Explicit Computation of the Quantum Phase of the Time Perturbed Harmonic Oscillator

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We consider a simple harmonic oscillator subject to a time-dependent, spatially homogeneous perturbation and show that it is canonically equivalent to a non-perturbed oscillator. The canonical transformation that relates both systems can be implemented as a unitary transformation mapping the perturbed system and the non-perturbed systems onto each other. This unitary transformation allows us to explicitly compute the time dependence of the states of the perturbed system, and in particular to compute the phase that affects all states. The phase turns out to be a classical action along the classical trajectory of the origin (in phase space), along its motion under the action of perturbation. In this simple system, the transition probabilities due to the perturbation can also be computed explicitly. Our approach is independent of the magnitude of the perturbation, and does not require an adiabaticity assumption.

topics: forced harmonic oscillator, canonical transformation, quantization of the forced oscillator, quantum phase

1. Introduction

The harmonic oscillator is a model of a system that comes up in very many situations. See [1] or [2] for anvexample. The higher dimensional harmonic oscillator appears when studying oscillations near equilibrium as described in [3] or [4]. In the quantum domain, the eigenstates of a simple and higher dimensional harmonic oscillator come up in many contexts in atomic and nuclear physics, see [5] or [6]. It comes up at the interface between probability theory and quantum mechanics [7] or in the harmonic analysis [8].

Nevertheless, there is still a lot to do with a simple harmonic oscillator. In [9] it is proved that a simple 1-dimensional oscillator, subject to a time-dependent but spatially constant force, is canonically equivalent to a simple non-perturbed oscillator. Furthermore, it is also shown there that a canonical transformation that realizes equivalence can be implemented as a unitary transformation between the corresponding quantized versions of the two systems.

In this work, we will make use of the computations carried out in [9] to address the problem considered by Berry in [10], where the general case is considered under the assumption that the perturbation is adiabatic. That work generated a very large body of literature. A cursory search with a few keywords yields lists ranging from tens of thousands to over hundreds of thousands. But neither the example that we work out below nor the method that we use to deal with it, it seems to have been considered before. Furthermore, the simplicity of the system is such that the adiabaticity of the perturbation is not required. The solution to the Schrödinger equation is obtained explicitly. It is shown that it acquires a global phase that has a clear physical interpretation. Namely, it is an action along a trajectory of the origin along the perturbed classical trajectory.

In Sect. 2, we explain how to relate the description of standard and perturbed harmonic oscillators by means of canonical transformation. This is the first step of relating the quantized versions of the forced and non-forced oscillators. The second step is to implement the canonical transformation as a unitary transformation between the corresponding Hilbert spaces.

Before describing the contents of Sect. 3, we mention the problem of describing canonical transformations in quantum mechanics, already present in the work of the founding fathers of quantum mechanics. To begin with, consider [11], where the relationship between the Heisenberg matrix formulation and the quantization rules is explored (and the harmonic oscillator comes up). Also in [12] and [13] the connection between canonical transformations and their corresponding quantum version is studied. (In the later one, Dirac's delta function makes its ap-

pearance.) The fact that canonical transformations have to be represented by unitary transformation was introduced in [14] after a probabilistic interpretation of the Schrödinger's wave function was proposed in [15][†1]. Apart from the fact that unitary and canonical transformations are an important tool, as exemplified in [16] and [6], the problem of establishing correspondence between classical observables and their realization as operators is still far from finished. See, for example [17], and especially [18], and the extensive literature cited there. Perhaps there is a bridge connecting the later ones to [19]. Both approaches use a phase-space approach to the quantization problem.

To continue, in Sect. 3, we prove that the unitary transformation realizing the classical canonical transformation maps the position, momentum and Hamiltonian operators as in the classical case. On the one hand, this means that the quantization rules behave consistently under canonical transformations, and on the other hand, that we can transform the solutions of the corresponding Schrödinger equations onto each other. Also, we indicate how we can compute the transition probabilities using the canonical mapping. There we show that the solution of the Schrödinger equation acquires a global phase and a local phase, but the phases are state-independent. As said above, the global phase happens to be an action along a classical trajectory of the origin under perturbation conditions. In Sect. 4 we show how to compute transition probabilities in the standard basis of a harmonic oscillator.

2. Canonical equivalence of the simple and perturbed harmonic oscillators

In the Hamiltonian formalism, the dynamics of a simple and a perturbed 1-dimensional harmonic oscillator are derived from the following two Hamiltonian functions

$$H_0(x,p)=\frac{1}{2}\eta^2+\frac{1}{2}\omega^2\xi^2, \tag{1}$$

$$H(x,p)=\frac{1}{2}p^2+\frac{1}{2}\omega^2x^2-xk(t). \tag{2}$$

The Hamilton equations of motion of the system modeled by $H(x,p)$ are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\p\end{pmatrix}&=\begin{pmatrix}\frac{\partial H}{\partial p}\\-\frac{\partial H}{\partial x}\end{pmatrix}=\begin{pmatrix}p\\-\omega^2x+k(t)\end{pmatrix}=\\&\begin{pmatrix}0&1\\-\omega^2&0\end{pmatrix}\begin{pmatrix}x\\p\end{pmatrix}+\begin{pmatrix}0\\k(t)\end{pmatrix}=\\&\mathbb{H}_0\begin{pmatrix}x\\p\end{pmatrix}+\begin{pmatrix}0\\k(t)\end{pmatrix}.\end{aligned} \tag{3}$$

[†1] From the almost simultaneity of the papers [14] and [15], these seem to have been hectic days of friendly and collaborative competition

The initial conditions are $x(0)=x_0, p(0)=p_0$. The equations of motion of an unperturbed oscillator are obtained by setting $k(t)=0$. The solution of this system is

$$\begin{pmatrix}x(t)\\p(t)\end{pmatrix}=U(t)\begin{pmatrix}x_0\\p_0\end{pmatrix}+\int_0^t \mathrm{d}s\,U(t-s)\begin{pmatrix}0\\k(s)\end{pmatrix}. \tag{4}$$

Here it is clear that the time dependence of the forcing term is arbitrary, subject to the mathematical requirement that the integrals on the right-hand side of (4) are defined. See concluding remarks in Sect. 5 for more on this issue. In (4), we introduced the following notations

$$U(t)=\begin{pmatrix}\cos(\omega t)&\frac{1}{\omega}\sin(\omega t)\\-\omega\sin(\omega t)&\cos(\omega t)\end{pmatrix}. \tag{5}$$

Clearly $U(t)$ satisfies $U(t+s)=U(t)U(s)$ or $U(t-s)=U(t)U(-s)$ for all s and t. To simplify the notations in what comes below, let us denote the coordinates (x,p) by $\boldsymbol{z}$ (thought as a column vector) and introduce $\boldsymbol{k}(s)=(0,k(s))^{\mathrm{T}}$. With these notations, we write the solution (4) to the system (3) as

$$\begin{aligned}\boldsymbol{z}(t)=U(t)\,\boldsymbol{z}(0)+\int_0^t \mathrm{d}s\;U(t-s)\,\boldsymbol{k}(s)=\\ \boldsymbol{z}_h(t)+\boldsymbol{z}_{nh}(t).\end{aligned} \tag{6}$$

The subscript h stands for *homogeneous* and nh stands for *non-homogeneous*.

If we put $H_0(x,p)=\frac{1}{2}(\omega^2x^2+p^2)$, then $\boldsymbol{z}_h(t)$ solves (3) with $H_0(x,p)$ instead of $H(x,p)$. Or, if one prefers, $\boldsymbol{z}_{nh}(t)$ is just the particular solution to (3) with zero initial conditions. Observe as well that $\boldsymbol{z}_h(t)=\boldsymbol{z}(t)-\boldsymbol{z}_{nh}(t)$ describes the motion of a simple harmonic oscillator, which is consistent with the fact that

$$\begin{aligned}\big\langle\big(\boldsymbol{z}(t)-\boldsymbol{z}_{nh}(t)\big),\mathbb{H}_0\big(\boldsymbol{z}(t)-\boldsymbol{z}_{nh}(t)\big)\big\rangle=\\ \text{constant}=\big\langle\boldsymbol{z}(0),\mathbb{H}_0\boldsymbol{z}(0)\big\rangle.\end{aligned} \tag{7}$$

This follows easily from the fact that the matrix $\mathbb{H}_0$ introduced in (3) satisfies

$$U^{\dagger}(t)\,\mathbb{H}_0\,U(t)=\mathbb{H}_0. \tag{8}$$

This means that we might think of the two terms on the right-hand side of (6) as follows: (i) interpret $\boldsymbol{z}_{nh}(t)$ as the motion of the center of mass of the coordinates of "laboratory", which undergoes a non-uniform motion, and (ii) interpret $\boldsymbol{z}_h(t)$ as the motion relative to a coordinates system in which there is no external force, i.e., the motion described in the coordinate system moving with the laboratory. In order to make this change of coordinates consistent with the Hamiltonian framework, we must implement the change of coordinates as a canonical transformation that preserves the canonical equations of motion.

We shall think of the coordinates $\boldsymbol{z}=(x,p)^{\mathrm{T}}$ as the coordinates in the laboratory system and the new coordinates $\boldsymbol{\zeta}=(\xi,\eta)^{\mathrm{T}}$ as position and

momentum in a moving coordinate system. Denote the coordinates $\boldsymbol{z}_{nh}(t)$ by $x_{nh}(t)$ and $p_{nh}(t) = m\dot{x}_{nh}(t)$. As usual, the dot denotes the derivative with respect to time.

Let us write $H_0(x,p) = \frac{1}{2}(p^2+\omega^2x^2) - xk(t)$ and $H(\xi,\eta) = \frac{1}{2}(\eta^2+\omega^2\xi^2)$ for two Hamiltonian functions of interest. We consider the following canonical transformation to realize the change of coordinates because it can be easily implemented as a unitary transformation. The transformation is given by

- either

$$F_1(x,\eta,t) = \big(x-x_{nh}(t)\big)\,\eta\dot{x}_{nh}(t) + A(t), \tag{9}$$

- or

$$F_2(\xi,p,t) = \big(\xi+x_{nh}(t)\big)\,\big(p-\dot{x}_{nh}(t)\big) + A(t), \tag{10}$$

depending on which coordinates we want to regard as old or new. The equations that relate the old to new coordinates are (see [3] or [4])

$$\xi(t) = \frac{\partial F_1}{\partial \eta}, \tag{11}$$

$$p(t) = \frac{\partial F_1}{\partial x}, \tag{12}$$

$$H = H_0 + \frac{\partial F_1}{\partial t}. \tag{13}$$

Note that the transformation equations yield that $\xi(t) = x(t) - x_{nh}(t)$ and $p = \eta + \dot{x}_{nh}(t)$. It is understood that in (13) the partial derivative with respect to t is carried out and the old coordinates are substituted for the new ones after solving (11)–(12). To see that $H(x,p)$ becomes $H_0(\xi,\eta)$ when we apply (11)–(13) to (9), we have to make use of the fact that

$$\ddot{x}_{nh}(t) + \omega^2 x_{nh}(t) - k(t) = 0 \tag{14}$$

and that $A(t)$ is to be chosen so that

$$\dot{A}(t) - \frac{1}{2}\dot{x}_{nh}^2(t) + \frac{\omega^2}{2}x_{nh}^2(t) - x_{nh}(t)\,k(t) = 0. \tag{15}$$

A similar procedure is followed for the passage from $H_0(\xi,\eta)$ to $H(x,p)$. The physical interpretation of $A(t)$ is clear. Note that the Lagrangian $L(x,v,t)$ corresponding to $H(x,p,t)$ is

$$L(x,v,t) = \frac{1}{2}v^2 - \frac{\omega^2}{2}x^2 + x\,k(t). \tag{16}$$

Therefore, it follows from (15) that $A(t)$ is the action computed along the trajectory of the moving origin of coordinates.

3. Unitary equivalence of simple and perturbed harmonic oscillators

Consider two classical systems with the phase space coordinates labeled by (x,p) and (ξ,η) and with the dynamics determined by the Hamiltonians $H_0(x,p)$ and $H(\xi,\eta)$. We shall denote by $\Psi(x)$ and $\Phi(\xi)$ the states in their Schrödinger representation. Let us denote by $\mathcal{H}$ and $\mathcal{K}$ the corresponding Hilbert state spaces provided with the usual scalar product.

Note that the canonical transformations involve the momentum variables of one system and the coordinate variables of the other, the unitary transformations are defined to act on the momentum representation of the states to yield the states in the coordinate representations. The description of the states in terms of momentum representation is given by taking the Fourier transforms, which is

$$\Psi(p) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}px}\,\Psi(x) \tag{17}$$

and

$$\Phi(\eta) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\xi\ \mathrm{e}^{-\mathrm{i}\eta\xi}\,\Phi(\xi). \tag{18}$$

Consider (9) and (10). In each case, the transform depends on the "new" momentum and the "old" coordinates. Thus, the unitary transformations induced by F_1 and F_2 go in the opposite directions and are defined as follows

$$\widetilde{\Phi}(x,t) = U_{F_1}(t)\,\Phi(x,t) = \int \frac{\mathrm{d}\eta}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}\,F_1(x,\eta)}\,\Phi(\eta,t), \tag{19}$$

and

$$\widetilde{\Psi}(\xi,t) = U_{F_2}(t)\,\Psi(\xi,t) = \int \frac{\mathrm{d}p}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}\,F_2(\xi,p)}\,\Psi(p,t). \tag{20}$$

The tilde is a notational reminder of the fact that a state is obtained by applying a canonical transformation. Keep in mind that the transformation is time-dependent and applied to the state dynamically. Also note that at $t = 0$ the transformations reduce to the connection between momentum and coordinate representations. The explicit computation of these transforms is simple

$$\widetilde{\Phi}(x,t) = \mathrm{e}^{\mathrm{i}\,\theta(x,t)}\,\Phi\big(x-x_{nh}(t),t\big), \tag{21}$$

$$\widetilde{\Psi}(\xi,t) = \mathrm{e}^{\mathrm{i}\,\theta'(\xi,t)}\,\Psi\big(\xi+x_{nh}(t),t\big). \tag{22}$$

Here we put

$$\theta(x,t) = \big(x-x_{nh}(t)\big)\,\dot{x}_{nh}(t) + A(t), \tag{23}$$

$$\theta'(\xi,t) = -\big(\xi+x_{nh}(t)\big)\,\dot{x}_{nh}(t) + A(t). \tag{24}$$

Clearly, these transformations are unitary. To use a notation resembling that of Berry [10], we rewrite (21) as, for example

$$\widetilde{\Phi}(x,t) = \mathrm{e}^{\mathrm{i}\,A(t)}\,\Phi(x,\boldsymbol{z}_{nh},t), \tag{25}$$

where

$$\Phi(x,\boldsymbol{z}_{nh},t) = \mathrm{e}^{\mathrm{i}\,(x-x_{nh}(t))\dot{x}_{nh}(t)}\,\Phi\big(x-x_{nh}(t),t\big). \tag{26}$$

This means that the solution of the non-perturbed oscillator is shifted to the new origin of coordinates and acquires the global phase $(A(t))$ plus the local phase $\big((x-x_{nh}(t))\,\dot{x}_{nh}(t)\big)$, which comes from the passage of a static coordinates system to a system of coordinates moving according to $\boldsymbol{z}_{nh}(t)$. Note that in this simple example, the phase can be exactly determined regardless of the adiabaticity of the perturbation.

3.1. Transformation of the position and momentum operators

Let us now verify that the position and momentum operators are transformed consistently with the correspondence rules. Let us begin by recalling that using (17) and (18), one verifies that

$$\hat{\xi}\,\Phi(\eta) = -\,\mathrm{i}\,\frac{\partial\,\Phi(\eta)}{\partial\eta} \qquad \text{and} \qquad \hat{\eta}\,\Phi(\eta) = \eta\,\Phi(\eta). \tag{27}$$

Now, applying U_{F_1} to Φ as in (19) and (20), we obtain that the position operator transforms as

$$U_{F_1}\big(\hat{\xi}\,\Phi\big)(x,t) = \big(x - x_{nh}(t)\big)\,\widetilde{\Phi}(x,t) =$$

$$\big(\hat{x} - x_{nh}(t)\big)\,\tilde{\Phi}(x,t), \tag{28}$$

whereas the momentum operator transforms as

$$U_{F_1}\big(\hat{\eta}\,\Phi\big)(x,t) = \Big(-\,\mathrm{i}\,\frac{\partial}{\partial x} - \dot{x}_{nh}(t)\Big)\,\widetilde{\Phi}(x,t) =$$

$$\Big(\hat{p} - \dot{x}_{nh}(t)\Big)\,\widetilde{\Phi}(x,t). \tag{29}$$

To conclude, keep in mind that the Hamiltonian operators in the "laboratory" and "accelerated" systems respectively are given by

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2 x^2 - x k(t) \tag{30}$$

and

$$\hat{H}_0 = -\frac{1}{2}\frac{\partial^2}{\partial \xi^2} + \frac{1}{2}\omega^2 \xi^2. \tag{31}$$

3.2. The evolution in time transforms consistently

Let us now verify the claim that the time evolution with respect to one Hamiltonian transforms into a time evolution with respect to the other Hamiltonian. The result drops out of the fact that

$$\mathrm{i}\,\frac{\mathrm{d}\,\tilde{\Psi}(\xi,t)}{\mathrm{d}t} = \mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(U_{F_2}\,\Psi\Big)(\xi,t) =$$

$$\Big(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}U_{F_2}\Big)\,\Psi\big(\xi,t\big) + U_{F_2}\Big(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\Psi\Big)(\xi,t). \tag{32}$$

A look at the definition in (22) and the effect on Ψ explicitly shown in (20) actually proves that the claim follows from the fact that

$$\mathrm{i}\,\frac{\mathrm{d}\,\tilde{\Psi}(\xi,t)}{\mathrm{d}t} = \mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathrm{e}^{\,\mathrm{i}\,\theta'(\xi,t)}\,\Psi\big(\xi + x_{nh}(t),t\big)\Big] =$$

$$\Big[\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\,\mathrm{i}\,\theta'(\xi,t)}\Big]\,\Psi\big(\xi + x_{nh}(t),t\big)$$

$$+\mathrm{e}^{\,\mathrm{i}\,\theta'(\xi,t)}\Big[\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\Psi\big(\xi + x_{nh}(t),t\big)\Big]. \tag{33}$$

Notice that

$$\Big[\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\Psi\big(\xi + x_{nh}(t),t\big)\Big] = \mathrm{i}\,\dot{x}_{nh}(t)\frac{\partial\,\Psi(\xi + x_{nh}(t),t)}{\partial x}$$

$$+\mathcal{H}\,\Psi\big(\xi + x_{nh}(t),t\big). \tag{34}$$

It is just a matter of substituting (28)–(29), making use of (11)–(13), and the fact that neither $x_{nh}(t)$ nor $\dot{x}_{nh}(t)$ involves x or p, to verify that all necessary cancellations take place to conclude that when $\Psi(x,t)$ satisfies the Schrödinger's equation in a laboratory system, then

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{\Psi}(\xi,t) = \hat{H}_0\,\tilde{\Psi}(\xi,t), \tag{35}$$

with

$$\tilde{\Psi}(\xi,0) = \Psi(\xi,0) = \Psi(x,0), \tag{36}$$

where $\tilde{\Psi}(\xi,t)$ satisfies the Schrödinger's equation in a moving system. An exactly analogous argument proves that if $\Phi(\xi,t)$ satisfies the Schrödinger's equation with the Hamiltonian operator $\hat{H}_0$, then $\tilde{\Phi}(x,t) = \big(U_{F_2}\,\Phi\big)(x,t)$ satisfies the Schrödinger's equation with the Hamiltonian $\hat{H}$. Note that if at $t = 0$ the system was prepared in the eigenstate $\Phi_n(\xi)$ of $\mathcal{K}$ of the energy E_n, then according to (21) or (25)–(26) the state of the perturbed system is

$$\widetilde{\Phi}_n(x,t) = \mathrm{e}^{-\,\mathrm{i}\,tE_n}\,\mathrm{e}^{\,\mathrm{i}\,A(t)}\,\Phi_n(x, \boldsymbol{z}_{nh}, t). \tag{37}$$

This result is in agreement with [10].

3.3. What if the perturbation ceases at T?

In order to better compare with the problem treated in [10], let us suppose that $k(t) \equiv 0$ for $t \geq T$. Then from time T, it happens that:

1. At the classical level, the system evolves according to the standard (non-perturbed) Hamiltonian H_0, and the initial condition from time T is $\boldsymbol{z}(T)$.

2. At the quantum level, the system evolves according to the standard (non-perturbed) Hamiltonian, and the initial state from time T is $\widetilde{\Phi}(T)$.

In order to compute the state of the system for times $t > T$, it is convenient to expand $\widetilde{\Phi}(T)$ in the basis $\{\Phi_m : m \geq 0\}$ of eigenstates of $\hat{H}_0$. So, let us suppose that at $t = 0$ the system was prepared to be in the eigenstate Φ_n. At time T it will be at the state $\widetilde{\Phi}(T)$ given by (37), that is

$$\widetilde{\Phi}_n(x,T) = \mathrm{e}^{-\,\mathrm{i}\,TE_n}\,\mathrm{e}^{\,\mathrm{i}\,A(T)}\,\Phi_n(x, \boldsymbol{z}_{nh}, T) =$$

$$\mathrm{e}^{-\,\mathrm{i}\,TE_n}\,\mathrm{e}^{\,\mathrm{i}\,A(T)}\,\mathrm{e}^{\,\mathrm{i}\,(x - x_{nh}(t))\dot{x}_{nh}(t)}\,\Phi(x - x_{nh}). \tag{38}$$

To carry out the expansion, we need to compute

$$\big\langle\,\Phi_m,\,\widetilde{\Phi}_n(x,T)\big\rangle \tag{39}$$

with $\widetilde{\Phi}_n(x,T)$ as above. Therefore, at time $t \geq T$ the state of the system becomes

$$\Psi(t,x) = \sum_{m=0}^{\infty}\,\mathrm{e}^{-\,\mathrm{i}\,(t-T)E_m}\,\big\langle\,\Phi_m,\,\widetilde{\Phi}_n(x,T)\big\rangle\,\Phi_m(x). \tag{40}$$

Clearly, the computation (39) amounts to computing the transition amplitudes at time t and set $t = T$. The transition probabilities can then be computed as

$$P_{n,m}(T) = \big|\big\langle\,\Psi_m,\,\Psi_n(T)\big\rangle\big|^2. \tag{41}$$

Notice that the global phase $\mathrm{e}^{\mathrm{i}\,A(T)}$, which is present in (40), disappears when we compute the transition probability. This is because the global phase is independent of the initial state and depends only on the transformation connecting the perturbed and non-perturbed states.

4. Computation of transition probabilities

Let us now use those results to compute the transition probabilities. Suppose once more that at $t = 0$ the oscillator was in the n-th eigenstate Ψ_n of the energy E_n and the perturbation is turned on. We are interested in computing $P_{n,m}(t)$ at time t.

According to (21) and (22), we have $\widetilde{\Phi_n}(x,t) = \mathrm{e}^{\mathrm{i}\,\theta(x,t)}\,\Phi_n(x{-}x_{nh}(t),t)$ and $\Phi_n(x{-}x_{nh}(t),t) = \mathrm{e}^{-\mathrm{i}\,E_n t}\,\Phi_n(x{-}x_{nh}(t))$, respectively. From (23), we obtain

$$\big\langle\, \Psi_m,\, \Psi_n(t)\big\rangle = \mathrm{e}^{-\mathrm{i}\,E_n t}\,\mathrm{e}^{\mathrm{i}\,A(t)} \times \int \mathrm{d}x\; \Psi_m(x)\,\mathrm{e}^{\mathrm{i}\,(x-x_{nh}(t))\,\dot{x}_{nh}(t)}\,\Psi_n(x{-}x_{nh}(t)). \tag{42}$$

The phase in front of the integral disappears when we consider the absolute value, so we forget about it when computing the integral. For this, we recall some basics.

The eigenstate $\Psi_n(x)$ is expressed in terms of the Hermite polynomials as

$$\Psi_n(x) = \frac{\omega^{\frac{1}{4}}}{\sqrt{2^n\sqrt{\pi}\,n!}}\,H_n\left(\sqrt{\omega}x\right)\mathrm{e}^{-\omega x^2/2}. \tag{43}$$

The constants are such that $\langle \Psi_n,\, \Psi_m\rangle = \delta_{n,m}$. The Hermite polynomials are obtained from their generating function as follows

$$G(x,u) = \mathrm{e}^{2xu-u^2} = \sum_{n=0}^{\infty}\frac{u^n}{n!}H_n(x). \tag{44}$$

To complete the recall that for any complex s, we have

$$\int\limits_{\mathbb{R}} \mathrm{d}x\;\; \mathrm{e}^{sx}\,\mathrm{e}^{-x^2} = \sqrt{\pi}\,\mathrm{e}^{s^2/4}. \tag{45}$$

If we leave the factor $C(m,n) = (2^{m+n}m!n!\pi)^{-\frac{1}{2}}$ aside and make a change of the variables $\sqrt{\omega}x \to x$ to calculate transition probabilities, we just end up needing to compute

$$\int\limits_{\mathbb{R}} \mathrm{d}x\, H_m(x)H_n(x)\,\mathrm{e}^{\mathrm{i}\,(x-a)b}\,\mathrm{e}^{-\frac{1}{2}(x-a)^2}\,\mathrm{e}^{-\frac{1}{2}x^2}, \tag{46}$$

where $a = \sqrt{\omega}x_{nh}(t)$ and $b = \dot{x}_{nh}(t)/\sqrt{\omega}$. To use (43), we multiply the integral by $v^m/m!$ and $u^n/n!$, sum over n and m from 0 to ∞. Instead of evaluating this integral for each m and n, we make use of (43) and evaluate instead

$$\int\limits_{\mathbb{R}} \mathrm{d}x\; G(x,v)G(x{-}a,u)\,\mathrm{e}^{\mathrm{i}\,(x-a)b}\,\mathrm{e}^{-\frac{1}{2}(x-a)^2}\,\mathrm{e}^{-\frac{1}{2}x^2}. \tag{47}$$

To obtain the desired integral, we compute $\partial^{m+n}/(\partial v^m \partial u^n)$ at $v{=}u = 0$. Next, we expand the exponents in (47), collect the powers of x in the exponent and use (44) to obtain

$$\int\limits_{\mathbb{R}} \frac{\mathrm{d}x}{\sqrt{\pi}}\,G(x,v)G(x{-}a,u)\,\mathrm{e}^{\mathrm{i}\,(x-a)b}\,\mathrm{e}^{-x^2} = \exp\left(2uv + (u+v)(\mathrm{i}b+a) - \frac{1}{4}(b^2+a^2) - \frac{1}{2}\mathrm{i}ab\right). \tag{48}$$

Before looking further into (48), note that it is symmetric in v and u. Since the phases are independent of the state, this means that (46) and (42) are symmetric with respect to the exchange of initial and final states. This is a nice reversibility condition preserved by the perturbation.

To continue with (48), let us replace a and b with $\sqrt{\omega}x_{nh}(t)$ and $b = \dot{x}_{nh}(t)/\sqrt{\omega}$. The next to the last term can be written as

$$\frac{1}{2\omega}\mathcal{E}_{nh}(t) := \frac{1}{2\omega}\left[\frac{1}{2}\left(\omega x_{nh}^2(t) + \dot{x}_{nh}^2(t)\right)\right] \tag{49}$$

which is the energy of the non-homogeneous part of the classical solution, i.e., the energy originating from the action of the force. The last term is

$$\frac{1}{2\omega}L_{nh} := \frac{1}{2\omega}x_{nh}(t)\dot{x}_{nh}(t), \tag{50}$$

which is the "angular momentum" of the motion of the origin in units of ω (recall that we are using $\hbar = 1$). If the external force were 0, this would be the derivative of the action with respect to time.

The factor (50) is the phase and does not contribute to the probability, but the factor

$$\exp\left(-\frac{1}{2\omega}\mathcal{E}_{nh}(t)\right) \tag{51}$$

stays put as valid. It is a global damping term present in all transition probabilities. When the perturbation lasts for time T and it happens that $x_{nh}(T) = \dot{x}_{nh}(T) = 0$, then $\mathcal{E}_{nh}(T) = 0$ and the damping term disappears.

5. Conclusions

To sum up, the fact that the perturbation can be eliminated by means of an appropriate canonical transformation allows us to get rid the adiabaticity and the smallness of the perturbation. Then we obtain an exact solution to the problem and determine the phase that the solution acquires.

Notice that the time dependence of the forcing term is arbitrary, subject to the requirement that $\int_0^t \mathrm{d}s\;|k(s)|$ is finite for all $t > 0$. Actually, at the expense of complicating the mathematical formalism by bringing in some stochastic calculus, random forcing terms could be considered.

We mention as well that by vectorizing the notations, the formalism presented above can be extended to the many-dimensional case, and that in the limit $\omega \to 0$, we obtain the case of a particle in constant time-dependent field.

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