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On the Density of Complex Eigenvalues of Wigner Reaction Matrix in a Disordered or Chaotic System with Absorption

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In an absorptive system, the Wigner reaction K-matrix (directly related to the impedance matrix in acoustic or electromagnetic wave scattering) is non-selfadjoint, hence its eigenvalues are complex. The most interesting regime arises when the absorption, taken into account as an imaginary part of the spectral parameter, is of the order of the mean level spacing. I will show how to derive the mean density of the complex K-matrix eigenvalues for the M-channel reflection problem in disordered or chaotic systems with broken time-reversal invariance. The computations have been done in the framework of the nonlinear σ-model approach, assuming fixed M and the dimension of the underlying Hamiltonian matrix $N \to \infty$. Some explicit formulas are provided for zero-dimensional quantum chaotic system as well as for a semi-infinite quasi-1D system with fully operative Anderson localization.

topics: quantum chaotic scattering, random matrix theory, non-Hermitian matrices.

1. Introduction

Consider the problem of wave scattering from a piece of random medium confined to a spatial domain $\mathcal{D}$ and described by a self-adjoint Hamiltonian H, e.g.,

$$H = \sum_{\boldsymbol{r}\in\boldsymbol{\Lambda}} V(\boldsymbol{r})\big|\boldsymbol{r}\rangle\langle\boldsymbol{r}\big| + \sum_{\boldsymbol{r}\sim\boldsymbol{r}'}\Big[t_{\boldsymbol{r}\boldsymbol{r}'}\big|\boldsymbol{r}\rangle\langle\boldsymbol{r}'\big| + t_{\boldsymbol{r}'\boldsymbol{r}}\big|\boldsymbol{r}'\rangle\langle\boldsymbol{r}\big|\Big], \tag{1}$$

where the second sum runs over nearest neighbours on a lattice $\boldsymbol{\Lambda}$ (assumed to be confined to the domain $\mathcal{D}$). The parameters $t_{\boldsymbol{r}'\boldsymbol{r}}$ are in general complex, satisfying $t^*_{\boldsymbol{r}\boldsymbol{r}'} = t_{\boldsymbol{r}'\boldsymbol{r}}$ to ensure the hermiticity of the Hamiltonian, $H = H^\dagger$, where we use t^* to denote complex conjugation of t and $H^\dagger$ for Hermitian conjugation of H. The disordered nature of the medium is taken into account by choosing the on-site potentials $V(\boldsymbol{r})$ and/or hopping parameters $t_{\boldsymbol{r}'\boldsymbol{r}}$ to be random variables. Such construction is known in the literature as the Anderson model and provides the paradigmatic framework to study single-particle localization phenomena. Note that form (1) can also be used for modelling a quantum particle motion on any graph $\boldsymbol{r} \in \mathfrak{G}$, with $t_{\boldsymbol{r}\boldsymbol{r}'}$ being the elements of the adjacency matrix of graph $\mathfrak{G}$.

The tight-binding representation is convenient as it allows one to think of such a Hamiltonian as described by a large $N \times N$ random matrix H, with N being the number of sites in the lattice or graph. Alternatively, one may think of its continuum analogue, $H = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r})$, $\boldsymbol{r} \in \mathcal{D}$, with the appropriate (e.g., Dirichlet) conditions at the boundary of $\mathcal{D}$. In fact, under appropriate conditions, the essentially random nature of wave scattering can be generated by an irregularly shaped boundary of the domain $\mathcal{D}$, without any intrinsic potential disorder. This is the standard case in the so-called wave billiards, the paradigmatic toy systems for studying the effects of quantum or wave chaos, see, e.g., [1–4]. In such a case, the famous Bohigas–Giannoni–Schmit conjecture [5] allows one to describe universal features of such systems efficiently by replacing the Hamiltonian H with a random $N \times N$ matrix from Gaussian ensembles: Gaussian orthogonal (GOE), Gaussian symplectic (GSE), or Gaussian unitary (GUE), depending on the presence or absence of time-reversal symmetry (and/or other relevant symmetries) in the system.

A very convenient framework for describing the scattering of classical or quantum waves from the disordered or chaotic medium has been formulated in [6] (see, e.g., [7] for more detail). Within such a framework, which is frequently called in the literature the "Heidelberg model", one constructs the unitary $M \times M$ energy-dependent scattering matrix $S(E)$ describing the scattering of waves incident on a random medium at some energy E and then exiting it via M open scattering channels, numbered by $c = 1, \ldots, M$ (see Fig. 1). Unitarity reflects the flux conservation, i.e., the vectors $\boldsymbol{a} = (a_1, \ldots, a_M)$ of incoming and $\boldsymbol{b} = (b_1, \ldots, b_M)$ of outgoing amplitudes are linearly related via $\boldsymbol{b} = S(E)\,\boldsymbol{a}$ and have the same norm.

The relation between $S(E)$ and the medium Hamiltonian H is then provided by the following expression

$$S(E)=\frac{\mathbb{1}-\mathrm{i}K(E)}{\mathbb{1}+\mathrm{i}K(E)},\quad K(E)=W^{\dagger}\frac{1}{E-H}W, \tag{2}$$

where columns W_c ($c=1,\ldots,M$) of an $N\times M$ matrix W of *coupling amplitudes* to M open scattering channels can be taken as fixed vectors satisfying the orthogonality condition

$$\sum_{i=1}^{N} W^{*}_{ci}W_{bi}=\gamma_c\delta_{cb}, \tag{3}$$

with $\gamma_c>0$ $\forall c=1,\ldots,M$ determining the "bare" strength of coupling of a given channel to the scattering system. The resulting $M\times M$ Hermitian matrix $K(E)$ is known in the literature as the Wigner reaction K-matrix. It is the hermiticity of K which implies S-matrix unitarity and, hence, implies the flux conservation. Note that the Wigner K-matrix is experimentally measurable in microwave scattering systems, as it is directly related to the systems impedance matrix, see, e.g., [8–10].

One of the serious challenges related to the theoretical description of scattering characteristics, however, is related to the fact that experimentally measured quantities suffer from inevitable energy losses (absorption), e.g., due to damping in resonator walls and other imperfections. Such losses violate the unitarity of the scattering matrix and are important for interpretation of experiments, hence considerable efforts were directed towards incorporating them into the Heidelberg approach [11]. At the level of the model (1), the losses can be taken into account by allowing the spectral parameter E to have a finite imaginary part by replacing $E\to(E+\mathrm{i}\alpha)\in\mathbb{C}$ with some $\alpha>0$. This replacement violates the hermiticity of the Wigner matrix $K(E+\mathrm{i}\alpha)$; in particular, entries of K now become complex even for real symmetric choice of H and real W. The most interesting, difficult, and experimentally relevant regime occurs when absorption parameter α is comparable with the mean separation $\Delta(E)$ between neighbouring eigenvalues of the wave-chaotic Hamiltonian H. For example, if one uses the Gaussian random matrix model for H, normalized to have the mean eigenvalue density given by Wigner semicircle $\nu(E)=\frac{1}{(2\pi)}\sqrt{4-E^2}$ in a finite interval $|E|<2$, one has $\Delta(E)=(\nu(E)N)^{-1}$ as $N\to\infty$. The statistics of the real and imaginary parts of K-matrix entries in such a regime have been the subject of a considerable number of theoretical papers [12–16] and are by now well-understood and measured experimentally with good precision for systems with preserved time-reversal invariance in microwave cavities [8–10, 17] and microwave simulations of quantum graphs [18–21]. More recently, experimental results for K-matrices in systems with broken time-reversal invariance [22, 23] and eventually symplectic symmetry [24] have been also reported.

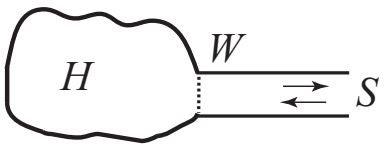

Fig. 1. A sketch of a chaotic wave scattering from a region schematically represented by a cavity and assumed to contain a random medium inside. An operator governing wave dynamics in such a cavity decoupled from the channels is assumed to be effectively described by a large random matrix H. An infinite lead is assumed to support M propagating channels in the considered energy range and is coupled to the cavity region by a matrix/operator W. The ensuing $M\times M$ unitary scattering matrix S can be related to H and W in the framework of the Heidelberg approach and is given by (1).

In the present paper, we will be interested in yet another characteristic of the non-Hermitian Wigner matrix $K(E+\mathrm{i}\alpha)$, the mean density of its complex eigenvalues $K_c=\mathrm{Re}(K_c)-\mathrm{i}\,\mathrm{Im}(K_c)$, $\forall c=1,\ldots,M$, defined as

$$\rho_M(u,v;y)=\left\langle\sum_{c=1}^{M}\delta\big(u-\mathrm{Re}(K_c)\big)\,\delta\big(v-\mathrm{Im}(K_c)\big)\right\rangle, \tag{4}$$

where we suppressed the energy dependence for simplicity, indicating instead explicit dependence on the appropriately scaled absorption parameter $y=\frac{2\pi\alpha}{\Delta}$. Here and henceforth, the angular brackets $\langle\ldots\rangle$ indicate the averaging over ensemble of random Hamiltonians H. Note that selecting the coupling vectors W_c coinciding with the first M basis vectors in N-dimensional space, i.e., $W_1=\boldsymbol{e}_1=(1,0,\ldots,0)$, $W_2=\boldsymbol{e}_2=(0,1,0,\ldots,0)$, etc., converts the K-matrix to $M\times M$ top left corner of the $N\times N$ resolvent matrix $(E+\mathrm{i}\alpha-H)^{-1}$. Physically, this corresponds to M perfectly coupled channels attached to the first M sites. From that angle, we aim to characterize the eigenvalue density for the corner resolvent minor at complex values of the spectral parameter, which is an interesting and potentially rich mathematical problem. We are not aware of any systematic studies in that direction.

Note that in a fully chaotic, zero-dimensional system, the positions of channel attachment do not play any role due to inherent ergodicity. In a more general non-ergodic situation, which may arise due to the presence of Anderson localization phenomena, one may think of such an arrangement as corresponding to a wave reflection problem. In such a setting, the density (4) has appeared recently in paper [25] as an important quantity facilitating the computation of the mean density of S-matrix poles, also known as *resonances*, in the complex energy plane. The latter density is experimentally measurable in wave-chaotic system [26, 27] and is a subject of long-standing theoretical interest, see, e.g., [28–34]. Clearly, the density (4) is also experimentally measurable in principle, provided that accurate experimental data can be sampled for the whole K-matrix. The paper [25] included, without

a proper derivation, an explicit expression for such density, valid for a general class of disordered systems with broken time-reversal invariance, namely for those that can be mapped on the so-called supersymmetric nonlinear σ-model (see [35] and discussions in [25] for more information). The present paper aims to fill that gap by providing a detailed derivation, which involves several steps that are only relatively briefly described in the available literature.

To begin with, for the special simplest case $M = 1$, the K-matrix consists of a single element, and finding the density (4) is equivalent to computing the joint probability density of the real and complex parts of such an element. Such density has been originally addressed in [36] via quite tedious calculations in the σ-model approximation. A much more efficient approach has been proposed later in [14] (see also an account in [11]). Our goal in this paper is to show how to generalize that approach to any number of open channels M, on an example of systems with broken time-reversal invariance. Along these lines, we also try to elucidate some features of the method which were omitted in the exposition in [11, 14].

2. Derivation of the main results

2.1. General exposition of the method

Given two real parameters $p \in \mathbb{R}$ and $q > 0$, we start with defining the following object

$$C_\alpha(p,q) := \Big\langle \mathrm{Tr}\left(z - K\big(E+\mathrm{i}\alpha\big)\right)^{-1} \mathrm{Tr}\left(\overline{z} - K\big(E-\mathrm{i}\alpha\big)\right)^{-1} \Big\rangle, \tag{5}$$

where we denoted $z = p + \mathrm{i}q$, $\overline{z} = p - \mathrm{i}q$ and assumed the real energy E and the absorption parameter $\alpha > 0$ to be fixed. As eigenvalues of the matrices $K(E+\mathrm{i}\alpha)$ and $K(E-\mathrm{i}\alpha)$ are complex conjugates of each other, one can write each trace in terms of $K_c = \mathrm{Re}(K_c) - \mathrm{i}\mathrm{Im}(K_c) := u_c - \mathrm{i}v_c$, with $v_c > 0$, representing (5) as a sum of diagonal and off-diagonal contributions

$$C_\alpha(p,q) = C_\alpha^{(\mathrm{diag})}(p,q) + C_\alpha^{(\mathrm{off})}(p,q), \tag{6}$$

where

$$C_\alpha^{(\mathrm{diag})}(p,q) := \Big\langle \sum_{c=1}^{M} \frac{1}{|z-K_c|^2} \Big\rangle = \Big\langle \sum_{c=1}^{M} \frac{1}{(p-u_c)^2+(q+v_c)^2} \Big\rangle, \tag{7}$$

whereas $C_\alpha^{(\mathrm{off})}(p,q)$ is given by

$$C_\alpha^{(\mathrm{off})}(p,q) := \Big\langle \sum_{c\neq c'}^{M} \frac{1}{(z-K_c)(\overline{z}-\overline{K_{c'}})} \Big\rangle = \Big\langle \sum_{c\neq c'}^{M} \frac{1}{u_c-u_{c'}-\mathrm{i}\,(2q+v_c+v_{c'})} \times \Big[\frac{1}{p-u_c+\mathrm{i}\,(q+v_c)} - \frac{1}{p-u_{c'}-\mathrm{i}\,(q+v_{c'})}\Big] \Big\rangle. \tag{8}$$

In the next step, let us introduce the Fourier transform in variable p,

$$\tilde{C}_\alpha(k,q) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p\ \mathrm{e}^{\mathrm{i}pk}\, C_\alpha(p,q). \tag{9}$$

Taking into account $q > 0$, $v_c \geq 0\ \forall c$, we get

$$\tilde{C}_\alpha^{(\mathrm{diag})}(k,q) = \Big\langle \frac{1}{2}\sum_{c=1}^{M} \frac{\mathrm{e}^{\mathrm{i}ku_c - |k|(q+v_c)}}{q+v_c} \Big\rangle, \tag{10}$$

whereas the Fourier-transformed off-diagonal part $\tilde{C}_\alpha^{(\mathrm{off})}(k,q)$ now reads

$$\Big\langle \sum\nolimits_{c\neq c'}^{M} (-\mathrm{i}) \Big[u_c - u_{c'} - \mathrm{i}(2q+v_c+v_{c'})\Big]^{-1} \times \Big[\theta(-k)\,\mathrm{e}^{\mathrm{i}ku_c + k(q+v_c)} + \theta(k)\,\mathrm{e}^{\mathrm{i}ku_{c'} - k(q+v_{c'})}\Big] \Big\rangle, \tag{11}$$

where $\theta(k) = 1$ for $k \geq 0$ and zero otherwise.

The next step is to continue analytically in the parameter q from positive real values to the whole complex plane slit along the negative real line $q = -v$, $v > 0$, and evaluate the jump across the slit, defined as

$$\delta\tilde{C}_\alpha(k, v{>}0) := \lim_{\epsilon\to 0}\Big(\tilde{C}_\alpha(k, -v - \mathrm{i}\epsilon) - \tilde{C}_\alpha(k, -v + \mathrm{i}\epsilon)\Big). \tag{12}$$

For the diagonal part one finds after straightforward algebra

$$\delta\tilde{C}_\alpha^{(\mathrm{diag})}(k, v>0) = \mathrm{i}\lim_{\epsilon\to 0}\Big\langle \sum_{c=1}^{M} \mathrm{e}^{\mathrm{i}ku_c + |k|(v-v_c)} \left(\frac{\epsilon\cos(\epsilon|k|)}{\epsilon^2+(v-v_c)^2} - \frac{\sin(\epsilon|k|)(v-v_c)}{\epsilon^2+(v-v_c)^2}\right) \Big\rangle, \tag{13}$$

which upon using

$$\lim_{\epsilon\to 0}\left[\frac{\epsilon\cos(\epsilon|k|)}{\epsilon^2+(v-v_c)^2} - \frac{\sin(\epsilon|k|)(v-v_c)}{\epsilon^2+(v-v_c)^2}\right] = \pi\,\delta(v-v_c) \tag{14}$$

reduces the diagonal contribution to

$$\delta\tilde{C}_\alpha^{(\mathrm{diag})}(k, v{>}0) = \mathrm{i}\pi\Big\langle \sum_{c=1}^{M} \mathrm{e}^{\mathrm{i}ku_c}\,\delta(v-v_c) \Big\rangle. \tag{15}$$

At the same time, straightforward computations show that assuming that the eigenvalues of the K-matrix are all distinct, i.e., $u_c - \mathrm{i}v_c \neq u_c' - \mathrm{i}v_{c'}$ for $c \neq c'$, the off-diagonal part does not generate any non-vanishing jump across the slit at $q = -v$, $v{>}0$, that is $\delta\tilde{C}_\alpha^{(\mathrm{off})}(k, v{>}0) = 0$. Finally, applying in (15) the inverse Fourier transform in the variable k and comparing with the definition (4) provides the expression for the density of complex eigenvalues of the K-matrix in the form

$$\rho_M(u,v;y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\mathrm{i}\pi^2}\ \mathrm{e}^{-\mathrm{i}ku}\,\delta\tilde{C}_{\alpha=\frac{y\Delta}{2\pi}}(k, v{>}0). \tag{16}$$

In this way, the problem of computing the density $\rho_M(u,v;y)$ is reduced to the ability to evaluate explicitly the correlation function $C_\alpha(p, q > 0)$ in (5) and perform the required Fourier transforms

and jump evaluation. Below, we show how this program is executed for those disordered or chaotic systems with broken time-reversal invariance, which can be mapped onto the corresponding nonlinear σ-model.

2.2. Computations for systems with broken time-reversal invariance

Referring the interested reader to [25] and references therein for a detailed discussion of physical assumptions behind such mapping, we only mention here that it provides the most powerful and systematic approaches to addressing universal single-particle features of wave propagation in a disordered medium, including Anderson localization phenomena. Developed in the seminal works by Efetov [35] building on earlier ideas of Wegner [37], the model is defined by specifying a weight function $\mathrm{e}^{-\mathcal{S}[Q]}$, with the action $\mathcal{S}[Q]$ describing interaction between supermatrices $Q(\boldsymbol{r})$ (i.e., matrices with Grassmann/anticommuting/fermionic and ordinary/commuting/bosonic entries) associated to every site $\boldsymbol{r} \in \tilde{\boldsymbol{\Lambda}}$ located on an auxiliary lattice $\tilde{\boldsymbol{\Lambda}}$. The size of the supermatrices involved depends on the underlying symmetries of the Hamiltonian H and in the simplest case of the Hamiltonians with fully broken time-reversal symmetry, denoted in the standard nomenclature as class A with Dyson parameter $\beta = 2$, the supermatrices are of the size 4×4. Physically, such a model provides, in a certain sense, a coarse-grained description of the original microscopic Anderson model or its continuous equivalent, with non-universal features on scales smaller than the mean free path l being effectively integrated out. In such a picture, every (super)matrix $Q(\boldsymbol{r})$ associated with a single lattice site in $\tilde{\boldsymbol{\Lambda}}$ "lumps together" behaviour of the microscopic model on scales of the order of the mean free path l. From this point of view, the billiards in the quantum chaotic regime, where essentially l is of the same order as the system length L, are effectively characterized by nonlinear σ-models with a single matrix Q without any spatial dependence. Such a limit is traditionally called "zero-dimensional". At the same time, all effects of the Anderson localization require considering extended lattices of interacting Q-matrices.

One of the central objects of such theory turns out to be the so-called "order parameter function" (OPF) $F_{\boldsymbol{r}}(Q)$, which is formally defined [38] by integrating the weight $\mathrm{e}^{-\mathcal{S}[Q]}$ over all but one supermatrix $Q(\boldsymbol{r})$. Due to global symmetries of the action, the OPF can be shown to actually depend on only a few real Cartan variables parametrizing Q matrices. In particular, for systems with broken time-reversal symmetry, one has $F_{\boldsymbol{r}}(Q) := \mathcal{F}(\lambda, \lambda_1)$, with $\lambda \in [-1, 1]$ and $\lambda_1 \in [1, \infty]$ being the compact and non-compact coordinates, respectively (we omitted spatial dependence on $\boldsymbol{r}$ for brevity). Note that the OPF characterizes the *closed* system, which (in the absence of absorption) conserves the number of particles, whereas allowing particles/waves at a given energy to be sent via the lead to the random medium and then collecting the reflected waves renders the medium *open*. However, if one makes an assumption of "locality" of the lead, whose transverse extent is assumed to be much smaller than the mean free path l in the disordered medium, then the coupling with it is effectively point-wise at the level of σ-model description. Still, even such point-wise lead may support arbitrarily many propagation channels M, though we will always assume that M remains negligible to the number of sites in the underlying microscopic lattice $\boldsymbol{\Lambda}$.

The power of nonlinear σ-model description in our case lies in our ability to provide an explicit representation for the correlation function $C_\alpha(p, q)$ defined in (5) in terms of the OPF $\mathcal{F}(\lambda, \lambda_1)$ at the point of lead attachment. For systems with broken time-reversal invariance, such computation has already been performed in [7], albeit formally only in the "zero-dimensional" limit, with OPF taking an especially simple form $\mathcal{F}(\lambda, \lambda_1) = \mathrm{e}^{-y(\lambda_1 - \lambda)}$, where, as before, $y = 2\pi\alpha/\Delta$ is the effective absorption parameter. It is, however, straightforward to adapt the calculation for arbitrary nonlinear sigma-model (see Appendix B of [11]), the result being given by the sum of two contributions, the disconnected one

$$C_\alpha^{(\mathrm{disc})}(p, q) = \sum_{c=1}^{M} \frac{1}{p - \gamma_c \frac{E}{2} - \mathrm{i}\left(q + \pi\,\nu(E)\gamma_c\right)} \sum_{b=1}^{M} \frac{1}{p - \gamma_c \frac{E}{2} + \mathrm{i}\left(q + \pi\,\nu(E)\gamma_b\right)} \tag{17}$$

and the connected one

$$C_\alpha^{(\mathrm{con})}(p, q) = \int_{-1}^{1} \mathrm{d}\lambda \int_{1}^{\infty} \mathrm{d}\lambda_1 \, \frac{\mathcal{F}(\lambda, \lambda_1)}{(\lambda_1 - \lambda)^2} \, \mathcal{R}_M(p, q|\lambda, \lambda_1), \tag{18}$$

where the last factor in (18) is given by

$$\mathcal{R}_M(p, q|\lambda, \lambda_1) := \mathcal{L}_{p,q} \prod_{c=1}^{M} \frac{\left(p - \gamma_c \frac{E}{2}\right)^2 + q^2 + 2\pi\,\nu(E)\gamma_c\lambda + \left(\pi\,\nu(E)\,\gamma_c\right)^2}{\left(p - \gamma_c \frac{E}{2}\right)^2 + q^2 + 2\pi\,\nu(E)\,\gamma_c\lambda_1 + \left(\pi\,\nu(E)\gamma_c\right)^2} \tag{19}$$

with the coupling coefficients γ_c defined in (3) and the differential operator $\mathcal{L}_{p,q} := \frac{1}{4}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right)$.

These expressions provide the basis for implementing the analytic continuation procedure described above. For simplicity, we consider below explicitly only the case $E = 0$, so that $\pi\nu(E) = 1$, and largely concentrate on the simplest, yet important, case of equivalent channels, namely $\gamma_c = \gamma$, $\forall c = 1, \ldots, M$[†1]. The analytic continuation procedure for the disconnected part amounts to a straightforward repetition of our derivation of (16) and yields $\rho^{(\mathrm{disc})}(u,v) = \sum_{c=1}^{M} \delta(u)\,\delta(v-\gamma_c)$. The connected contribution to the density is much less trivial, and we consider it below.

One starts with rewriting (19) in the form

$$\mathcal{R}_M(p,q|\lambda,\lambda_1) = \mathcal{L}_{p,q}\left(1-\tfrac{2q\gamma(\lambda_1-\lambda)}{p^2+q^2+2\gamma_c\lambda_1+\gamma_c^2}\right)^M, \tag{20}$$

which after expanding the binomial reduces to

$$\mathcal{R}_M(p,q|\lambda,\lambda_1) = -\sum_{l=1}^{M}\left(\begin{array}{c} M \\ l \end{array}\right)\tfrac{(\lambda_1-\lambda)^l}{(l-1)!} \times \tfrac{\partial^{l-1}}{\partial\lambda_1^{l-1}}\mathcal{L}_{p,q}\,\tfrac{2q\gamma}{p^2+q^2+2\gamma_c\lambda_1+\gamma_c^2}. \tag{21}$$

The latter form makes it an easy task to perform the Fourier transform in the variable p assuming $q > 0$, which essentially amounts to making in (21) the replacement

$$\mathcal{L}_{p,q}\,\tfrac{2q\gamma}{p^2+q^2+2\gamma_c\lambda_1+\gamma_c^2} \longrightarrow \phi(k,q),$$

$$\phi(k,q) = \tfrac{\pi\gamma}{2}\left(\tfrac{\partial^2}{\partial q^2}-k^2\right) q\,\tfrac{\exp\left[-|k|\sqrt{q^2+2\gamma\lambda_1 q+\gamma^2}\right]}{\sqrt{q^2+2\gamma\lambda_1 q+\gamma^2}}. \tag{22}$$

Following the procedures described in (12), we now continue analytically in the parameter q from positive real values to the whole complex plane slit along the negative real line $q = -v$, $v > 0$ and we evaluate the associated jump across the slit

$$\delta\phi(k,v{>}0) := \lim_{\epsilon\to 0}\Big(\phi(k,-v-\mathrm{i}\epsilon)-\phi(k,-v+\mathrm{i}\epsilon)\Big), \tag{23}$$

which is easily found to be equal to

$$\delta\phi(k,v{>}0)=\pi\gamma\big(\tfrac{\partial^2}{\partial v^2}-k^2\big)\,\tfrac{v\,\cos\left[k\sqrt{2\gamma\lambda_1 v-v^2-\gamma^2}\right]}{\sqrt{2\gamma\lambda_1 v-v^2-\gamma^2}} \times\theta\big(2\gamma\lambda_1 v-v^2-\gamma^2\big). \tag{24}$$

Straightforward inversion of the Fourier transform in the variable k converts the above into

$$\delta\phi(u,v>0) = \frac{\gamma}{2}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) v\,\frac{\delta\left(u-\sqrt{2\gamma\lambda_1 v-v^2-\gamma^2}\right)+\delta\left(u+\sqrt{2\gamma\lambda_1 v-v^2-\gamma^2}\right)}{\sqrt{2\gamma\lambda_1 v-v^2-\gamma^2}}\;\theta(2\gamma\lambda_1 v-v^2-\gamma^2) = \frac{1}{2}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\delta\left(\lambda_1-x_\gamma\right), \tag{25}$$

where $x_\gamma := \frac{u^2+v^2+\gamma^2}{2\gamma\, v}$. Next we trade the derivatives over λ_1 for those over x_γ by the identity

$$\frac{\partial^{l-1}}{\partial\lambda_1^{l-1}}\delta\left(\lambda_1-x_\gamma\right) = (-1)^{l-1}\,\frac{\partial^{l-1}}{\partial x_\gamma^{l-1}}\delta\left(\lambda_1-x_\gamma\right) \tag{26}$$

and in this way arrive at replacing (21) with

$$\tilde{\mathcal{R}}_M(u,v|\lambda,\lambda_1) = -\tfrac{1}{2}\left(\tfrac{\partial^2}{\partial u^2}+\tfrac{\partial^2}{\partial v^2}\right) \times\sum_{l=1}^{M}\left(\begin{array}{c} M \\ l \end{array}\right)\tfrac{(-1)^{l-1}(\lambda_1-\lambda)^l}{(l-1)!}\,\tfrac{\partial^{l-1}}{\partial x_\gamma^{l-1}}\,\delta\left(\lambda_1-x_\gamma\right). \tag{27}$$

With this, (16) and (18) imply the density of K-matrix eigenvalues via

$$\rho^{(\mathrm{con})}(u,v)=\int_{-1}^{1}\frac{\mathrm{d}\lambda}{2\pi}\int_{1}^{\infty}\mathrm{d}\lambda_1\,\frac{\mathcal{F}(\lambda,\lambda_1)}{(\lambda_1-\lambda)^2}\,\tilde{\mathcal{R}}_M(u,v|\lambda,\lambda_1), \tag{28}$$

which upon substituting (27) into it and changing the order of integrations yields

$$\rho^{(\mathrm{con})}(u,v)=\frac{1}{4\pi}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\int_{-1}^{1}\mathrm{d}\lambda\,G_M(\lambda|x_\gamma). \tag{29}$$

Here we denoted

$$G_M(\lambda|x_\gamma) := \sum_{l=1}^{M}\left(\begin{array}{c} M \\ l \end{array}\right)\tfrac{(-1)^{l-1}}{(l-1)!} \times\tfrac{\partial^{l-1}}{\partial x_\gamma^{l-1}}\Big[(x_\gamma-\lambda)^l T(\lambda|x_\gamma)\Big], \tag{30}$$

with

$$T(\lambda|x_\gamma) = \frac{\mathcal{F}(\lambda,x_\gamma)}{(x_\gamma-\lambda)^2}. \tag{31}$$

Applying the Leibnitz formula

$$\tfrac{\partial^{l-1}}{\partial x_\gamma^{l-1}}\Big[(x_\gamma-\lambda)^l\,T(\lambda|x_\gamma)\Big] = \sum_{k=0}^{l-1}\left(\begin{array}{c} l-1 \\ k \end{array}\right)\tfrac{l!}{(k+1)!}(x_\gamma-\lambda)^{k+1}\,\tfrac{\partial^k}{\partial x_\gamma^k}\,T(\lambda|x_\gamma) \tag{32}$$

and substituting it back to (30) one may change the order of summation as

$$G_M(\lambda|x_\gamma) = \sum\nolimits_{l=1}^{M} A_l \sum\nolimits_{k=0}^{l-1} B_{k,l}V_k = \sum\nolimits_{-}k=0^{M-1}V_k\sum\nolimits_{l=k+1}^{M}A_l B_{k,l} \tag{33}$$

with $V_k := (x_\gamma-\lambda)^{k+1}\,\frac{\partial^k}{\partial x_\gamma^k}T(\lambda|x_\gamma)$ and

$$A_l := \left(\begin{array}{c} M \\ l \end{array}\right)\frac{(-1)^{l-1}}{(l-1)!},\quad B_{k,l} := \left(\begin{array}{c} l-1 \\ k \end{array}\right)\frac{l!}{(k+1)!}. \tag{34}$$

[†1] However, see expression (42) for two non-equivalent channels.

This gives

$$\sum_{l=k+1}^{M} A_l B_{k,l} = \frac{M!}{k!(k+1)!}\sum_{l=k+1}^{M}\frac{(-1)^{l-1}}{(M-l)!}\frac{1}{(l-k-1)!} = \frac{(-1)^k\, M!}{(M-k-1)!k!(k+1)!}\sum_{n=0}^{M-k-1}(-1)^n\binom{M-k-1}{n} = \frac{(-1)^{M-1}}{(M-1)!}\,\delta_{k,M-1} \tag{35}$$

using the Kronecker symbol $\delta_{k,k'}$, since the sum over n is vanishing for all $0 \le k < M-1$, and is equal to unity at $k = M-1$.

As a result, we get the final expression for the connected part of the mean density of K-matrix eigenvalues in the form

$$\rho_M^{(\mathrm{con})}(u,v) = \frac{1}{4\pi}\frac{(-1)^{M-1}}{(M-1)!}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \times \int_{-1}^{1} \mathrm{d}\lambda\, (x_\gamma-\lambda)^M \frac{\partial^{M-1}}{\partial x_\gamma^{M-1}}\frac{\mathcal{F}(\lambda,x_\gamma)}{(x_\gamma-\lambda)^2}. \tag{36}$$

A few remarks are here in order to help properly interpret and appreciate the content of (36).

Remark 1. Recalling from (25) that

$$x_\gamma = \frac{u^2+v^2+\gamma^2}{2\gamma\, v} \equiv \frac{u^2}{2\gamma\, v} + \frac{1}{2}\left(\frac{v}{\gamma}+\frac{\gamma}{v}\right) \ge 1, \tag{37}$$

one may straightforwardly check that for any smooth enough function $\Phi(x)$ holds

$$\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\Phi(x_\gamma) = \frac{1}{v^2}\frac{\partial}{\partial x_\gamma}(x_\gamma^2-1)\frac{\partial}{\partial x_\gamma}\Phi(x_\gamma) \tag{38}$$

for $x_\gamma > 1$. This exact form was used to represent the density in [25].

There is, however, a subtlety in (36) related to its content at $x_\gamma \to 1$. In our derivation, we tacitly assumed $x_\gamma > 1$. However, a more careful analysis shows that the integral in (36) should be pre-multiplied with the step-function factor $\theta(x_\gamma - 1)$ arising as the result of performing integration over $\lambda_1 \in [1,\infty)$ with the factor $\delta(\lambda_1 - x_\gamma)$. The presence of such a seemingly innocent θ-factor has, however, important consequences — when acted upon with the differential operator on the right-hand side of (38), it generates the δ-function factors exactly cancelling the contribution from the disconnected part, $\rho^{(\mathrm{disc})}(u,v) = \sum_{c=1}^{M}\delta(u)\,\delta(v-\gamma_c)$. As a result, the formula (36) as it is written (i.e., without θ-factor) in fact gives the full, properly normalized, eigenvalue density for the K-matrix in absorptive systems. A similar mechanism of cancellation of δ-terms has been first noticed in [39], and we explain in Appendix A how it works in our case using the simplest case of $M = 1$ as an example.

Remark 2. With hindsight, one may notice that one could have arrived at the same expression (36) by a much simpler procedure. Namely, by defining

$$\tilde{x} := \frac{p^2+q^2+\gamma^2}{2\gamma\, q}, \tag{39}$$

Fig. 2. A sketch of the "quasi-1D" model. The left part in grey represents an infinite-length ideal lead supporting M propagating modes. The disordered part is of a finite length L and contains finite concentration of random impurities inside.

rewriting (20) in the form

$$\mathcal{R}_M(p,q|\lambda,\lambda_1) = \frac{1}{4q^2}\frac{\partial}{\partial\tilde{x}}(\tilde{x}^2-1)\frac{\partial}{\partial\tilde{x}}\left(\frac{\tilde{x}+\lambda}{\tilde{x}+\lambda_1}\right)^M. \tag{40}$$

Then one must simply replace $u \to p$, $q \to -v - \mathrm{i}0$, implying $\tilde{x} \to -x_\gamma + \mathrm{i}0$, and calculate the associated jump across the cut using

$$\mathrm{Im}\left[\left(\frac{-x_\gamma+\mathrm{i}0+\lambda}{-x_\gamma+\mathrm{i}0+\lambda_1}\right)^M\right] = (x_\gamma-\lambda)^M\,\frac{(-1)^{M-1}}{(M-1)!} \times\frac{\partial^{M-1}}{\partial x_\gamma^{M-1}}\,\mathrm{Im}\left[\frac{1}{x_\gamma-\lambda_1-\mathrm{i}0}\right] = \pi(x_\gamma-\lambda)^M\,\frac{(-1)^{M-1}}{(M-1)!}\,\frac{\partial^{M-1}}{\partial x_\gamma^{M-1}}\,\delta(x_\gamma-\lambda_1). \tag{41}$$

Such a recipe is exactly the same as the one employed for $M = 1$ in [14], though without a proper explanation provided there or in the review [11].

Armed with such a recipe, one can easily apply it to the case of non-equivalent channels. General formulas in that case look quite complicated, but in the simplest case of two non-equivalent channels with coupling constants $\gamma_1 \ne \gamma_2$, one gets a relatively compact expression

$$\rho_{\gamma_1,\gamma_2}(u,v) = \frac{1}{4\pi}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \times\int_{-1}^{1}\mathrm{d}\lambda\left\{\left[\frac{\mathcal{F}(\lambda,x_1)}{x_2-\lambda_2}+\frac{\mathcal{F}(\lambda,x_2)}{x_1-\lambda_1}\right]-\frac{\left[\mathcal{F}(\lambda,x_1)-\mathcal{F}(\lambda,x_2)\right]}{x_1-x_2}\right\}, \tag{42}$$

where we defined

$$x_1 = \frac{u^2+v^2+\gamma_1^2}{2\gamma_1\, v}, \qquad x_2 = \frac{u^2+v^2+\gamma_2^2}{2\gamma_2\, v}. \tag{43}$$

Remark 3. It is clear that performing further analysis of (36) hinges on our ability to have a good understanding of the OPF $\mathcal{F}(\lambda,x)$ for the closed counterpart of the scattering system, which in general also depends on the (appropriately normalized) absorption parameter α. Such knowledge is currently available mainly in two cases: (i) the "zero-dimensional" limit, with OPF taking an especially simple form $\mathcal{F}^{(0\mathrm{d})}(\lambda,x) = \mathrm{e}^{-y(x-\lambda)}$, where, as before, $y = 2\pi\alpha/\Delta$, and (ii) in a (semi) infinite quasi-one-dimensional wire (see Fig. 2) of length $L \to \infty$, with one edge closed for the waves and second edge attached to an infinite waveguide with M propagating channels.

Such a wire is characterized by a classical microscopic diffusion constant D related to the localization length ξ of the quantum wave problem as

$\xi = 2\pi\nu D$, with ν being, as before, the mean eigenvalue density at a given energy. Note that mathematically such wires can be modelled by a large banded random matrix [40, 41]. In such a system, the OPF at points close to its edges has been originally found in [42] and takes the following form in terms of the modified Bessel functions $I_p(z), K_p(z)$,

$$\mathcal{F}^{(1D)}(\lambda, x) = b\, K_0(a)\, I_1(b) + a\, K_1(a)\, I_0(b), \tag{44}$$

with

$$a = \kappa\sqrt{(x+1)/2}, \qquad b = \kappa\sqrt{(\lambda+1)/2}, \tag{45}$$

where the parameter κ is related to the absorption α as

$$\kappa = \sqrt{8\alpha/\Delta_\xi}, \tag{46}$$

with an important energy scale $\Delta_\xi = (4\pi^2 D\nu^2)^{-1} = D/\xi^2$ giving the mean level spacing in the quasi-one-dimensional wires whose length L is equal to the localization length ξ.

In the "zero-dimensional" limit, due to a simple form of the order parameter function, one can relatively straightforwardly perform the required integrations and differentiations in (36) and get the explicit formulas, which we present below for the simplest cases $M = 1$ and $M = 2$ of equivalent channels

$$\rho_{0D,M=1}(u,v) = \frac{1}{2\pi v^2}\, \mathrm{e}^{-x_\gamma} \Big[y\cosh(y) - \sinh(y)(1 - y x_\gamma) \Big] \tag{47}$$

and

$$\rho_{0D,M=2}(u,v) = \frac{1}{2\pi v^2}\, \mathrm{e}^{-x_\gamma}\, \sinh(y) \big[y(x_\gamma^2 - 1) - 2x_\gamma \big] + \frac{1}{\pi v^2}\, \mathrm{e}^{-x_\gamma} \Big[y\cosh(y) - \sinh(y)\big(1 - y x_\gamma\big) \Big], \tag{48}$$

with the same definition of x_γ (see (37)). The formula equivalent to (47) appeared already in the literature (see Eq. (5) in [13]), but the two-channel case seems to be new. As to the quasi-1D system of infinite length, it turns out that again the results can be found explicitly in the general case. Below we present it only for the simplest case of a single attached channel, when the density acquires quite an elegant form after manipulations outlined in Appendix B of this paper

$$\rho_{1D,M=1}(u,v) = \tfrac{1}{2\pi v^2} P_0(x_\gamma),$$

$$P_0(x) = \tfrac{\kappa^2}{4} \left[I_2(\kappa)\, K_0\left(\kappa\sqrt{\tfrac{x+1}{2}}\right) + I_1(\kappa)\sqrt{\tfrac{x+1}{2}} \times K_1\left(\kappa\sqrt{\tfrac{x+1}{2}}\right) \right]. \tag{49}$$

As is shown in [13], for $M = 1$ and $\gamma = 1$ the variable $r = (x-1)/(x+1)$ is nothing else but the modulus of the reflection coefficient, which in the absorptive system is smaller than one. Correspondingly, the function $P_0(x)$ in (49) provides the distribution for x, hence for r, in a single-channel quasi-1D system with absorption. This complements a result for the same geometry in the case of no absorption inside the sample, but for the second edge of the sample being in contact with perfectly absorbing lead, see Eqs. (12)–(13) in [12]. Note also that it is not difficult to further integrate the variable u, getting an explicit formula for the distribution of variable v, known as the local density of states, corresponding to locations close to the edge of the sample. The latter is an important characteristic of disordered single-particle systems, see [36, 43, 44].

3. Conclusions

In conclusion, we derived the mean density of complex eigenvalues for random Wigner reaction K-matrices for absorptive disordered or chaotic systems with broken time-reversal invariance, in the σ-model approximation. Extension of these results to systems with preserved time-reversal invariance (and then eventually symplectic symmetry) is certainly possible along similar lines, generalizing $M = 1$ results presented in [11]. These subjects are left for future publications.

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Appendix A: Cancellation of the disconnected part

Our starting point is the formula (36) with included θ-factor, specified for simplicity and transparency to the case of a single channel $M = 1$ and $\gamma = 1$, so that $x_{\gamma=1} = x$. We write it in the form

$$\rho^{(\mathrm{con})}_{M=1}(u,v) = \frac{1}{4\pi v^2}\, \mathcal{L}_x \left[\theta(x-1)\, \Phi(x) \right], \qquad \Phi(x) = \int_{-1}^{1} \mathrm{d}\lambda\, \frac{\mathcal{F}(\lambda, x)}{(x-\lambda)}, \tag{50}$$

where we introduce the differential operator

$$\mathcal{L}_x := \frac{\partial}{\partial x}(x^2 - 1)\, \frac{\partial}{\partial x}. \tag{51}$$

Straightforward differentiation then gives

$$\mathcal{L}_x \big[\theta(x-1)\, \Phi(x) \big] = \theta(x-1)\, \mathcal{L}_x\, \Phi(x) + \delta(x-1) \Big[2x\, \Phi(x) + 2(x^2-1)\, \Phi'(x) \Big] + \delta'(x-1) \Big[(x^2-1)\, \Phi(x) \Big]. \tag{52}$$

Further using the integration by parts identity

$$\delta'(x-1) \big[(x^2-1)\, \Phi(x) \big] = -\delta(x-1)\, \frac{\mathrm{d}\left[(x^2-1)\, \Phi(x) \right]}{\mathrm{d}x} = -\delta(x-1) \Big[2x\, \Phi(x) + (x^2-1)\, \Phi'(x) \Big] \tag{53}$$

we conclude that

$$\mathcal{L}_x\big[\theta(x-1)\,\Phi(x)\big] = \delta(x-1)\,\big[(x^2-1)\,\Phi'(x)\big] + \theta(x-1)\mathcal{L}_x\,\Phi(x), \tag{54}$$

so it remains to evaluate $\lim_{x\to 1}[(x^2-1)\,\Phi'(x)]$. To this end, we notice that it can be generally shown that $\lim_{x\to 1}\mathcal{F}(\lambda,x)=1$, hence from (50) we have $\Phi(x\to 1)\approx\int_{-1}^{1}\mathrm{d}\lambda\,\frac{1}{(x-\lambda)}=\ln[\frac{x+1}{x-1}]$, which immediately implies $\lim_{x\to 1}[(x^2-1)\,\Phi'(x)]=-2$. This gives the singular contribution to the density (50) in terms of the variables u,v, as follows

$$-\tfrac{2}{4\pi v^2}\,\delta\left(\tfrac{u^2+v^2+1}{2v}-1\right)=-\tfrac{1}{\pi v}\,\delta\left(u^2+(v-1)^2\right)= -\delta(u)\,\delta(v-1), \tag{55}$$

which exactly cancels the contribution from the disconnected part.

Appendix B

In this appendix, we show how (44), when substituted into (36), implies (49). Throughout this appendix, we again use $x_\gamma = x$ and $\mathcal{L}_x := \frac{\partial}{\partial x}(x^2-1)\frac{\partial}{\partial x}$. First of all, we use the identity (45) from the paper [25], which claims that

$$\tfrac{\partial}{\partial\kappa}\mathcal{F}^{(1\mathrm{D})}(\lambda,x)=-\tfrac{\kappa(x-\lambda)}{2}K_0\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)I_0\left(\kappa\sqrt{\tfrac{\lambda+1}{2}}\right). \tag{56}$$

By differentiating both sides of (36) over κ and using (56) in the right-hand side yields

$$\tfrac{\partial}{\partial\kappa}\rho_{1\mathrm{D},M=1}(u,v) = -\tfrac{1}{8\pi v^2}\mathcal{L}_x\left[\kappa\,K_0\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)\times\int_{-1}^{1}\mathrm{d}\lambda\,I_0\left(\kappa\sqrt{\tfrac{\lambda+1}{2}}\right)\right] \tag{57}$$

and after performing the integral by substitution $\lambda=2z^2-1$, $z\in[0,1]$ find that

$$\tfrac{\partial}{\partial\kappa}\rho^{(\mathrm{con})}_{1\mathrm{D},M=1}(u,v)=-\tfrac{1}{2\pi v^2}\mathcal{L}_x\left[K_0\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)I_1(\kappa)\right]= -\tfrac{1}{2\pi v^2}\tfrac{\partial}{\partial x}\sqrt{\tfrac{x+1}{2}}\left[\tfrac{1-x}{2}\,\kappa\,I_1(\kappa)\,K_1\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)\right]. \tag{58}$$

In the next step, we employ the following identity (see Sect. 5.54 in p. 624 of [45])

$$\tfrac{1-x}{2}\,\kappa\,I_1(\kappa)K_1\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)=\tfrac{\partial}{\partial\kappa}\left[\kappa I_2(\kappa)K_1\left(\kappa\sqrt{\tfrac{x+1}{2}}\right) +\kappa\sqrt{\tfrac{x+1}{2}}I_1(\kappa)K_2\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)\right]. \tag{59}$$

Using the fact that $\rho^{(\mathrm{con})}_{1\mathrm{D},M=1}(u,v)\to 0$ as $\kappa\to\infty$, we then may conclude that (58) and (59) together imply

$$\rho_{1\mathrm{D},M=1}(u,v)=-\tfrac{1}{2\pi v^2}\tfrac{\partial}{\partial x}\sqrt{\tfrac{x+1}{2}}\kappa\,I_2(\kappa)K_1\left(\kappa\sqrt{\tfrac{x+1}{2}}\right) -\tfrac{1}{2\pi v^2}\tfrac{\partial}{\partial x}\tfrac{x+1}{2}\kappa\,I_1(\kappa)K_2\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)= -\tfrac{1}{2\pi v^2}\left\{\tfrac{I_1(\kappa)}{\kappa}\tfrac{\partial}{\partial x}\left[\kappa^2\tfrac{x+1}{2}K_2\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)\right] +I_2(\kappa)\tfrac{\partial}{\partial x}\left[\kappa\sqrt{\tfrac{x+1}{2}}K_1\left(\kappa\sqrt{\tfrac{x+1}{2}}\right)\right]\right\}. \tag{60}$$

Finally, introducing in the above the variable $z=\kappa\sqrt{\frac{x+1}{2}}$, using the chain rule and the identity (see Sect. 8.846.14 in [45])

$$\frac{\mathrm{d}}{\mathrm{d}z}\big(z^pK_p(z)\big)=-z^pK_{p-1}(z), \tag{62}$$

allows us to bring the density $\rho^{(con)}_{1\mathrm{D},M=1}(u,v)$ to the final form (49).

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