Quantum Phase Diffusion in Two-Mode Bose–Einstein Condensates with Particle Losses

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We study the effect of particle losses on dynamical phase diffusion in a two-mode Bose–Einstein condensate. Starting with an equal-population coherent state, the single-particle coherence is solved analytically by the Monte Carlo wave function method. We show that decoherence slows down the loss of single-particle coherence. Moreover, we find that the bigger phase diffusion is, the more sensitive to the noise it becomes. Finally, we discuss the effect of the losses on the spin squeezing and show that the squeezing angle is robust to the decoherence even though the value of the squeezing is greatly affected.

topics: phase diffusion, Bose–Einstein condensate, particle losses, Monte Carlo wave function

1. Introduction

Bose–Einstein condensates (BECs) of dilute gases with weak interaction are widely used to study various condensed-matter models. In addition to studying the small perturbations of the ground state, BECs provide a platform for exploring nonequilibrium dynamics with remarkable control. In the BECs system, the spin squeezing can be generated due to the internal self-interaction [1–17], where the effective spins are collective variables that can be defined in terms of two different internal states of the atoms [18] or two orthogonal bosonic modes [19]. Although the self-interaction can generate spin squeezing, it also leads to phase diffusion of the BEC. It indicates a decay of single-particle decoherence [20–29] and thus restricts the applications of the BEC systems in high-precision measurement and quantum information processing. In the current experiments, phase diffusion can be observed by measuring the fringe visibility in atomic interference experiments [30].

In the past few years, many schemes have been proposed to suppress phase diffusion [31, 32]. However, the decoherence, which is unavoidable, may always affect phase diffusion. Under the current experimental conditions, the BECs are usually regarded as an open system coupled with the environment since the collisions of condensed atoms with the noncondensed thermal clouds, which lead to atom losses, are unavoidable. Many theoretical works have investigated the dissipation-induced effects in the open two-mode BECs [33–36] and showed that some changes in the dynamical properties of the condensate could not be negligible anymore. However, the decoherence may result in the enhancement of the quantum effect. It is similar to the phenomenon of the quantum Zeno effect [37–41], which results from the continuous projection onto relative number states and thus prevents the corresponding phase from taking a definite value. For example, dephasing of the quasimomentum modes suppressing phase diffusion was observed in the two-site Bose–Hubbard model with quantum Zeno limit [42]. The maximum spin squeezing in two-mode BECs can also be reached in the presence of particle losses [43]. In addition, the dissipation could even lead to enhancement of coherence in the open two-mode BEC with some specific conditions [44, 45].

In this paper, we investigate the phase diffusion in the two-mode BECs with particle losses. We discuss the dynamic behavior of the first-order coherence and the von Neumann entropy to describe the behavior of the phase diffusion. We give an analytical result of the single-particle coherence by the Monte Carlo wave function method and show that the presence of particle losses suppresses the phase diffusion. We also discuss the effect of the particle losses on the spin squeezing angle, which plays an important role in squeezing measurement and quantum metrology. We find the squeezing angle is robust to the particle losses even though the spin squeezing is greatly affected.

The paper is organized as follows: Sect. 2 introduces two-mode BECs and the master equation of the system with the particle losses. In Sect. 3, we discuss the single-particle coherence using the Monte Carlo wave function method. The effect of the particle losses on the spin squeezing and the squeezing angle are discussed in Sect. 4. Summary and conclusions are presented in the last section.

2. Theoretical model

We consider a two-component weakly interacting BEC consisting of N particles. The second quantized Hamiltonian of the system in the single-mode approximation reads [43]

$$\hat{H}_0 = \frac{\chi}{4}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})^2 + f(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}), \tag{1}$$

where the first term is induced by atom-atom collisions with χ being the nonlinear interaction strength. The second one in $\hat{H}_0$ is some function of the total atom number. Here, we note that the second term commutes with the density operator $\hat{\rho}$ of the system and can be omitted. Introducing the angular momentum operators, $\hat{J}_x = (\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})/2$, $\hat{J}_y = (\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})/(2\mathrm{i})$, $\hat{J}_z = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$, which obey the SU(2) Lie algebra, the Hamiltonian in (1) reduces to $\hat{H}_0 = \chi\hat{J}_z^2$.

In the practical experiments of cold atoms, particle losses are the unavoidable source of decoherence. In such a case, the dynamics of the system will obey the Markovian kinetic master equation,

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}\left[\hat{H}_0, \hat{\rho}\right] + \hat{\mathcal{L}}, \tag{2}$$

where $\hat{\rho}$ is the density matrix of the system, $\hat{\mathcal{L}} \equiv -\frac{1}{2}\sum_{\epsilon=a,b}\gamma_\epsilon(\hat{c}_\epsilon^\dagger\hat{c}_\epsilon\hat{\rho} + \hat{\rho}\hat{c}_\epsilon^\dagger\hat{c}_\epsilon - 2\hat{c}_\epsilon\hat{\rho}\hat{c}_\epsilon^\dagger)$ is the dissipation with γ_ϵ being the dissipation velocity of component ϵ. Here $\hat{c}_a = \hat{a}$ and $\hat{c}_b = \hat{b}$. For simplicity, we set $\gamma_a = \gamma_b = \gamma$ in the following discussion.

It is well known that the self-interaction χJ_z^2, the so-called one-axis twisting (OAT) effect, induces spin squeezing [1–4]. In fact, the self-interaction also leads to phase diffusion, which indicates a decay of the single-particle coherence. Such a kind of coherence can be measured by off-diagonal elements of the reduced single-particle density matrix elements

$$R_{ij} = \frac{1}{\langle\hat{N}\rangle}\left(\frac{\langle\hat{N}\rangle\mathbf{1}}{2} + \langle\hat{J}_x\rangle\sigma_x + \langle\hat{J}_y\rangle\sigma_y + \langle\hat{J}_z\rangle\sigma_z\right)_{ij}, \tag{3}$$

where $\mathbf{1}$ and $\sigma_{x,y,z}$ are the identity and Pauli matrices, respectively, $\hat{N} = \hat{a}^\dagger a + \hat{b}^\dagger b$, and $i, j = 1, 2$. Such coherence can be observed in the experiment by extracting the visibility of the Ramsey fringes [30]. In this paper, we focus on the effect of the particle losses on the first-order coherence R_{12}. We assume the initial state is prepared as the equal-population coherent states

$$|\Psi(0)\rangle = \frac{\left(\hat{a}^\dagger + \mathrm{e}^{\mathrm{i}\varphi}\hat{b}^\dagger\right)^N}{\sqrt{2^N N!}}|0\rangle, \tag{4}$$

where $|0\rangle$ is a vacuum state. Such an initial state can also be written as a coherent spin state $|\vartheta = \pi/2, \varphi\rangle$, which is defined as $|\vartheta,\varphi\rangle = \exp\left[\frac{\mathrm{i}}{2}\vartheta(\hat{J}_x\sin(\varphi) - \hat{J}_y\cos(\varphi))\,|j,j\rangle\right]$. For simplicity, we assume the initial relative phase $\varphi = 0$, and so that $\langle\Psi(0)|\hat{J}_x|\Psi(0)\rangle = N/2$, which means the collective spin initially along the x axis. In the absence of particle losses ($\gamma = 0$), we have [46, 47]

$$R_{12} = \frac{1}{2}\cos^{N-1}(\chi t) \simeq \frac{1}{2}\mathrm{e}^{-(t/t_d)^2}. \tag{5}$$

The diffusion time t_d is given by $\chi t_d = \sqrt{N/2}$. According to (5), the coherence R_{12} will decay to zero at $t_0 = \pi/(2\chi)$ and revive to maximum at $t_r = \pi/\chi$. Such a behavior exactly reflects the binomial Gaussian-like distribution of the occupation shown in (4).

3. Phase diffusion with Monte Carlo wave function approach

In the presence of the losses, the exact solutions of the master equation in (2) can be found by expanding it in the number Fock state basis and numerical integration. In order to get an analytical result of R_{12}, we adopt the Monte Carlo wave function approach [43, 48]. Firstly, we rewrite the master equation in the interaction picture with respect to $\hat{H}_0$, and then we get

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -\frac{1}{2}\sum_{\epsilon=a,b}\gamma\left(\tilde{c}_\epsilon^\dagger\tilde{c}_\epsilon\tilde{\rho} + \tilde{\rho}\tilde{c}_\epsilon^\dagger\tilde{c}_\epsilon - 2\tilde{c}_\epsilon\tilde{\rho}\tilde{c}_\epsilon^\dagger\right), \tag{6}$$

where $\tilde{\rho} = \mathrm{e}^{\mathrm{i}\hat{H}_0 t}\hat{\rho}\mathrm{e}^{-\mathrm{i}\hat{H}_0 t}$ and $\tilde{c}_\epsilon = \mathrm{e}^{\mathrm{i}\hat{H}_0 t}\hat{c}_\epsilon\mathrm{e}^{-\mathrm{i}\hat{H}_0 t}$. Next, we define an effective non-Hermitian Hamiltonian and the jump operator $\hat{J}_\epsilon$ as, respectively,

$$\hat{H}_{\mathrm{eff}} = -\frac{\mathrm{i}\gamma}{2}\left(\tilde{c}_a^\dagger\tilde{c}_a + \tilde{c}_b^\dagger\tilde{c}_b\right), \tag{7}$$

and

$$\hat{J}_\epsilon = \sqrt{\gamma}\,\tilde{c}_\epsilon. \tag{8}$$

The master equation in (6) reduces to

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -\mathrm{i}\left(\hat{H}_{\mathrm{eff}}\tilde{\rho} - \tilde{\rho}\hat{H}_{\mathrm{eff}}\right) + \sum_{\epsilon=a,b}\hat{J}_\epsilon\tilde{\rho}\hat{J}_\epsilon^\dagger, \tag{9}$$

and the last term is usually interpreted as the one responsible for the so-called quantum jumps [43, 48].

In the process of the time evolution, we assume that a small fraction of atoms will be lost, and thus we consider the parameters χ and γ as constants. Under the Monte Carlo wave function approach, the state evolution in a single quantum trajectory is a sequence of random quantum jumps at times t_k with the non-unitary Hamiltonian evolutions of duration τ_k, the detailed process of the state evolution is given by [43]

$$|\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}(t-t_k)}J_{\epsilon_k}(t_k)\mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}\tau_k J_{\epsilon_k-1}}(t_{k-1})$$
$$\times \ldots J_{\epsilon_{k-1}}(t_1)\mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}\tau_1}|\Psi(0)\rangle, \tag{10}$$

where $|\Psi(0)\rangle$ is the initial state of the system. For an arbitrary observable operator $\hat{A}$, the expectation value can be obtained by averaging all possi-

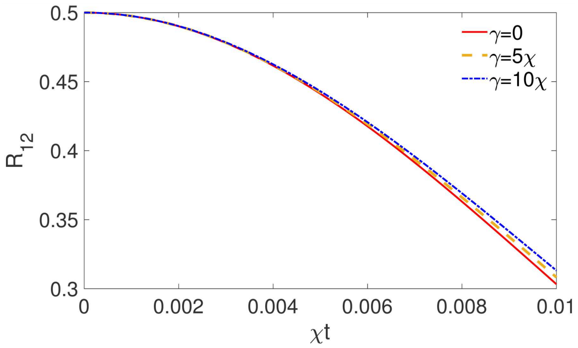

Fig. 1. The single-particle coherence R_{12} as a function of rescaled time χt, starting from the coherent state $|\pi/2, 0\rangle$ with different γ. The initial particle number of the system is $N = 10^4$.

ble stochastic realizations, including the times and number of quantum jumps. In the calculation, each trajectory is weighted by its probability, and then we get

$$\langle \hat{A} \rangle = \sum_k \int_{\substack{0<t_1<t_2< \\ \dots t_k<t}} \mathrm{d}t_1\,\mathrm{d}t_2 \dots \mathrm{d}t_k \sum_k \langle \Psi(t)|\hat{A}\,|\Psi(t)\rangle\,. \tag{11}$$

With the initial state $|\pi/2, 0\rangle$ and constant loss rate approximation, we get $\langle \hat{N} \rangle = N\,\mathrm{e}^{-\gamma t}$ and the exact result of the single-particle coherence

$$R_{12} = \frac{1}{2}\left[\frac{\mathrm{e}^{-\gamma t}\Big(\gamma\chi\sin(\chi t){+}\chi^2\cos(\chi t)\Big) + \gamma^2}{\gamma^2 + \chi^2}\right]^{N-1}. \tag{12}$$

In the limit of $N(\chi t)^2 \ll 1$, a more insightful expression for R_{12} is obtained as

$$R_{12} \approx \frac{1}{2}\cos^{N-1}(\chi t) + \frac{1}{6}N\gamma\chi^2 t^3 = R_{12}^{(0)}\left[1 + \frac{N\gamma\chi^2 t^3}{3R_{12}^{(0)}}\right], \tag{13}$$

where $R_{12}^{(0)} = \frac{1}{2}\cos^{N-1}(\chi t)$ is the single-particle coherence in the absence of particle losses. The second term describes the noise added to the diffusion. This shows that (i) the fact that only a small fraction of atoms are lost at a short time does not imply that the correction on the phase diffusion due to losses is small; (ii) the bigger phase diffusion is, the more sensitive R_{12} is to the losses. As shown in Fig. 1, we plot R_{12} as a function of χt for different γ, and we can see that R_{12} becomes large as γ increases. It means that the presence of the particle losses lightly suppresses the phase diffusion.

To further investigate the phase diffusion, the single-particle coherence is also evaluated by using the von Neumann entropy of the state ρ, which is defined as

$$S \equiv -\mathrm{Tr}\big(\rho\ln(\rho)\big). \tag{14}$$

With the reduced single-particle density matrix, S becomes $S = -\mathrm{Tr}(R\ln(R))$. According to (3), we obtain

$$S = -\frac{1}{2}\ln\left[\frac{(1{+}2R_{12})^{(1+2R_{12})}(1{-}2R_{12})^{(1-2R_{12})}}{4}\right]. \tag{15}$$

In Fig. 2, we plot the von Neumann entropy as a function of χt with different γ. It clearly shows that S decreases a bit as the parameter γ increases, which indicates that the presence of the losses slightly slows down the phase diffusion.

4. Effect of the particle losses on the spin squeezing

It is well known that the system with the Hamiltonian $\hat{H}_0$ can generate spin-squeezed states, which is a good resource for quantum metrology. Even though the spin squeezing was intensively studied in the past, the spin squeezing angle is rarely discussed. In fact, the squeezing angle is very important for the measurement of squeezing and in quantum metrology. Starting with the initial state $|\pi/2, 0\rangle$, the mean spin direction along the x-direction and the spin squeezing is quantified by the parameter [3, 17]

$$\xi^2 = \frac{\langle \hat{N} \rangle \min(\Delta\hat{S}^2_{n_\perp})}{\langle \hat{S}_x \rangle^2}, \tag{16}$$

where $n_\perp$ refers to an axis on the (y, z) plane and the minimization is over all directions $n_\perp$. The operator $\hat{S}_{n_\perp}$ is defined as $\hat{S}_{n_\perp} = \cos(\theta)\hat{S}_y + \sin(\theta)\hat{S}_z$ with θ being the squeezing angle and given by

$$\theta = \tfrac{1}{2}\arctan\big(\mathcal{B}/\mathcal{A}\big), \tag{17}$$

where $\mathcal{A} = \langle \hat{S}_y^2 - \hat{S}_z^2 \rangle$ and $\mathcal{B} = \langle \hat{S}_y\hat{S}_z + \hat{S}_z\hat{S}_y \rangle$. According to the Monte Carlo wave function method, we obtain

$$\mathcal{A} = \frac{N(N-1)}{8}\,\mathrm{e}^{-2\gamma t}\left[1 - \left(\frac{\gamma^2 + \mathrm{e}^{-\gamma t}\big(4\chi^2\cos(2\chi t) + 2\gamma\chi\sin(2\chi t)\big)}{\gamma^2 + 4\chi^2}\right)^{N-2}\right] \tag{18}$$

and

$$\mathcal{B} = \frac{N(N-1)}{2}\sin(\chi t)\,\mathrm{e}^{-2\gamma t}\left(\,)\frac{\gamma^2 + \mathrm{e}^{-\gamma t}\big(\chi^2\cos(\chi t) + \gamma\chi\sin(\chi t)\big)}{\gamma^2 + \chi^2}\right)^{N-2}. \tag{19}$$

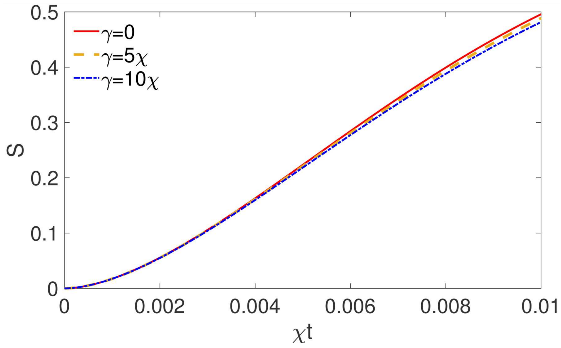

Fig. 2. Single-particle von Neumann entropy S as a function of the rescaled time χt for different γ with the initial coherent state $|\pi/2, 0\rangle$ and $N = 10^4$.

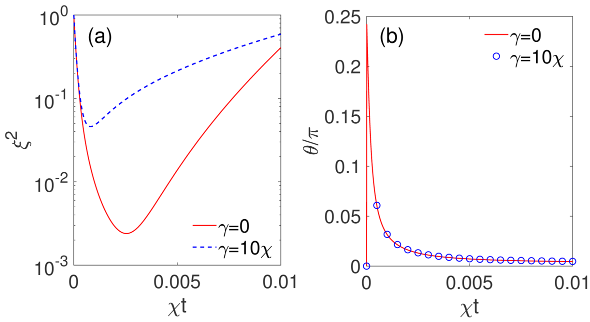

Fig. 3. (a) Spin squeezing and (b) spin squeezing angle as a function of the rescaled time, starting from the coherent state for different γ with $N = 10^4$.

In the limit of $N(\chi t)^2 \ll 1$, we obtain

$$\mathcal{B}/\mathcal{A} \approx \frac{2}{N\chi t} - \chi t + \frac{2}{3}\gamma\chi t^2. \tag{20}$$

Here, we notice that the last term in (20) is much smaller than the first and second terms. Thus the particle losses will not affect the squeezing angle. In Fig. 3, we plot ξ^2 and the squeezing angle θ as a function of χt. Obviously, the spin squeezing is greatly affected by particle losses (see Fig. 3a), while in the case of the squeezing angle, we can find it is robust to such decoherence.

5. Conclusion

In summary, we have studied the collisional phase diffusion in a two-mode Bose–Einstein condensate with particle losses. We employ the Monte Carlo function method to analytically solve the single-particle coherence and von Neumann entropy. We show that the particle losses suppress the phase diffusion. In addition, we also find that the bigger phase diffusion is, the more sensitive the single-particle coherence is to the losses. Finally, we discuss the effect of the particle losses on the spin squeezing angle and reveal that the squeezing angle is robust to the particle losses.

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