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On Magnetic Boundary Control for Metric Graphs

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It is investigated how magnetic boundary control can be used to solve inverse problems for Schrödinger operators on metric graphs. Explicit examples show that such reconstruction is sometimes possible, starting from a single contact vertex in the graph.

topics: metric graphs, inverse problems, spectral analysis, boundary control

1. Introduction

The current note is devoted to the boundary control method [1–6] applied to solve inverse problems for magnetic Schrödinger operators on metric graphs [7, 8]. It is well-known that the precise form of the magnetic potential cannot be reconstructed from the spectrum — standard magnetic Schrödinger operators with equal fluxes through the cycles and the same electric potentials are unitary equivalent and hence isospectral. Moreover, the inverse problems for operators with zero magnetic potential cannot always be solved uniquely [9–18] (see also numerous papers devoted to isospectral discrete graphs, e.g., [19–23]). It was proposed in [24] to use magnetic fluxes to enrich the set of spectral data in order to get a unique solution to the inverse problem. The spectral data are given by the spectrum for different values of the magnetic fluxes through the cycles. It was proposed to call this method magnetic boundary control (MBC-method) to underline the role of the magnetic field in the solution of the inverse problem.

In the case of trees (graphs without cycles), the magnetic potential can be removed completely, hence the spectrum is independent of the magnetic potential. The inverse problem is uniquely solvable if the M-function (energy-dependent Dirichlet-to-Neumann map) associated with all degree one vertices is known [25, 26]. The MBC-method for standard operators on graphs with one cycle was considered in [24]. It was proven that the solution to the inverse problem is not unique if the cycle is given by a loop. On the other hand, if the cycle is not a loop, then the solution is unique in the generic case[†1]. Our goal today is to study the MBC-method for graphs with several cycles. It appears that the inverse problem can sometimes be uniquely solved using this new method. We have illustrated our discoveries with a few explicit examples showing both the power of the method and its limitations.

[†1] One should mention that for the lasso graph with other than standard vertex conditions, the inverse problem is uniquely solvable by the MBC-method [27].

2. Metric graphs and standard Schrödinger operators

A finite compact metric graph Γ can be seen as a collection of compact intervals $E_n = [x_{2n-1}, x_{2n}]$, $n = 1, 2, \ldots, N$, called *edges* with the set of end points $\boldsymbol{V} = \{x_j\}_{j=1}^{2N}$ divided into equivalence classes V^m $(m = 1, 2, \ldots, M)$ called *vertices*, so that $\boldsymbol{V} = \bigcup_{m=1}^{M} V^m$ and $V^{m_1} \bigcap V^{m_2} = \emptyset$, provided $m_1 \neq m_2$. Then the metric graph Γ is the union of edges $\Gamma = \bigcup_{n=1}^{N} E_n$ with the end points belonging to the same vertex identified [8].

The corresponding Hilbert space of square-integrable functions on Γ coincides with the orthogonal sum of the spaces of functions on the edges

$$L_2(\Gamma) = \bigoplus_{n=1}^{N} L_2(E_n). \tag{1}$$

Let $q \in L_2(\Gamma)$ and $a \in C(\Gamma \setminus \boldsymbol{V})$ be real-valued electric and magnetic potentials on the edges. Then the standard magnetic Schrödinger operator in $L_2(\Gamma)$ is defined by the differential expression

$$\tau_{q,a} := \left(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}x} + a(x) \right)^2 + q(x) \tag{2}$$

on the functions from

$$W_2^2(\Gamma) = \bigoplus_{n=1}^{N} W_2^2(E_n) \tag{3}$$

satisfying standard vertex conditions at every vertex V^m $(m = 1, 2, \ldots, M)$

$$\begin{cases} u(x_j) = u(u_l), \quad \text{provided} \quad x_j, x_l \in V^m \\ \quad \text{— continuity condition,} \\ \sum_{x_j \in V^m} \partial u(x_j) = 0 \quad \text{— Kirchhoff condition.} \end{cases} \tag{4}$$

Here, $u(x_j)$ and $\partial u(x_j) = (-1)^{j-1} u'(x_j)$ denote the limiting values of the function $u \in W_2^2(\Gamma)$ and its first oriented derivative, respectively. Note that the derivates are taken in the direction pointing inside the corresponding edge and, therefore, are independent of the chosen orientation of the edges.

The main results presented in this paper hold even for potentials $q \in L_1(\Gamma)$, but our presentation will be limited to $L_2(\Gamma)$ potentials for the sake of clarity. The general case has been treated in [8]. We shall also avoid discussing other than standard vertex conditions — the reason is that the elimination of the magnetic potential leads to a change in vertex conditions.

Let us denote the corresponding self-adjoint operator by $L_{q,a}(\Gamma)$. Its spectrum is discrete [7, 8].

The spectrum of $L_{q,a}(\Gamma)$ does not depend on the particular form of the magnetic potential a, but on the magnetic fluxes — the integrals of the magnetic potential along non-trivial cycles C_j in Γ, i.e.,

$$\Phi_j = \int_{C_j} \mathrm{d}y \; a(y), \qquad j = 1, 2, \ldots, \beta_1, \tag{5}$$

where β_1 is the first Betti number — the number of independent cycles in Γ.

3. M-function: short introduction

The (scalar) Titchmarsh–Weyl M-function was first introduced to describe spectral properties of one-dimensional Schrödinger operators [28]. For many years, its matrix generalisation has been used to solve inverse problems for operators on metric graphs [25, 29]. It is not our aim to give a rigorous introduction to the theory of M-functions for metric graphs — it is given in [8].

Among all vertices in Γ, we choose a non-empty subset $\partial\Gamma$, which we call the *contact set*. It should be understood that the graph Γ can be approached only via this set. Of course, we are interested in the case when this set is small compared to the set of all vertices in the graph. For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ consider solutions to the stationary magnetic Schrödinger equation on the edges

$$-\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\, a(x)\Big)^2 \psi(\lambda, x) + q(x)\psi(\lambda, x) = \lambda \psi(\lambda, x), \tag{6}$$

satisfying standard conditions (4) at all *internal vertices* $V^m \in \boldsymbol{V} \setminus \partial\Gamma$ and just continuous at the contact vertices $V^m \in \partial\Gamma$. Every such solution is uniquely determined by its values on the contact set. Consider the matrix-valued function, called M-function for the graph Γ and contact set $\partial\Gamma$,

$$M_\Gamma(\lambda) : \boldsymbol{\psi}|_{\partial\Gamma} \mapsto \partial\boldsymbol{\psi}|_{\partial\Gamma}, \tag{7}$$

connecting the values of the solution at the contact vertices $\psi(V^m)$ to the sums of oriented derivatives

$$\partial\boldsymbol{\psi}(V^m) := \sum_{x_j \in V^m} \partial\psi(x_j). \tag{8}$$

This is a matrix-valued Herglotz–Nevanlinna function in λ, i.e., it is analytic outside of the real axis and has a positive imaginary part in the upper half-plane

$$\Im\big(\lambda\big) > 0 \Rightarrow \Im(M_\Gamma(\lambda) := \frac{\big(M_\Gamma(\lambda) - M_\Gamma^*(\lambda)\big)}{2\mathrm{i}} \geq 0. \tag{9}$$

In what follows, we shall also use the Dirichlet Schrödinger operator $L_{q,a}^{\mathrm{D}}(\Gamma)$ — the operator in $L_2(\Gamma)$ defined by the same differential expression, standard vertex conditions at the internal vertices, and Dirichlet conditions at the contact vertices. This operator is again self-adjoint and has a discrete spectrum.

The singularities of the M-function coincide with the spectrum of the Dirichlet operator $L_{q,a}^{\mathrm{D}}(\Gamma)$, while some of the spectrum of $L_{q,a}(\Gamma)$ can be identified using the secular equation $\det M_\Gamma(\lambda) = 0$. The last equation determines those eigenvalues of $L_{q,a}(\Gamma)$, which are not simultaneously eigenvalues of $L_{q,a}^{\mathrm{D}}(\Gamma)$.

The eigenvalues and the normalised (in the original Hilbert space $L_2(\Gamma)$) eigenfunctions of $L_{q,a}(\Gamma)$ and $L_{q,a}^{\mathrm{D}}(\Gamma)$ will be denoted by λ_n, ψ_n and λ_n^{D}, ψ_n^{D}, respectively.

The structure of M-functions for the graph is best seen from the following two explicit formulas [30, 31]

$$M_\Gamma(\lambda) = -\left[\sum_{n=1}^{\infty} \frac{\big\langle \psi_n|_{\partial\Gamma}, \cdot \big\rangle_{\ell_2(\partial\Gamma)} \psi_n|_{\partial\Gamma}}{\lambda_n - \lambda}\right]^{-1}, \tag{10}$$

$$M_\Gamma(\lambda) - M_\Gamma(\lambda') = \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^{\mathrm{D}} - \lambda)(\lambda_n^{\mathrm{D}} - \lambda')} \times \big\langle \partial\psi_n^{\mathrm{D}}|_{\partial\Gamma}, \cdot \big\rangle_{\ell_2(\partial\Gamma)} \partial\psi_n^{\mathrm{D}}|_{\partial\Gamma}. \tag{11}$$

In the above formulas, $\psi_n|_{\partial\Gamma}$ and $\partial\psi_n^{\mathrm{D}}|_{\partial\Gamma}$ denote the function values and the oriented derivative values at the contact vertices, respectively, and $\lambda' \neq \lambda$ is any complex number not lying in the spectra of $L_{q,a}(\Gamma)$ and $L_{q,a}^{\mathrm{D}}(\Gamma)$.

The first formula determines $M_\Gamma(\lambda)$ directly, while the second formula determines only the difference between the values of M-function at two different points. To use the second formula, it is enough to know the degrees d_m of the contact vertices since the M-function possesses the asymptotics

$$M(-s^2) = -s\,\mathrm{diag}\big\{d_m\big\} + \mathcal{O}(1), \quad s \to \infty. \tag{12}$$

To understand formula (12), consider the boundary control for the Laplace operator ($q(x) = a(x) \equiv 0$). If the boundary control is applied at a degree d_m vertex, then for sufficiently small times the vertex acts as a collection of degree one vertices. The boundary control creates outgoing waves in the

edges joined at the vertex, hence for small times the response is equal to the sum of responses from the degree one vertices. Connectivity between these edges starts to play a role only when the travelling waves return after reflection from the neighbouring vertices. Adding electric and magnetic potentials does not affect the asymptotics.

4. Dissolving a vertex

By solving the inverse problem using the MBC-method, one dissolves high-degree contact vertices. In this section, we shall discuss whether the M-function known for different values of the magnetic fluxes can be used to reconstruct the M-function for the graph where one of the vertices is substituted with several (d_m) degree one vertices.

Definition 1. We say that the metric graph Γ_1 is obtained from a metric graph Γ by dissolving a certain vertex V^0 in Γ if:

- the metric graphs Γ and Γ_1 share the same set of edges $\{E_n\}_{n=1}^N$,
- the end points connected at V^0 in Γ form degree one vertices in Γ_1,
- all other vertices in Γ and Γ_1 coincide.

Let us see Fig. 1, where the dissolving procedure is presented schematically. The green area represents the part of the graph that is not affected by the procedure. The degree four vertex V^0 is substituted with four degree one vertices $V^1, \dots, V^4$.

We restrict our presentation to connected graphs (both Γ and Γ_1 are connected). Then, the number of broken cycles is given by

$$\beta_1(\Gamma) - \beta_1(\Gamma_1) = d_0 - 1. \tag{13}$$

Our goal is to compare the M-functions corresponding to Γ and Γ_1. These functions depend not only on the spectral parameter λ, but on the magnetic fluxes as well. We shall indicate dependence on the fluxes through the broken cycles, assuming that the other fluxes (through preserved cycles) are fixed.

By $V^1, \dots, V^{d_0}$ we denote the pendant vertices in Γ_1 coming from the vertex V^0 in Γ and let C_j be a path connecting V^{d_0} to V^j, $j = 1, 2, \dots, d_0 - 1$. These paths on Γ_1 correspond to the cycles in Γ that are broken under the dissolution. The corresponding fluxes are

$$\Phi_j = \int_{C_j} \mathrm{d}y\ a(y) = \int_{V^{d_0}}^{V^j} \mathrm{d}y\ a(y), \tag{14}$$

where $j = 1, 2, \dots, d_0 - 1$. These fluxes form the vector $\boldsymbol{\Phi}$. It will be convenient to view $\boldsymbol{\Phi}$ as an element of $\mathbb{R}^{d_0}$ despite the fact that only $d_0 - 1$ of its coordinates may be non-zero

$$\boldsymbol{\Phi} = \big(\Phi_1, \Phi_2, \dots, \Phi_{d_0-1}, 0\big). \tag{15}$$

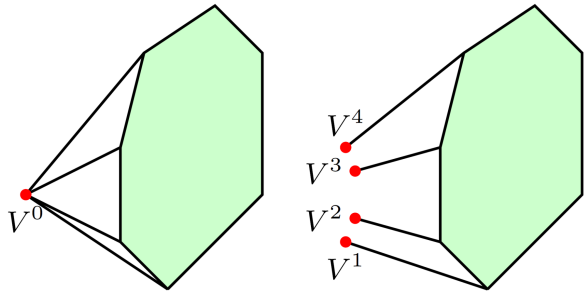

Fig. 1. Dissolving a vertex.

To reconstruct the M-function for Γ_1, it is enough to consider the fluxes equal to 0 and π, therefore we introduce the signs

$$\mu_j := \mathrm{e}^{\mathrm{i}\,\Phi_j}, \qquad j = 1, 2, \dots, d_0;$$
$$\boldsymbol{\mu} = \big(\mu_1, \mu_2, \dots, \mu_{d_0}\big) = \mathrm{e}^{\mathrm{i}\,\boldsymbol{\Phi}}, \tag{16}$$

and consider the M-functions depending on the signs μ_j instead of the phases Φ_j. To get the corresponding spectral data, it is enough to consider the standard operators with zero magnetic potential and additional signing conditions

$$\begin{pmatrix} u(y_j+0) \\ u'(y_j+0) \end{pmatrix} = -\begin{pmatrix} u(y_j-0) \\ u'(y_j-0) \end{pmatrix} \tag{17}$$

introduced at certain points y_j ($j = 1, 2, \dots, d_0$) on the pendant edges. The sign conditions can be seen as a singular magnetic potential concentrated at the point y_j [32]. These operators will be marked with $L_q^{\mathrm{sign}}(\Gamma)$ and called *signed Schrödinger operators*. We have 2^{d_0-1} different signed operators.

Our first step is to establish the relation between the diagonal element of the M-function associated with Γ and the vertex V^0

$$M_\Gamma^{00}(\lambda, \boldsymbol{\mu}) =: \mathbb{M}(\lambda, \boldsymbol{\mu}) \tag{18}$$

and the diagonal $d_0 \times d_0$ block of the M-function associated with the graph Γ_1 and the degree one vertices coming from V^0. We shall find an explicit relation between the scalar Herglotz–Nevanlinna function $\mathbb{M}(\lambda, \boldsymbol{\mu})$ and the $d_0 \times d_0$ matrix-valued Herglotz–Nevanlinna function $\mathbb{M}_1(\lambda, \boldsymbol{\mu}) := \{M_{\Gamma_1}^{ij}(\lambda, \boldsymbol{\mu})\}_{i,j=1}^{d_0}$.

The dependence of $\mathbb{M}_1(\lambda, \boldsymbol{\mu})$ upon $\boldsymbol{\mu}$ is trivial, namely

$$\mathbb{M}_1(\lambda, \boldsymbol{\Phi}) = \mathrm{diag}\,\{\mu_j\}\,\mathbb{M}_1(\lambda, \mathbf{1})\ \underbrace{\mathrm{diag}\,\{\mu_j\}^{-1}}_{=\mathrm{diag}\,\{\mu_j\}}, \tag{19}$$

where $\mathbf{1} = \big(1, 1, \dots, 1\big)$. To see this, it is enough to eliminate the magnetic potential starting from V^{d_0} using the transformation

$$f(x) \mapsto g(x) = \exp\left[-\mathrm{i}\int_{V^{d_0}}^{x} \mathrm{d}y\ a(y)\right] f(x). \tag{20}$$

The scalar function $\mathbb{M}(\lambda, \boldsymbol{\mu})$ is simply equal to the sum of all entries in $\mathbb{M}_1(\lambda, \boldsymbol{\mu})$

$$\underbrace{\mathbb{M}(\lambda, \boldsymbol{\mu})}_{=M_{\Gamma}^{00}(\lambda,\boldsymbol{\mu})} = \sum_{i,j=1}^{d_0} \mu_i \mu_j \underbrace{\mathbb{M}_1^{ij}(\lambda, \mathbf{1})}_{=M_{\Gamma_1}^{ij}(\lambda,\mathbf{1})}. \tag{21}$$

This formula determines the M-function for any signed operator on Γ through the M-function for Γ_1.

Let ψ_n^{D} denote the eigenfunction corresponding to zero fluxes through the broken cycles. These eigenfunctions can be chosen to be real-valued. Then, the normal derivatives of the Dirichlet eigenfunctions for non-zero fluxes are given by $\mu_j \partial\psi_n^{\mathrm{D}}(V^j)$, implying in particular that the normal derivative at V^0 is

$$\sum\nolimits_{j=1}^{d_0} \mu_j \partial\psi_n^{\mathrm{D}}(V^j). \tag{22}$$

It follows that the singularity of $\mathbb{M}(\lambda, \boldsymbol{\mu})$ is of the form

$$\mathbb{M}(\lambda, \boldsymbol{\mu}) \underset{\lambda\to\lambda_n^{\mathrm{D}}}{\sim} \frac{1}{\lambda_n^{\mathrm{D}}-\lambda} \sum_{i,j=1}^{d_0} \mu_i\mu_j\, \partial\psi_n^{\mathrm{D}}(V^i)\partial\psi_n^{\mathrm{D}}(V^j) = \frac{1}{\lambda_n^{\mathrm{D}}-\lambda}\left[\sum_{i=1}^{d_0}\left(\partial\psi_n^{\mathrm{D}}(V^i)\right)^2 + \sum_{\substack{i,j=1,\\ i\neq j}}^{d_0} \mu_i\mu_j\, \partial\psi_n^{\mathrm{D}}(V^i)\partial\psi_n^{\mathrm{D}}(V^j)\right], \tag{23}$$

where we have used that $\partial\psi_n^{\mathrm{D}}$ are real-valued.

Introducing the notation $a_j := \partial\psi_n^{\mathrm{D}}(V^j)$, we are faced with the following trivial problem — determine a_j if the numbers

$$(\pm a_1 \pm a_2 \pm \cdots \pm a_{d_0-1} + a_{d_0})^2 \tag{24}$$

are known for all possible combinations of the signs. It is clear that this reconstruction is possible only up to the multiplication of all a_j by -1, which corresponds to the multiplication of the corresponding eigenfunctions by -1.

The sum of the squares can be obtained by averaging over all possible signs

$$\sum_{i=1}^{d_0} a_j^2 = \sum_{\boldsymbol{\mu}\in(\{1,-1\}^{d_0-1},1)} \frac{\left(\mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_{d_0-1} a_{d_0-1} + a_{d_0}\right)^2}{2^{d_0-1}}. \tag{25}$$

Hence, we are able to determine the following combinations of a_j's

$$\sum_{\substack{i,j=1,\\ i\neq j}}^{d_0} \mu_i\mu_j\, a_i a_j = \left(\sum_{i=1}^{d_0} \mu_i a_i\right)^2 - \sum_{i=1}^{d_0} a_j^2. \tag{26}$$

We recover the products by averaging a second time

$$a_k a_l = \frac{1}{2^{d_0-1}} \sum_{\substack{\boldsymbol{\mu}\in(\{1,-1\}^{d_0-1},1),\\ \mu_k=\mu_l}} \left(\sum_{\substack{i,j=1,\\ i\neq j}}^{d_0} \mu_i\mu_j\, a_i a_j\right), \tag{27}$$

for $k \neq l$. The product $a_k a_l = a_l a_k$ appears in the double sum precisely 2^{d_0-1} times, while all other products cancel since $\mu_i\mu_j$ attains $+1$ and -1 equally many times.

If at least three of the coefficients are non-zero, then the squares a_j^2 are determined as

$$a_i^2 = \frac{(a_i a_j)\,(a_i a_l)}{(a_j a_l)}, \tag{28}$$

provided $a_j, a_l \neq 0$. We are able to recover one non-zero a_j up to a sign, but then all other non-zero coefficients are determined from the products $a_j a_i$. We conclude that if the squared sums $(\sum_{j=1}^{d_0} \mu_j a_j)^2$ are known for all $\boldsymbol{\mu}$ of the form $\boldsymbol{\mu} \in (\{1,-1\}^{d_0-1}, 1)$, then the coefficients a_j are determined up to a common sign.

It follows that the diagonal element $\mathbb{M}(\lambda, \boldsymbol{\mu})$ known for all $\boldsymbol{\mu} \in (\{1,-1\}^{d_0-1}, 1)$ determines the vector

$$\partial\boldsymbol{\psi}_{n\,2}^{\mathrm{D}} := \left(\partial\psi_n^{\mathrm{D}}(V^1), \partial\psi_n^{\mathrm{D}}(V^2), \ldots, \partial\psi_n^{\mathrm{D}}(V^{d_0})\right), \tag{29}$$

up to the common sign, hence the singular part of $\mathbb{M}_1(\lambda, \vec{0})$ is determined, which as before allows us to reconstruct it up to the constant matrix $\mathbb{A}$, yielding

$$\mathbb{M}_1(\lambda, \mathbf{0}) = \mathbb{A} + \sum_{\lambda_n^{\mathrm{D}}(\Gamma_1)} \frac{\lambda-\lambda'}{(\lambda_n^{\mathrm{D}}-\lambda)(\lambda_n^{\mathrm{D}}-\lambda')} \left\langle \boldsymbol{\psi}_{n\,2}^{\mathrm{D}}, \cdot \right\rangle \partial\boldsymbol{\psi}_{n\,2}^{\mathrm{D}}. \tag{30}$$

To determine $\mathbb{A}$, we remember that the M-function possesses the asymptotics (12).

We conclude that the M-function for Γ_1 can be recovered, provided the M-functions of all signed operators on Γ are known and the following generically satisfied conditions are fulfilled:

- the spectrum of $L_q^{\mathrm{D}}(\Gamma_1)$ is simple;
- the corresponding eigenfunctions ψ_n^{D} on Γ_1 are either invisible from the involved degree one vertices (all $\partial\psi_n^{\mathrm{D}}(V^j) = 0$, $j = 1, 2, \ldots, d_0$), or at least three normal derivatives $\partial\psi_n^{\mathrm{D}}(V^j)$, $j = 1, 2, \ldots, d_0$, are different from zero.

5. First examples

In this section, we discuss how to apply the MBC-method to solve inverse problems for metric graphs. We start by presenting examples where the whole graph can be reconstructed starting from a single vertex.

5.1. Example 1

Consider the graph presented in Fig. 2 and assume that the contact set consists of the single vertex V. Let us dissolve the vertex V. The described procedure allows us to determine the M-function associated with the new graph and all its degree one vertices. Conventional boundary control allows us to determine the lengths of the pendant edges and the potential q on them [25]. Then these edges can be peeled away, and we can reduce the inverse problem to a smaller graph, where contact vertices are indicated by red points. By repeating the procedure by dissolving the vertices V' and V'', the inverse problem is reduced to a tree with all pendant vertices in the contact set (see the upper sequence in Fig. 2). The MBC-method allows us to solve the inverse problem for this graph. Note that starting from a single vertex, we recovered both the metric graph Γ and the electric potential q on it.

The inverse problem for this graph can be solved by dissolving the vertices V, V^*, and V^{**} instead (see the lower sequence in Fig. 2). The resulting graph is the cycle with 3 contact points — the inverse problem can again be solved by dismantling the cycle into three intervals.

This example shows that the MBC-method allows us to solve the inverse problem for rather complicated graphs with an arbitrary number of cycles and very few contact points.

5.2. Example 2

Figure 3 presents another graph with a single contact vertex V. After dissolving V and removing the pendant edges, we get the graph with three vertices. We may dissolve only the vertex V' because the remaining two contact vertices have degree two.

This leads to a graph with three contact vertices, namely two degree two vertices and one bottleneck vertex V'' — the dissolution of this vertex would disconnect the graph. The inverse problem for the remaining graph cannot be solved by dismantling it, since the corresponding trees are not independent. Note that the original graph in this example is a slight modification of the graph presented in Fig. 2.

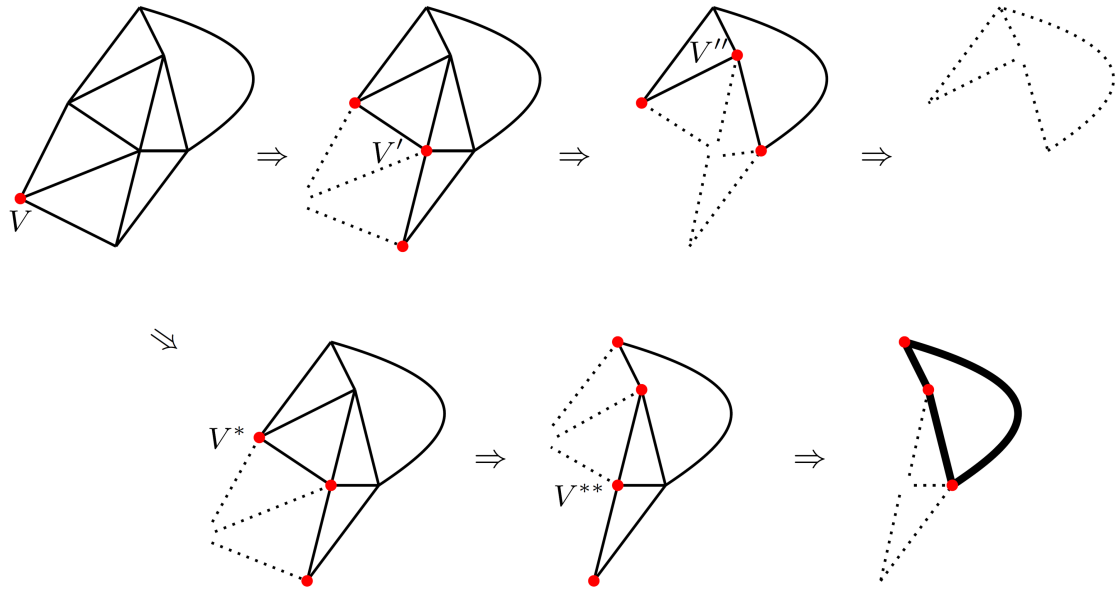

Fig. 2. Reconstruction of the whole graph using the MBC-method.

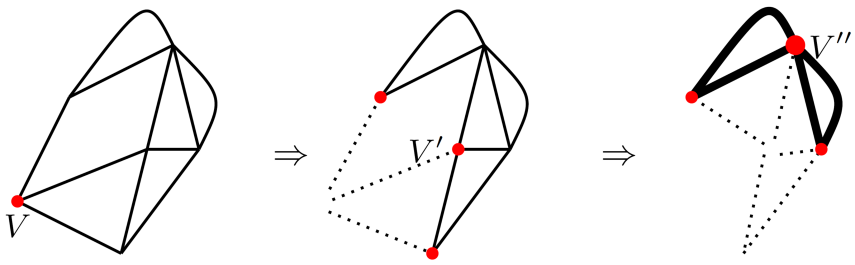

Fig. 3. Reconstruction terminated by the bottleneck.

It is not surprising that not all pendant-free graphs may be reconstructed starting from a single contact vertex — the described procedure may terminate immediately or after a few steps. As the last example shows, there are two reasons for the termination:

- degree two contact vertices,
- bottlenecks.

6. Conclusions

It is shown how magnetic boundary control can be applied to solve inverse problems for Schrödinger equations on metric graphs. It remains to characterise all metric graphs together with contact sets that guarantee solvability of the inverse problem. One may prove explicit theorems characterising the minimal contact sets.

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