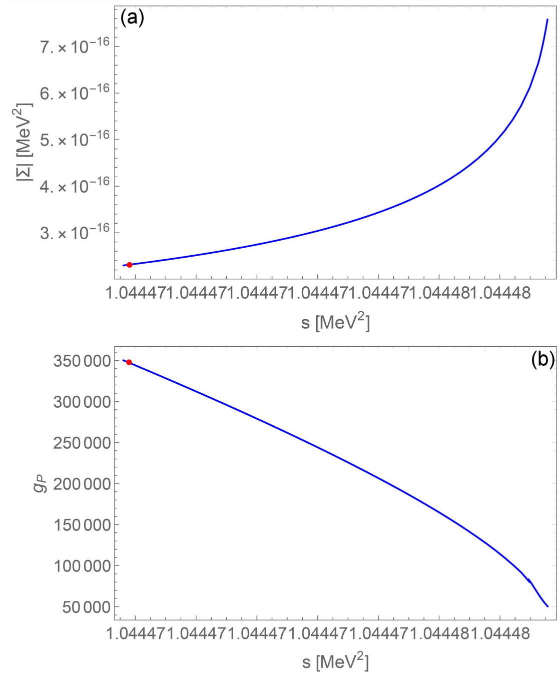

Fig. 3. (a) Dependence of the loop function Σ on the running square mass of the positronium $s = M_P^2$. (b) Dependence of coupling constant g_P on $s = M_P^2$. Red dots represent the physical case.

Moreover, at first glance it also seems that in (13) the covariance is broken. This is not necessarily the case, since we can interpret the vertex function as resulting from the Lorentz-invariant object [25]

$$\frac{-(pq)^2 + p^2 q^2}{p^2} = \boldsymbol{q}^2, \tag{14}$$

thus in the rest frame of the decaying particle, in which the four-momentum of the positronium is $p = (\sqrt{s}, \boldsymbol{0})$, reduces to

$$\mathcal{F}(p,q) = \mathcal{F}\left(\frac{-(pq)^2 + p^2 q^2}{p^2}\right) = \mathcal{F}_{RF}(\boldsymbol{q}^2). \tag{15}$$

As a consequence of this setup, in the rest reference frame there is no q^0 dependence (and no additional pole) resulting from the vertex function.

Following [23] (see also the scalar model in [22]) let us now consider another possibility for the vertex function, constructed as follows

$$\mathcal{F}(\boldsymbol{q}^2) = \gamma^4\left(\boldsymbol{q}^2 + \gamma^2\right)^{-1} \tag{16}$$

with $\gamma^2 = m^2 - \frac{1}{4}M_P^2$. First, in Fig. 3 we present the loop function $\Sigma(s{=}M_P^2)$ and the coupling constant for this choice. It is visible that with increasing the variable s the value of the loop function $\Sigma(s)$ also increases and reaches a cusp at the threshold located at the energy $2m_e$. Consequently, the coupling constant g_P as a function of $s = M_P^2$ decreases with increasing s and vanishes at the threshold.

In Table II we list the result of the decay width of the process p−Ps $\to \gamma\gamma$ obtained in our model by making use of the vertex function of (16).

TABLE II

Pole contribution to the $\Gamma \mathrm{p-Ps} \to \gamma\gamma$).

Pole(s) contribution	Result [μs^{-1}]
Pole 1	7968.15
Pole 1 + pole 2	7995.34
Pole 1 + pole 2+ pole 3	7920.26

It is visible that the result involving the contribution of all three poles is now closer to the experimental value 7990.9(1.7) μs^{-1} than the lowest-order Wheeler–Pirenne result. In this respect, it can be shown that in the non-relativistic limit, our model correctly reduces to $\alpha^5 m_e/2$.

Moreover, it is interesting to study the role of each pole, so in (10) we split $I = I_1 + I_2 + I_3$, where I_k arises from the pole R_k. The contribution to the total decay rate from the first pole is by far the dominant one. Namely, the ratio of the amplitude contribution of the second pole with respect to the first one is 0.00170446, while the contribution of the third with respect to the first pole is −0.00471415. Interestingly, the third pole gives a negative contribution to the decay width. This goes in the right direction, but the contribution is even too strong, delivering a theoretical result smaller than the experiment.

Note, the result found in [23] for the same vertex function is 7952.7 μs^{-1}, the difference being due to the Weinberg compositeness condition implemented in our approach. Still, we agree with the interpretation that QFT approaches implicitly contain the resummation of a certain class (but not all) of QED diagrams, see the discussion in [26].

3. Conclusions

After a brief recall of the theoretical and experimental results on the para-positronium decay rate into two photons, we introduced a QFT composite model to describe this decay. Upon evaluating the triangle diagram with virtual electrons and using the Weinberg compositeness condition, we calculated the $\gamma\gamma$ decay width. We showed that the role of the vertex function is important. The choice made in [23] delivers quite good results, but further improvement is needed to make the finding agree with the very precise experimental result.

In the future, one should test other choices for the vertex function, such as the promising covariant Ansatz of [27]. In general, the connection to the bound-state of quarks can also be useful [19, 21, 26, 28]. Also, studies of the decay of radially excited states of para-positronium into photons, as well as the decay of ortho-positronium into three photons, represent an outlook of our model.

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