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Cumulative Distribution Functions as Hysteresis Models

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The cumulative distribution functions can be used as the basis for hysteresis models. Here it is described how, using only 3 parameters, including one representing the shape, hysteresis curves can be constructed using symmetric distribution functions. The model is useful in the interpretation of magnetic Barkhausen noise data. The model also has a clear physical meaning because it represents the distribution of coercivity inside the sample. An isotropic Stoner–Wohlfarth hysteresis was partially modelled by a three-parameter cumulative distribution function of Gaussian hysteresis for the 1st and 3rd quadrants. Asymmetric distributions will provide better hysteresis adjustment, but these are four-parameter models.

topics: hysteresis, magnetic Barkhausen noise (MBN), Stoner–Wohlfarth, cumulative distribution function (CDF)

1. Introduction

Hysteresis modelling has been the subject of many recent studies [1–7] because it allows, among other things, a better understanding of the processes related to the reversal of magnetization. The main idea of this study is to use a function representing the shape of hysteresis. Cumulative distribution functions (CDF) are a very reasonable choice for describing the hysteresis form, as discussed in this paper.

The inspiration for using the cumulative distribution function comes from several sources, especially from the Benitez model [8] for the magnetic Barkhausen noise (MBN). This model [8] assumes that the envelope of the MBN signal can be divided into two Gaussians and has previously been applied to study the grain size effect [9]. The physical basis for the Benitez model [8] is that MBN can be interpreted as a differential $\mathrm{d}B/\mathrm{d}t$ [10]; see, for example, the paper by H.J. Williams, W. Shockley, and C. Kittel [11].

In the case of a soft magnetic material, even in a quasi-static condition with a frequency near zero, several different processes take place inside the hysteresis [12]: (i) irreversible rotation of domains, (ii) irreversible domain wall displacement, (iii) creation and annihilation of domain walls, (iv) elimination of "90° closure domains", associated with magnetostrictive effects. Thus, it is difficult to cover all these different processes within a single model. Besides, the fact that soft magnetic materials have 3 easy axes (iron) or 4 easy axes (nickel) makes it very difficult to evaluate the magnetization processes, also due to the difficulty of evaluating the demagnetizing field [13, 14].

The sigmoidal shape of the hysteresis is due to magnetocrystalline anisotropy. Otherwise, the hysteresis would be an ellipsoid, as assumed, for example, in the superellipse model [15]. Cumulative distribution functions well reflect the sigmoidal shape, such as the error function, which is used to study atomic diffusion processes [16].

One of the objectives here is to find a model with a small number of parameters. For example, the Jiles–Atherton (JA) model has 5 adjusting parameters, as described in the Sablik–Jiles model for plastic deformation [17]. However, even with 5 parameters, the JA model was not able to fit or obtain the experimental hystereses, which were modified due to the plastic deformation in electrical steels [18]. The reason is simple, namely the JA model imposes a Langevin function as the "skeleton" of the curve. However, this is an unrealistic assumption. Thus, the JA model can be considered a purely phenomenological model. In other words, the JA model only gives a geometric description of hysteresis. Therefore, other functions can be considered as those representing the "hysteresis skeleton". One of such functions is the Gaussian distribution [19].

The Stoner–Wohlfarth (SW) model has been successful in describing the complete hysteresis of 2:17 type SmCo magnets [20, 21], and the reason is that the only reversal process in that case is coherent rotation. Thus, as a starting point, the Gaussian CDF

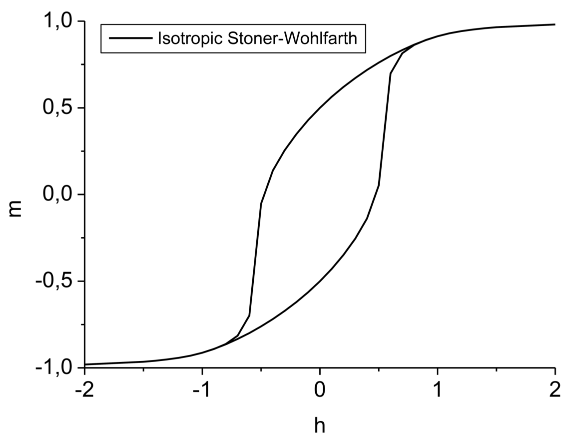

Fig. 1. Isotropic Stoner–Wohlfarth model.

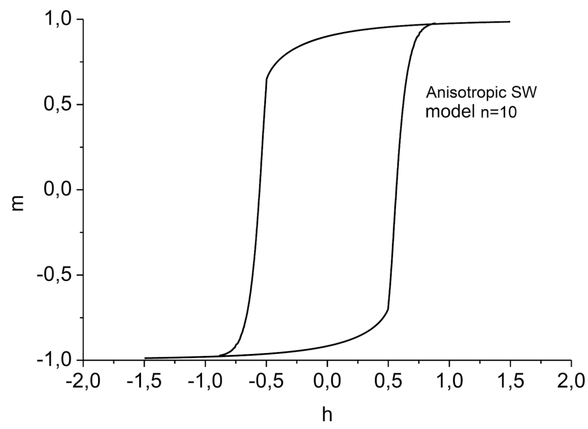

Fig. 3. Textured Stoner–Wohlfarth model; n=10.

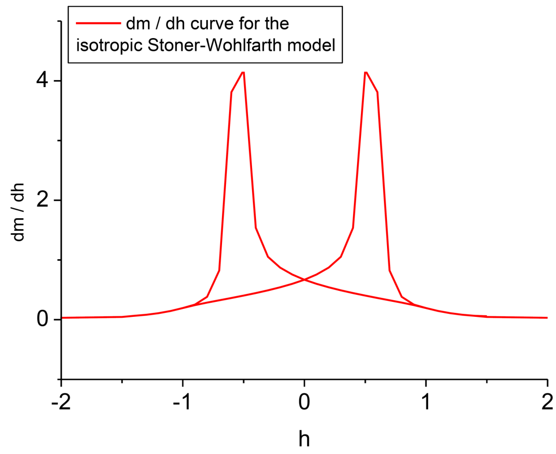

Fig. 2. The dm/dh curve for the isotropic Stoner–Wohlfarth model.

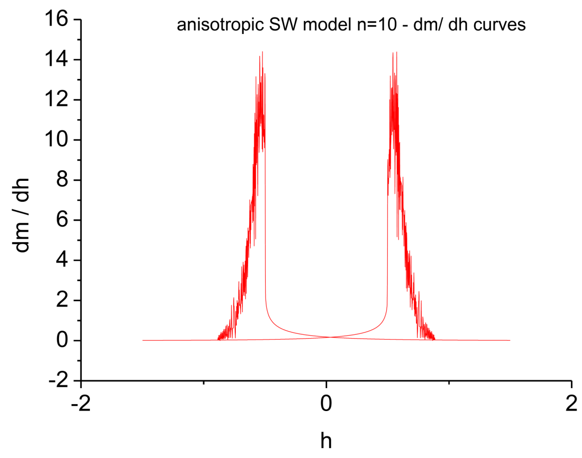

Fig. 4. The dm/dh curve for textured isotropic Stoner–Wohlfarth model; $n = 10$.

model will be compared with isotropic SW hysteresis. The derivative of any hysteresis is a distribution function, and it represents portions of the sample with reversal of magnetization promoted by a given applied field.

2. Isotropic and anisotropic Stoner–Wohlfarth model and its derivative

The isotropic Stoner–Wohlfarth (SW) model [22, 23] is a case where there are only 3 parameters; 2 are the scale parameters related to the abscise and ordinate, and the shape of the hysteresis is due to texture — isotropic in the case of Fig. 1. It is observed, even for this very simple case, that the dM/dH curves only could be represented by an asymmetrical distribution (see Fig. 2).

In other words, the distributions obtained with the dM/dH curves define the shape of the hysteresis. Then, by observing the derivative of experimental hysteresis, a compatible distribution can be chosen, and thus, the hysteresis can be better modelled. In this paper, M is the magnetization, and H is the applied field, while m is the reduced magnetization $m = M/M_s$, and h is the reduced field $h = H/H_A$, where H_A is the anisotropy field, and M_s is the saturation magnetization. The Stoner–Wohlfarth model uses dimensionless parameters m and h.

Figures 3 and 4 show the hysteresis calculated with $n = 10$ and its derivative, respectively. As aforementioned, this is a 3-parameter model, with the texture given by $n = 10$ and $M_r/M_s = 0.917$, because $M_r/M_s = n + 1/n + 2$. In the case of the isotropic SW model, $f(\alpha) = 1$ [22]. For the anisotropic SW model, the magnetization $m(h)$ needs to be altered according to the distribution, i.e.,

$$m_{\parallel} = \frac{\int_0^{2\pi} \mathrm{d}\alpha\ f(\alpha)\cos(\alpha-\varphi)\sin(\alpha)}{\int_0^{2\pi} \mathrm{d}\alpha\ f(\alpha)\sin(\alpha)}. \quad (1)$$

Here, in Figs. 3 and 4, n=10 [23] was used with the distribution given by

$$f(\alpha) = \cos^n(\alpha). \tag{2}$$

The conclusion from both Figs. 2 and 4 is that the distributions $\mathrm{d}M/\mathrm{d}H$ obtained from SW hysteresis are asymmetrical. Thus, it is quite possible that the Gaussian function is not the best option for representing the hysteresis in Figs. 1 and 3 because Gaussian instead has a symmetrical distribution.

As a title of curiosity, a "perfect square" hysteresis would be obtained with $n = \infty$. By making $\mathrm{d}M/\mathrm{d}H$, then for $n = \infty$ instead of the distribution, the line would appear at the point $h = 1$. Thus, by increasing n, the squareness of the hysteresis increases, as well as the sharpness of the corresponding distribution. Thus, by differentiating the hysteresis curve, it is possible to establish methods for determining the squareness, which is a relevant parameter in some applications [24].

3. Cumulative distribution function: Gaussian case

The integral of a distribution function is its cumulative distribution function (CDF), here denoted by Φ,

$$\Phi = \int \mathrm{d}x\ \mathrm{e}^{-x^2}. \tag{3}$$

The most commonly used distribution is the "normal" or Gaussian distribution. But there are many others, such as Cauchy–Lorentz, which is also symmetrical. In the case of the Gaussian distribution (3), there is a solution for the infinite integration interval given by

$$\Phi = \int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-x^2} = \sqrt{\pi} \tag{4}$$

— a famous result first obtained by Laplace. Note that (3) can be solved numerically by means of a Taylor series expansion.

Then, if the integral is to be equal to unity or 100% by definition, it is necessary to divide it by $\sqrt{\pi}$. If a constant a multiplies x^2, as in the following expression

$$\Phi = \int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{5}$$

then for obtaining the normalization $\int \mathrm{d}x\ f(x) = 1$, it is necessary to divide the integral (5) by $\sqrt{\pi/a}$. As a consequence, the error function is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t\ \mathrm{e}^{-t^2}. \tag{6}$$

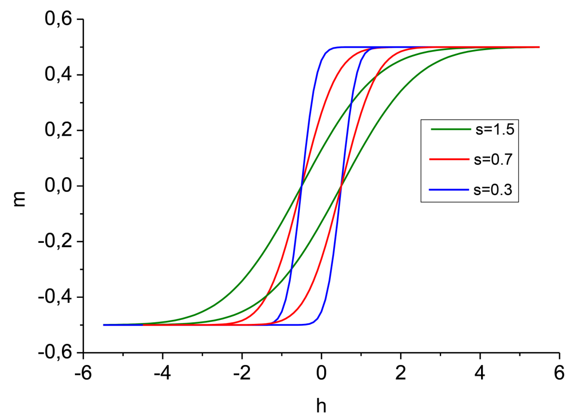

Fig. 5. Effect of parameter s on the hysteresis curve of the CDF Gaussian hysteresis model. All curves used the same scale parameter p = 0.5. Thus, all hysteresis have the same H_c.

Therefore, the cumulative distribution of a Gaussian is given by the error function (erf), as follows

$$y = \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right). \tag{7}$$

Cumulative distribution functions can be used as hysteresis models, especially if the applied field allows for proximity of sample saturation and, thus, the hysteresis has a sigmoidal shape. In the case of applied fields distant from saturation, the hysteresis has an ellipsoid shape, and then the sigmoidal hysteresis models are not valid. Thus, the CDF model may not be suitable for describing minor loops. When plotted on the graph, (7) has a sigmoidal shape. For (7), the center of hysteresis at $(0,0)$ is obtained for $y_1 = y_2 = y - 0.5$, and $x_1 = x - 0.5$ and $x_2 = x + 0.5$.

The standard deviation s of the Gaussian function can be taken into account, as seen in

$$y = \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{s\sqrt{2}}\right). \tag{8}$$

Also, the shift parameter p promotes an alteration in the hysteresis shape, and it is defined as $x_p = x + p$ and $x_p = x - p$, as seen in (8). Coercivity is related to the parameter p, whereas permeability is associated with the parameter s. For example, increasing s has the effect of reducing permeability, as observed in Fig. 5. The effect of the p parameter on hysteresis is depicted in Fig. 6. This is only a 3-parameter model (in contrast with the JA 5-parameter hysteresis). A mean-field parameter, as in the SW–CLC[†1] model [20], can also be included if necessary, increasing the number of parameters to 4. It should be noted that in this model, for the same p, the coercivity is the same, as can be observed in Figs. 5 and 6.

[†1] CLC — Callen–Liu–Cullen

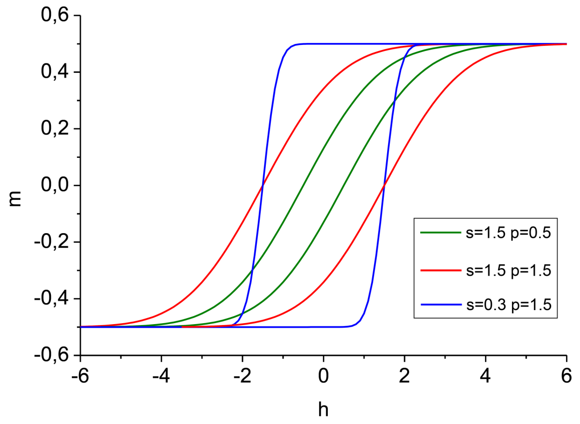

Fig. 6. Effect of parameter p on the hysteresis shape of the CDF Gaussian hysteresis model. It should be noted that, for same p, same coercivity.

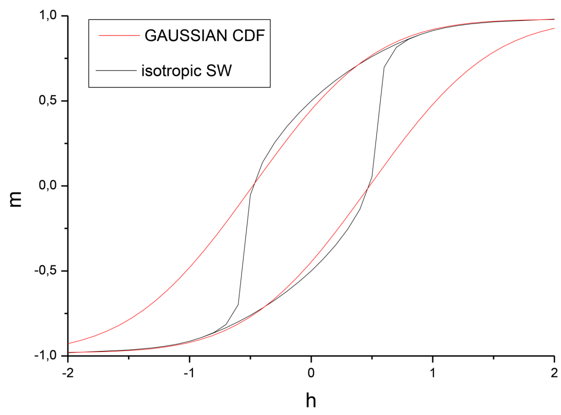

Fig. 7. SW isotropic hysteresis compared with the CDF Gaussian hysteresis model. Model parameters: $s = 0.79$, $\Psi = 1.96$, and $p = 0.48$.

In the CDF Gaussian hysteresis model, the scale parameter of the abscise is p, whereas the shape parameter is s. Thus, a third parameter related to the ordinate — Ψ — is defined in

$$y = \Psi \frac{1}{2} \operatorname{erf}\left(\frac{x}{s\sqrt{2}}\right). \tag{9}$$

The model is useful for application in the analysis of Barkhausen magnetic noise data. Any other probability function, such as, for example, Voigt or Lorentzian, can also be used as the basis for similar hysteresis models. The physical interpretation of the parameter p is that it represents the coercive field. The model, therefore, has a clear physical meaning, namely, it gives the distribution of the coercive force inside the sample, which may concern different regions (groups of grains) or individual grains.

Another possibility for a symmetrical distribution is the raised cosine distribution [25], i.e.,

$$y = \frac{1}{2S_c}\left[1 + \cos\left(\frac{x}{S_c}\pi\right)\right], \tag{10}$$

and the respective CDF distribution given as

$$\Phi = \frac{1}{2}\left[1 + \frac{x}{S_c} + \frac{1}{\pi}\sin\left(\frac{x}{S_c}\pi\right)\right]. \tag{11}$$

Here, S_c denotes the hysteresis shape parameter for the raised cosine distribution. The Laplace distribution [26] is also a possibility, i.e.,

$$y = \frac{1}{2S_L}\exp\left(-\frac{|x|}{S_L}\right), \tag{12}$$

$$\Phi = \frac{1}{2}\exp\left(\frac{x}{S_L}\right) \quad \text{for} \quad x \le 0, \tag{13}$$

$$\Phi = 1 - \frac{1}{2}\exp\left(-\frac{x}{S_L}\right) \quad \text{for} \quad x \le 0. \tag{14}$$

It should be noted that the CDF of the Laplace distribution is different for $x < 0$ and for $x > 0$ (see (13) and (14)). For the Laplace distribution, S_L is the shape parameter. Both the raised cosine distribution and the Laplace distribution are easy to integrate and do not present a very complicated CDF.

4. Models comparison

In Fig. 7, the comparison of the CDF Gaussian hysteresis model with the isotropic SW hysteresis is presented. The fitting parameters are $s = 0.79$ and $\Psi = 1.96$. The parameter p was set to 0.48, because in the SW isotropic model, the coercivity is 0.48. In Fig. 7, it is noted that the adjustment is only partial. However, by making the p parameter flexible, it was possible to model the 1st and 3rd quadrants of the SW isotropic curve, as seen in Fig. 8.

5. Additional comments

The comparison of the two models in Fig. 7 shows the limitation of the CDF Gaussian hysteresis model. However, a reasonable fitting was presented in Fig. 8 for the 1st and 3rd quadrants of the hysteresis.

Modelling a hysteresis is a very laborious process that involves trial and error to see if a given distribution can fit the experimental data. Instead of fitting the hysteresis, fitting the derivative $\mathrm{d}M/\mathrm{d}H$ can be a more rapid method for finding the hysteresis parameters.

Other functions can be considered. Unfortunately, asymmetrical distribution functions have at least 2 shape parameters, and this increases the number of parameters to four. Even so, this is better than the JA model with 5 parameters.

Alternative possibilities for the symmetric Gaussian distribution are skewed distributions. Especially the asymmetric Laplace distribution [27]

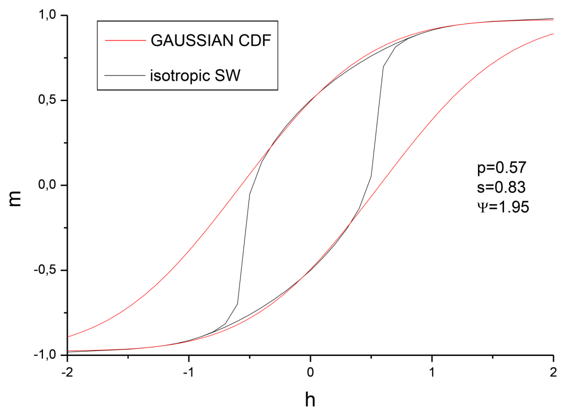

Fig. 8. SW isotropic hysteresis compared with the CDF Gaussian hysteresis model. Model parameters: $s = 0.83$, $\Psi = 1.95$, and $p = 0.57$.

could solve the problem of modelling hysteresis. Another possibility is the Weibull distribution [28]. The Gamma distribution is also an alternative [29].

The Gamma function was established by Euler after studying the Wallis formula for π. This gave its name to a family of integrals, i.e., the Wallis integrals, which are solved using the Gamma function [30].

6. Conclusions

The cumulative distribution functions (CDFs) can be used as the basis for hysteresis models. Here it is described how, using only two additional constant parameters, a sigmoidal hysteresis curve can be constructed. The model is useful in the interpretation of magnetic Barkhausen noise data. The model also has a clear physical meaning because it represents the distribution of the coercive field inside the sample. The model is first presented for a Gaussian distribution function, but it can be easily extended to Voigt, Lorentzian, or any other distribution.

An isotropic Stoner–Wohlfarth hysteresis was partially modelled by a three-parameter CDF Gaussian hysteresis, but only in the 1st and 3rd quadrants. Asymmetric distributions will provide better hysteresis adjustment, but these are 4-parameter models.

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