Bose–Einstein Condensation of q-Deformed Bosons Harmonically Trapped on Sierpiński Carpet and Menger Sponge

I.A. Sadiq* and M.A.Z. Habeeb

Department of Physics, College of Science, Al-Nahrain University, Jaderyia, Baghdad, Iraq

Received: 16.06.2023 & Accepted: 13.09.2023

Doi: 10.12693/APhysPolA.144.234 *e-mail: ibrahim.sadiq@nahrainuniv.edu.iq

Bose–Einstein condensation, as a fifth state of matter, can only occur under certain conditions. One of those conditions is the spatial dimensions confining the bosonic systems. We investigated Bose–Einstein condensation for a finite number of harmonically trapped bosons on fractal structures. The investigation involves two approaches; one belongs to standard Bose–Einstein statistics, and the other belongs to the theory of q-deformed bosons. The properties of Bose–Einstein condensates in the two approaches are computed by performing the sum over the energy states. From these two approaches, we attempt to gain insight into the possibility of using q-numbers to assign fractal dimensions via Bose–Einstein condensation. In this endeavor, the bosons are considered ideal to emphasize that the parameter q only represents the fractal dimension of the structures confining the bosons. The results reveal that a condensate of q-deformed bosons with $q = 0.74$ is adequate to represent a condensate of standard bosons on a Sierpiński carpet. The results also reveal that a condensate of q-deformed bosons with $q = 0.33$ is adequate to represent a condensate of standard bosons on a Menger sponge. We also suggest an expression for using the parameter q to measure the interaction between bosons harmonically trapped on fractal structures, which may also help to study the effect of porosity or fractal dimension on the interaction between bosons.

topics: Bose–Einstein condensation, q-deformed bosons, Tsallis nonextensive parameter, fractal dimensions

1. Introduction

The abrupt collapse of a system of bosons into the ground state energy below a certain temperature is known as Bose–Einstein (BE) condensation. This abrupt collapse was understood as a phase transition of matter. This phase transition is responsible for several phenomena like superfluidity and superconductivity [1]. There are different aspects restricting the occurrence of this phase transition according to the conditions of the bosonic system. The spatial dimension of the space confining the bosons is one of the major aspects of BE condensate (BEC) formation. Historically, Einstein's prediction of BEC was for an ideal, homogeneous Bose gas within the thermodynamic limit [2]. The formation of BEC on an ideal bosonic system is possible for a gas with uniform spatial density (homogeneous) and within the thermodynamic limit. This formation of BEC is for a 3D bosonic system because the bosonic gas occupies a volume (3D space).

At low dimensions ($D < 3$), statistical mechanics excluded the occurrence of BEC if the ideal gas is homogeneous and confined in a space of dimensions $D \leq 2$ [3]. This general result was also asserted [4] for ideal homogeneous gases in two dimensions or one dimension. Theoretical prediction of BEC formation in one or two dimensions suggested inhomogeneous bosonic systems that can be attainable in the presence of external fields or a rotational Bose liquid [5]. Therefore, the research on the possibility of BEC occurrence in low dimensions mainly concerned Bose systems trapped in external potentials [6–12]. These theoretical works confirmed that the occurrence of BEC in a gas confined in a certain dimension (D) can occur for sufficiently confining potentials. About this concern, one can refer to the general result $(D/2 + D/\eta) > 1$, where η is the power-law exponent of the potential trap [13]. The aforementioned works also showed that when the thermodynamic limit condition holds, inhomogeneous bosonic systems do not condense in low dimensions for some potential traps. The possibility of BEC occurring in one dimension ($D = 1$) was thought to be impossible even for a Bose gas in a harmonic trap [7].

The role of a finite number of bosons was revealed through the experimental realization of BEC [14–16]. Those experiments revealed that the theoretical value of condensation temperature in an isotropic 3D harmonic potential when thermodynamic limit holds

$$T_0^{3\mathrm{D}} = \frac{\hbar\omega}{k_{\mathrm{B}}}\left(\frac{N}{\zeta(3)}\right)^{1/3}, \tag{1}$$

where N, k_{B}, and ζ are, respectively, the total number of bosons, Boltzmann constant, and Riemann zeta function, is not accurate. This is because the condensation temperature in those experiments $T_c^{3\mathrm{D}}$ witnessed a downward shift. This downward shift led theoretical works to adopt the treatment of Bose gas with a finite number of particles [8–19], and the relative correction of the downward shift was determined by [8, 17]

$$\frac{T_0^{3\mathrm{D}} - T_c^{3\mathrm{D}}}{T_0^{3\mathrm{D}}} = -0.7275\ N^{-1/3}. \tag{2}$$

Concerning the case of a finite number of bosons confined to an isotropic 2D harmonic trap, the condensation temperature was determined by [10, 12]

$$T_c^{2\mathrm{D}} = \frac{\hbar\omega}{k_{\mathrm{B}}}\left(\frac{N}{\Gamma(2)\,\zeta(2)}\right)^{1/2} = \frac{\hbar\omega}{k_{\mathrm{B}}}\sqrt{\frac{6N}{\pi^2}}. \tag{3}$$

These corrections, (2) and (3), are indeed in excellent agreement with BEC experimental findings. Consequently, all thermodynamic properties of BE condensates, such as the condensate fraction and the heat capacity, on a 3D or 2D harmonic trap should follow the corrections (2) and (3), respectively. The adoption of a finite number of particles also led to the prediction of BEC formation in 1D harmonic trap [17]; BEC formation with atomic gases in 1D was predicted to occur in highly anisotropic harmonic traps [18], and the crossover of BEC of the 3D confinement into BEC in 2D or 1D confinement was observed in the highly anisotropic potential trap [19].

BEC, as a phase transition, takes place not only in Bose gases but also in other bosonic systems. Signatures of a sharp heat capacity and temperature transition, similar to that of BEC formation in Bose gases, were recorded in thin ^{4}He films that are adsorbed on porous structures [20]. More recent works related to BEC in porous media concern local BE condensates in nanopores [21] and positronium BEC in micro-sized cavities [22]. The remarkable observation concerning the surface of porous glass was its fractal nature with fractal dimension $D > 2$ [23]. The dimension of a fractal means that D is non-integer. This fact led to the suggestion of analyzing the fractal nature of such structures by finding an analogy with fractal dimensions [24, 25]. A similar suggestion was also made for bosonic systems in external potentials [7, 13]. Different theoretical approaches were introduced to treat BEC and its thermodynamic properties in fractal media; one of those approaches was the q-deformed bosons [26–30].

In this work, BEC of a finite number of bosons harmonically trapped in fractal structures is investigated via two approaches. Section 2 of this paper is devoted to the first approach, which belongs to the standard Bose–Einstein statistics and is suitable to deal with bosons harmonically trapped in fractal structures or non-integer dimensions. Section 3 is devoted to q-deformed bosons trapped on isotropic 3D and 2D harmonic traps. In these two approaches, the bosons are considered ideal or non-interacting, for we aim to assign numerical values of q that are counterpart to the fractal dimensions of Sierpiński carpet and Menger sponge in Sect. 4.

2. Finite number of bosons harmonically trapped on non-integer dimensions (fractal structures)

BEC is generally characterized by the condensation temperature T_c, which indicates the onset of the phenomenon, and the two signatures, i.e., the condensate fraction and a sharp and continuous heat capacity. In this section, we intend to evaluate T_c and the two signatures of BEC starting with the total number of particles N for an ideal Bose gas in the framework of grand canonical ensemble given by [31]

$$N = N_o + \sum_{n=1}^{\infty} \frac{g_n}{z^{-1}\ \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - 1}, \qquad N_o = \frac{z}{1-z}, \tag{4}$$

where N_o is the ground state occupation number, and the sum term is the number of bosons in higher energy states. The factor g_n is the degeneracy of the harmonic oscillator, while z and ε_n are, respectively, the fugacity and the single-particle energy spectrum. In the case of isotropic harmonic traps, the single-particle energy spectrum is $\varepsilon_n = n\hbar\omega$, where n is a running integer (energy state label), and ω is the trap frequency. The degeneracy of isotropic 1D, 2D, or 3D harmonic traps (D is a positive integer number) is given by the discrete numbers [31]

$$g_n = \frac{(D+n-1)!}{(D-1)!\,n!}. \tag{5}$$

Generally, analytical treatments to evaluate T_c and BEC thermodynamic properties use different techniques. The most frequent technique was converting the sum of (4) into integration over the proper expression of the density of states [8, 17, 26] or using the Euler–Maclaurin summation technique [9, 31, 32].

Our treatment to evaluate T_c and BEC thermodynamic properties for a finite number of particles consists in performing the sum over the energy states. To introduce continuous or non-integer (fractal) dimensions, generalizing the factorial function to the Gamma function is required. Next, degeneracy factors of (5) are generalized for continuous or fractal dimensions by [31]

$$g_n = \frac{\Gamma(D+n)}{\Gamma(D)\,\Gamma(n+1)}. \tag{6}$$

Then, the total number of bosons harmonically trapped in fractal dimensions is expressed by [31]

$$N = N_o + \frac{z}{\Gamma(D)} \sum_{n=1}^{\infty} \frac{\Gamma(n+D)}{\Gamma(n+1)} \frac{1}{\mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - z}. \tag{7}$$

Here, N_o is the ground state population, and the sum term is the number of bosons in higher energy states. The evaluation of T_c requires applying the conditions $N_o = 0$ and $z = 1$ and solving (7) for T_c. Once T_c is determined, fugacity temperature dependence $z(T)$ can be determined by solving (7) for z. The condensate fraction (N_o/N) and the total energy E are functions of temperature T and fugacity z. The temperature dependence of N_o/N and E for these Bose systems are obtained, respectively, via

$$\frac{N_0}{N} = 1 - \frac{1}{N} \left[\frac{z}{\Gamma(D)} \sum_{n=1}^{\infty} \frac{\Gamma(n+D)}{\Gamma(n+1)} \frac{1}{\mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - z} \right] \tag{8}$$

and

$$E_{z,T} = \frac{z}{\Gamma(D)} \sum_{n=1}^{\infty} \varepsilon_n \frac{\Gamma(n+D)}{\Gamma(n+1)} \frac{1}{\mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - z}. \tag{9}$$

The total energy E computed via (9) has the units of the harmonic oscillator energy $(\hbar\omega)$, and the heat capacity defined by $C_V = \partial E/\partial T$ is obtained by differentiating (9) numerically.

3. The q-deformed bosons on harmonic traps

The approach of q-deformed bosons is based on deforming creation and annihilation operators of the standard algebra. The deformation parameter q is a real positive number that represents the extent of deviation from a standard quantum mechanical model. In this section, we deal with q-deformed bosons (or q-deformed oscillators) whose commutation relation of the creation and annihilation operators b_q and $b_q^\dagger$ [26] is

$$\left[b_q, b_q^\dagger\right] = b_q b_q^\dagger - q\, b_q^\dagger b_q = 1. \tag{10}$$

This type of deformation is called mathematical or non-symmetric because the deformation in (10) is non-symmetric under the transformation $q \to q^{-1}$. The mean value of the occupation number of the s state of non-interacting free q-deformed bosons, within the grand canonical ensemble treatment, is given by [26]

$$N_s = \frac{1}{\ln(q)} \ln\left(\frac{z^{-1}\, \mathrm{e}^{\varepsilon_s/(k_{\mathrm{B}}T)} - 1}{z^{-1}\, \mathrm{e}^{\varepsilon_s/(k_{\mathrm{B}}T)} - q} \right). \tag{11}$$

When the conditions of BEC of q-deformed bosons are established, the fugacity z reaches its maximum value z_q that satisfies the condition [26]

$$z \le z_q = \begin{cases} q^{-2}, & q > 1, \\ 1, & 0 < q < 1. \end{cases} \tag{12}$$

BEC formation and its thermal properties of free q-deformed bosons can be determined via the total number of q-deformed bosons [26]

$$N = N_o + \sum_{n=1}^{\infty} \frac{1}{\ln(q)} \ln\left(\frac{z^{-1}\, \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - 1}{z^{-1}\, \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - q} \right),$$

$$N_o = \frac{1}{\ln(q)} \ln\left(\frac{z^{-1} - 1}{z^{-1} - q} \right), \tag{13}$$

where N_o is the ground state population and the sum term is the number of bosons in higher energy states. The formalism of q-deformation provides that the q-deformed bosons system reduces to the standard quantum mechanical model when $q \to 1$, i.e., all q-deformed bosons relations reduce to the relations of standard bosons [26]. This also means that the standard Bose–Einstein statistics is a special case of the q-deformed bosons only when $q = 1$. The theory and applications of q-deformed analysis may represent different effects, like the interaction between quantum particles, impurities, or the fractality of space [26–30, 33–35].

Our adoption of q-deformed analysis aims to represent the fractality of the structure confining non-interacting bosons. In this concern, we indicate two remarkable proposals. One is the Tsallis proposal [36], which directly connects the parameter of Tsallis nonextensive statistical mechanics q_T to fractal dimensions given by $q_T = D/D_E$, where D is the fractal dimension of a porous medium, and D_E is the Euclidean (integer) dimension in which the fractal structure is embedded. Accordingly, fractal structures or porous media have $q_T < 1$ for $D < D_E$. The second proposal, [37] and references therein, is that the deformation parameter q of the deformed bosons theory is not only connected to the parameter q_T — it was also proposed that the two parameters should be the same. Based upon these two notable connections, we investigate the interval $0 < q < 1$, whereas q may represent a measure of the fractality of a fractal structure (porosity of a porous medium).

When q-deformed bosons are on isotropic harmonic traps, the degeneracy factors are the same as of (5), and the single-particle energy spectrum of the harmonic oscillator is the same as the non-deformed (standard) bosons and is also expressed by $\varepsilon_n = n\hbar\omega$. Hence, we can express the total number of q-deformed bosons in harmonic traps by

$$N = N_o + \sum_{n=1}^{\infty} \frac{g_n}{\ln(q)} \ln\left(\frac{z^{-1}\, \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - 1}{z^{-1}\, \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - q} \right). \tag{14}$$

The temperature dependence of condensate fraction N_o/N and the total energy E of the condensate, in $\hbar\omega$ units, are determined, respectively, by

$$\frac{N_o}{N} = 1 - \frac{1}{N} \sum_{n=1}^{\infty} \frac{g_n}{\ln(q)} \ln\left(\frac{z^{-1}\, \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - 1}{z^{-1}\, \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - q} \right) \tag{15}$$

and

$$E_{z,T} = \sum_{n=1}^{\infty} \varepsilon_n \frac{g_n}{\ln(q)} \ln\left(\frac{z^{-1}\, \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - 1}{z^{-1}\, \mathrm{e}^{\varepsilon_n/(k_{\mathrm{B}}T)} - q} \right). \tag{16}$$

We evaluate the condensation temperature of the q-deformed bosons, $T_c^{(q)}$, using (14), the condensate fraction N_o/N using (15), and the heat capacity via performing the sums over the energy states with the same procedure we illustrated in the previous section for the standard bosonic system.

4. Results and discussion

Our results of condensation temperature and the condensate properties are numerically computed by performing the sum over energy states for a finite number of bosons ($N = 1000$) in harmonic traps.

Figure 1 exhibits the evaluation of T_c using (7) for a finite number of bosons harmonically trapped on non-integer or fractal dimensions $1.8 \leq D \leq 3$ in $(\hbar\omega/k_{\rm B})$ units. This figure shows that the condensation temperature T_c is a decreasing function of the dimensions D of the harmonic trap. It also shows that the computed T_c on 3D and 2D harmonic traps, are, respectively, $T_c^{\rm 3D} = 8.7$ and $T_c^{\rm 2D} = 23.7$. These two results are in excellent agreement with the experimental corrections (2) and (3), and this can emphasize the robustness of T_c evaluation via performing the sum over the energy states for bosons harmonically trapped on fractal or non-integer dimensions ($1.8 \leq D \leq 3$).

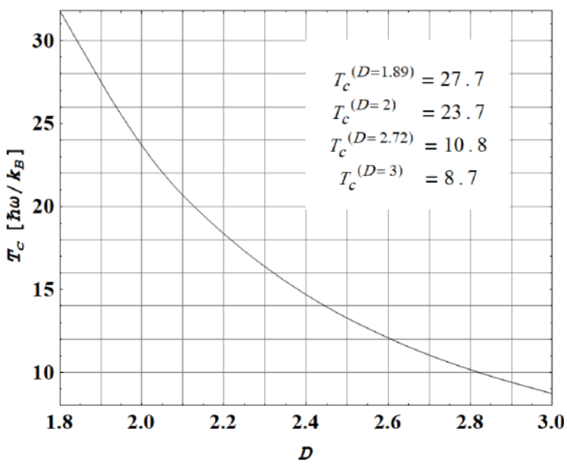

Fig. 1. Condensation temperature of 1000 bosons harmonically trapped on fractal dimensions.

Figure 2 shows the variation of the condensation temperature of q-deformed bosons, $T_c^{(q)}$ in $(\hbar\omega/k_{\rm B})$ units, on 3D and 2D harmonic traps against q via (14). This figure shows that for $0 < q < 1$, $T_c^{(q)}$ is a decreasing function of q, and $T_c^{(q)}$ reaches minimum when q =1. These results are consistent with the theory of q-deformed bosons and also with the results of [26]. Figure 2 also shows that when $q = 0.999$ ($q \to 1$), the values of $T_c^{(q)}$ on 3D and 2D harmonic traps are, respectively, 8.7 and 23.7.

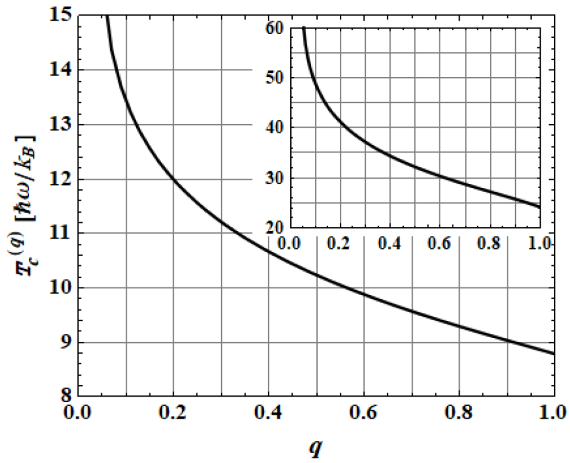

Fig. 2. Condensation temperature of 1000 q-deformed bosons on a 3D harmonic trap. The inset is for q-deformed bosons on a 2D harmonic trap.

It becomes clear that these results for the condensation temperature are the same as those we determined for the standard bosonic system confined on 3D and 2D harmonic traps shown in Fig. 1. This agreement in results of the condensation temperature in two bosonic systems, the standard and the q-deformed, may emphasize the validity of the thermal properties of the BE condensates we determined below for these two bosonic systems.

BEC formation of bosons harmonically trapped on arbitrary fractal dimensions ($D = 2.4, 2.6, 2.8$) are exhibited in Figs. 3 and 4 by computing the sums in (8) and (9). Figures 3 and 4 show the dependence of BEC formation of harmonically trapped bosons on the fractal dimension D.

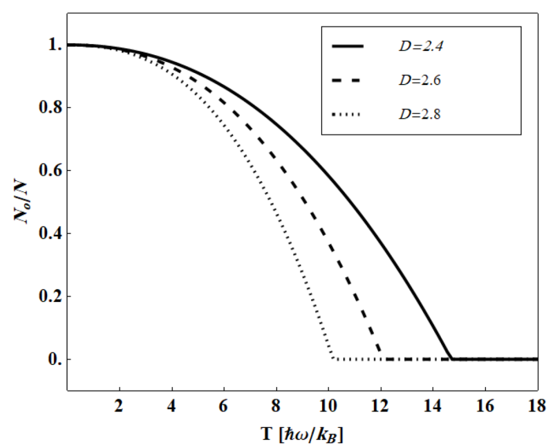

Fig. 3. Condensate fraction of 1000 bosons harmonically trapped on fractal dimensions.

Figure 4 also shows the explicit dependence of heat capacity on the degeneracy factors given by (6); bosons harmonically trapped on fractal structures with a larger fractal dimension D have higher heat capacities.

Concerning q-deformed bosons on 3D and 2D harmonic traps, the condensate fraction N_o/N dependence on q, evaluated using (15), are exhibited, respectively, in Figs. 5 and 6. These figures also show results (dotted curves) of condensate fractions of standard (non-deformed) bosons, which (8) gives for $D = 3$ and $D = 2$, respectively. It is clear that the plots we obtained for q-deformed bosons, whose $q = 0.999$ ($q \to 1$), via (15) are in excellent agreement with the plots we obtained for non-deformed (standard) bosons via (8).

The heat capacity plots on a 3D harmonic trap and 2D harmonic trap are, respectively, exhibited in Figs. 7 and 8. These figures show sharp signatures of heat capacity for q-deformed bosons for arbitrary values of q obtained by differentiating (16) numerically. Figures 7 and 8 also show plots (dotted curves) for heat capacities of standard (non-deformed) bosons, which are obtained by differentiating (9) numerically.

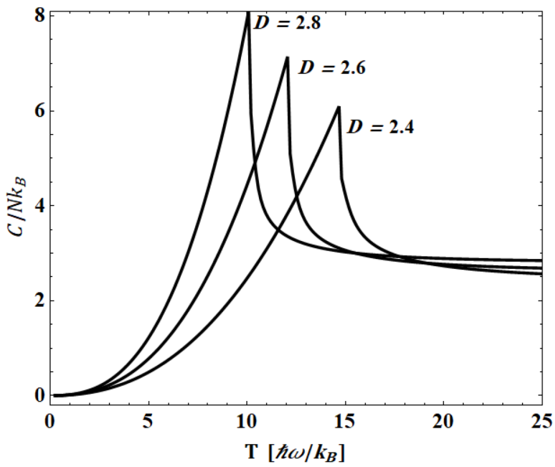

Fig. 4. Heat capacity of 1000 bosons harmonically trapped on fractal dimensions.

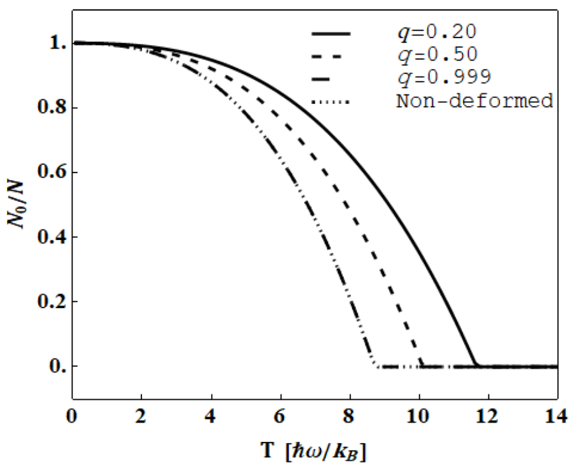

Fig. 5. The condensate fraction of 1000 q-deformed bosons for arbitrary values of q on a 3D harmonic trap.

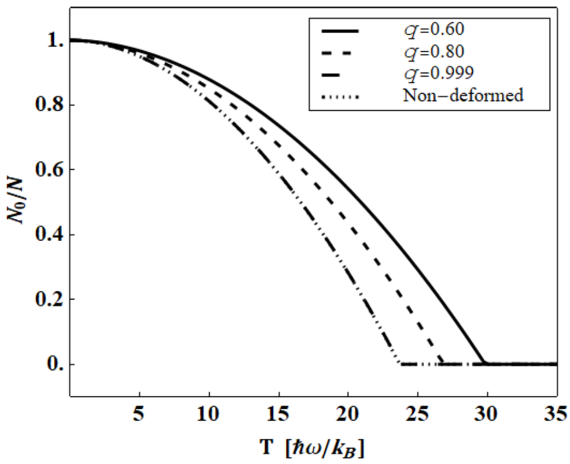

Fig. 6. The condensate fraction of 1000 q-deformed bosons for arbitrary values of q on a 2D harmonic trap.

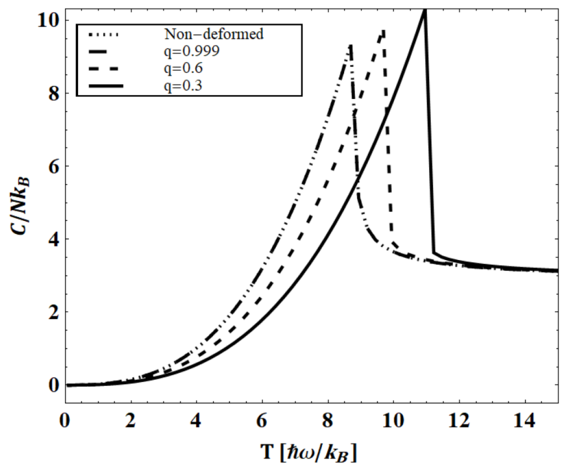

Fig. 7. Heat capacity of 1000 q-deformed bosons on a 3D harmonic trap.

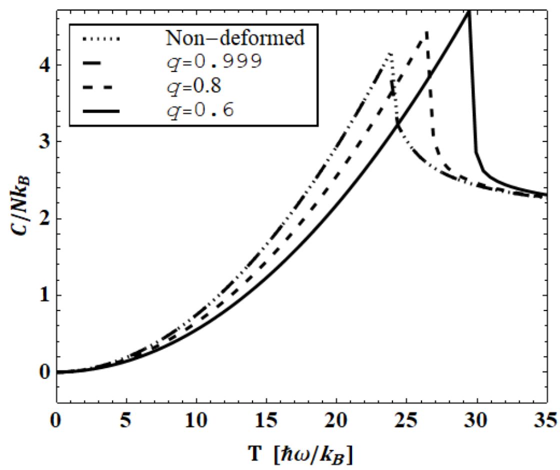

Fig. 8. Heat capacity of 1000 q-deformed bosons on a 2D harmonic trap.

It is clear that the plots of the heat capacity of q-deformed bosons, whose $q = 0.999$, obtained by differentiating (16) numerically, excellently agree with the plots of the heat capacity of non-deformed bosons obtained by differentiating (9) numerically. This agreement in the heat capacity for q-deformed bosons with standard bosons is also consistent with q-deformed bosons theory. From Figs. 7 and 8, we also notice that the plots of heat capacity of q-deformed bosons on a 3D harmonic trap are higher than those on a 2D harmonic trap. This is directly ascribed to the value of the harmonic oscillator degeneracy factors given by (5) because g_n on a 3D harmonic trap ($D = 3$) are larger than those on a 2D harmonic trap ($D = 2$).

The plots of the condensate fraction and the heat capacity that belongs to q-deformed bosons in harmonic traps (Figs. 5–8) explicitly show the utility of using the deformation parameter q to represent the fractality of the confining structures or media such as Sierpiński carpet and Menger sponge.

Now, our approach to assigning the values of q that may represent the fractal dimensions of the Sierpiński carpet and the Menger sponge is based upon the agreements in condensation temperature and thermal behavior of the two bosonic systems, i.e., the standard (non-deformed) system harmonically trapped on the Sierpiński carpet and the Menger sponge with the q-deformed bosonic system.

The Sierpiński carpet is a fractal structure that covers an area (2D space), and its Hausdorff fractal dimension is $D = \log(8)/\log(3) \approx 1.89$, while the Menger sponge is a fractal structure that fills a volume (3D space), and its Hausdorff fractal dimension is $D = \log(20)/\log(3) \approx 2.72$. Accordingly, the fractal dimension of a fractal structure (or porous medium) is less than the Euclidean (integer) dimension in which the fractal structure is embedded [31, 32].

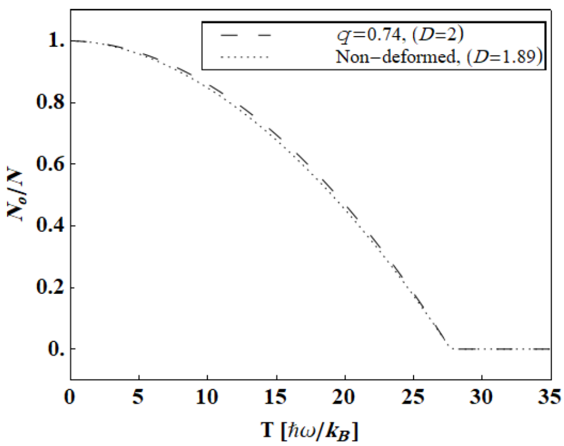

Fig. 9. Condensate fraction of 1000 bosons harmonically trapped on Sierpiński carpet.

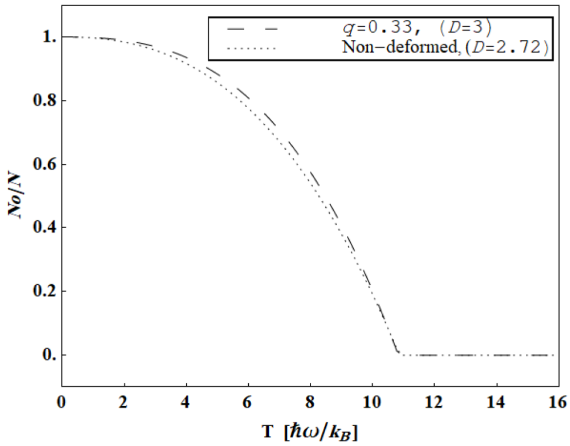

Fig. 10. Condensate fraction of 1000 bosons harmonically trapped on Menger sponge.

Concerning our attempt to assign a value of q that may represent the fractal dimension of the Sierpiński carpet ($D = 1.89$), we consider the case of q-deformed bosons on a 2D harmonic trap ($D = 2$). To assign a value of q that may represent the fractal dimension of the Menger sponge ($D = 2.72$), we consider the case of q-deformed bosons on a 3D harmonic trap ($D = 3$). In this context, our results of the condensation temperature T_c, plotted in Fig. 1, show that standard bosons harmonically trapped on Sierpiński carpet ($D = 1.89$) and Menger sponge ($D = 2.72$) are, respectively, 27.7 and 10.8. From the results in Fig. 2, and with two digits of accuracy for the value of q, it is found that the condensation temperature of q-deformed bosons on 3D harmonic trap $T_c^{(q)} = 10.8$ corresponds to $q = 0.33$, while the condensation temperature of q-deformed bosons on 2D harmonic trap $T_c^{(q)} = 27.7$ corresponds to $q = 0.74$. These two assigned values of the parameter q (0.33 and 0.74) are used to examine the thermal behavior of q-deformed bosons along with the thermal behavior of the standard bosonic systems on the Sierpiński carpet and Menger sponge.

Figures 9 and 10, respectively, exhibit results of the thermal behavior of the condensate fraction on the Sierpiński carpet and Menger sponge for the standard (non-deformed) bosonic system and the q-deformed bosonic system. These figures explicitly show the agreement in the results of the thermal behavior of the condensate fractions for the two bosonic systems.

Figures 11 and 12, respectively, exhibit results of the thermal behavior of the heat capacity on the Sierpiński carpet and Menger sponge for the standard (non-deformed) bosonic system and the q-deformed bosonic system. These figures also show that the heat capacities of q-deformed systems are larger than those of non-deformed systems. This discrepancy in the heat capacities is, indeed, due to the differences in the degeneracy factors g_n. According

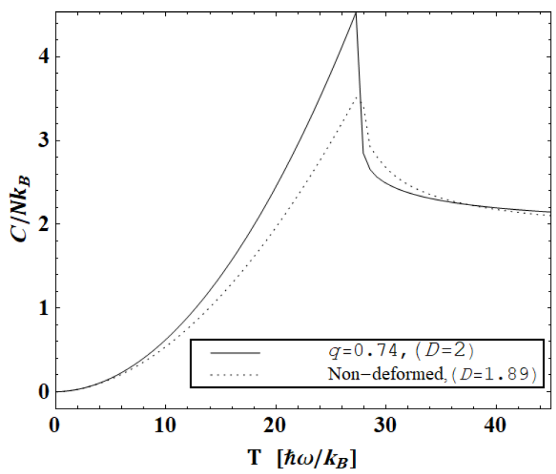

Fig. 11. Heat capacity of 1000 bosons harmonically trapped on Sierpiński carpet.

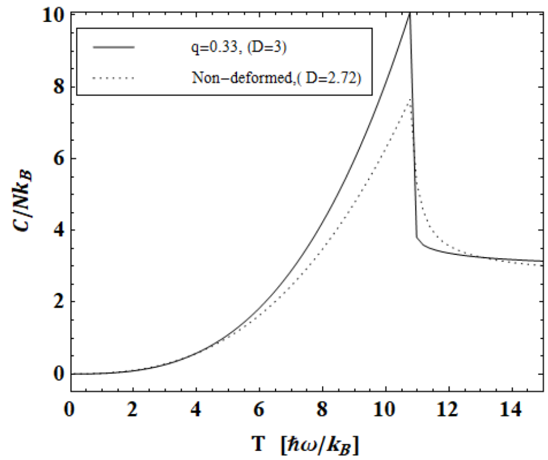

Fig. 12. Heat capacity of 1000 bosons harmonically trapped on Menger sponge.

to (6), g_n for $D = 3$ are larger than those for $D = 2.72$ and, similarly, g_n for $D = 2$ are larger than those for $D = 1.89$. Despite this discrepancy, in our opinion, the thermal behavior of the heat capacities in these two bosonic systems is similar, for the heat capacities of the two bosonic systems reach maximum simultaneously.

5. Conclusions

This work is an attempt to determine a value of q that can represent the fractality of a fractal structure or porosity of a porous medium. Our results show that q-deformed bosons in a 2D harmonic trap with $q = 0.74$ are adequate to represent condensates harmonically trapped on the Sierpiński carpet. Similarly, the q-deformed bosons on the 3D harmonic trap with $q = 0.33$ are adequate to represent condensates harmonically trapped on the Menger sponge. In other words, our results for the BEC phenomenon reveal that $q = 0.74$ is adequate to represent a Hausdorff fractal dimension $D = 1.89$, and $q = 0.33$ is adequate to represent a Hausdorff fractal dimension $D = 2.72$. Also, to the best of our knowledge, this attempt is the first determination of the Hausdorff fractal dimension via the parameter q. Verification of the values of q obtained in this work demands examining the formation of BEC on the Sierpiński carpet and Menger sponge with ideal free bosons or via phenomena other than BEC.

This work shows the utility of using q-deformed bosons to represent the fractality of media or structures possessing fractal dimensions, and it may emphasize that porous media or fractal structures have deformation values of q such that $0 < q < 1$. This result may show equivalence with the Tsalis non-extensive parameter for defining the fractal dimension of a porous material [36] via $q_T = (D/D_E)$, which gives $q_T < 1$. Consequently, our work assists the connection between the q-deformation parameter q and Tsalis non-extensive parameter $q_T$indicated by [37].

According to the later point, this leads to the thought that the BEC phenomenon on rough surfaces (surfaces whose Hausdorff fractal dimension $D > 2$) can be investigated with q-deformed bosons for $q > 1$. In our opinion, such investigation may have special significance on superconductivity and surface physics via q-deformation theory. According to [26], it is worth mentioning here that for $q > 1$, the condensation temperature of q-deformed bosons in harmonic traps $T_c^{(q)}$is an increasing function of q.

It is also useful to mention that parameter q can be also used to represent a parameter of interaction for non-ideal bosons. In this context, (13) can be used to determine the properties of interacting free bosons, and (14) can be used to determine the properties of interacting bosons on harmonic traps of integer dimensions. We also suggest an expression for interacting bosons harmonically trapped on fractal structures by inserting (6) into (14) to get

$$N = N_o + \frac{1}{\ln(q)\Gamma(D)} \times \sum_{n=1}^{\infty} \frac{\Gamma(n+D)}{\Gamma(n+1)} \ln\left(\frac{z^{-1}\,\mathrm{e}^{\varepsilon_n/(k_\mathrm{B}T)} - 1}{z^{-1}\,\mathrm{e}^{\varepsilon_n/(k_\mathrm{B}T)} - q}\right), \tag{17}$$

which may help to investigate the effect of the fractal dimension on the interaction between bosons, especially for BEC in microcavities [21] or nanopores structures [22].

Finally, this work exhibits the utility of using the approach of performing the sum over the energy states to evaluate the physical properties of systems of a finite number of particles. So, we do recommend this technique for systems in microcavities or nanopores for two reasons: (i) because these micro- or nano-sized pores actually can confine a finite number of particles, (ii) because this approach is the exact statistical treatment and there is no need to seek an appropriate density of states to accomplish the calculations.

Acknowledgments

This study did not receive any specific grant or funding from any funding source.

References

[1] C. Yang, *Rev. Mod. Phys.* **34**, 694 (1962).

[2] E. Pérez, T. Sauer, *Arch. Hist. Exact Sci.* **64**, 561 (2010).

[3] K. Huang, *Statistical Mechanics*, 1st ed., Wiley, New York 1963.

[4] S. Coleman, *Commun. Math. Phys.* **31**, 259 (1973).

[5] A. Widom, *Phys. Rev.* **176**, 254 (1968).

[6] V. Bagnato, D. Pritchard, D. Kleppner, *Phys. Rev. A* **35**, 4354 (1987).

[7] V. Bagnato, D. Kleppner, *Phys. Rev. A* **44**, 7439 (1991).

[8] S. Grossmann, M. Holthaus, *Phys. Lett. A* **208**, 188 (1995).

[9] T. Haugset, J. Haugerud, J. Andersen, *Phys. Rev. A* **55**, 2922 (1997).

[10] R.K. Pathria, *Phys. Rev. A* **58**, 1490 (1998).

[11] M. Li, L. Chen, J. Chen, Z. Yan, C. Chen, *Phys. Rev. A* **60**, 4168 (1999).

[12] F. Dalfovo, S. Giorgini, L. Pitaevskii, S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).

[13] L. Salasnich, *J. Math. Phys.* **41**, 8016 (2000).

[14] M. Anderson, J. Ensher, M. Matthews, C. Wieman, E. Cornell, *Science* **269**, 198 (1995).

[15] C. Bradley, C. Sackett, J. Tollet, R. Hulet, *Phys. Rev. Lett.* **75**, 1687 (1995).

[16] K. Davis, M. Mewes, M. Andrews, N. van Druten, D. Durfee, D. Kurn, W. Ketterle, *Phys. Rev. Lett.* **75**, 3969 (1995).

[17] W. Ketterle, N. van Druten, *Phys. Rev. A* **54**, 656 (1996).

[18] W. Ketterle, N.J. van Druten, *Phys. Rev. Lett.* **79**, 549 (1997).

[19] A. Gorlitz, J. Vogels, A. Leanhardt, C. Raman, T. Gustavson, J. Abo-Shaeer, A. Chikkatur, S. Gupta, S. Inouye, T. Rosenband, W. Ketterle, *Phys. Rev. Lett.* **87**, 130402 (2001).

[20] K. Yamamoto, Y. Shibayama, K. Shirahama, *Phys. Rev. Lett.* **100**, 195301 (2008).

[21] J. Bossy, J. Ollivier, H.R. Glyde, *Phys. Rev. B* **99**, 165425 (2019).

[22] M.X. Asaro, S. Herrera, M. Fuentes-Garcia, G.G. Cecchini, E.E. Membreno, R.G. Greaves, A.P. Mills Jr., *Eur. Phys. J. D* **76**, 107 (2022).

[23] A. Höhr, H.-B. Neumann, P.W. Schmidt, P. Pfeifer, D. Avnir, *Phys. Rev. B* **38**, 1462 (1988).

[24] S.-T. Nam, *J. Korean Phys. Soc.* **44**, 464 (2004).

[25] J. Kou, F. Wu, H. Lu, Y. Xu, F. Song, *Phys. Lett. A* **374**, 62 (2009).

[26] Q.-J. Zeng, Z. Cheng, J.-H. Yuan, *Physica A* **391**, 563 (2012).

[27] A. Lavagno, A.M. Scarfone, P.N. Swamy, *Eur. Phys. J. C* **47**, 253 (2006).

[28] A. Lavagno, *J. Phys. A: Math. Theor.* **41**, 244014 (2008).

[29] A. Lavagno, *Rep. Math. Phys.* **64**, 79 (2009).

[30] Y. Wang X.-M. Kong, *Mod. Phys. Lett. B* **24**, 135 (2010).

[31] A. Rovenchack, *Acta Phys. Pol. A* **118**, 531 (2010).

[32] J.P. Chen, arXiv:1202.1274 (2012).

[33] F.G. Garaschenko, D.S. Kruchinin, V.E. Novikov, *Int. J. Mod. Phys. B* **22**, 3923 (2008).

[34] I.S. Sogami, K. Koizumi, *Prog. Theor. Phys.* **107**, 1 (2002).

[35] A.A. Marinho, F.A. Brito, C. Chesman, *Physica A* **391**, 3424 (2012).

[36] C. Tsallis, *Fractals* **3**, 541 (1995).

[37] R.S. Johal, *Phys. Lett. A* **258**, 15 (1999).