Proceedings of the 4th Jagiellonian Symposium on Advances in Particle Physics and Medicine

Multichannel Decay: Alternative Derivation of the i-th Channel Decay Probability

F. Giacosa[a,b,*]

[a] *Institute of Physics, Jan Kochanowski University, Uniwersytecka 7, 25-406, Kielce, Poland*
[b] *Institute for Theoretical Physics, J.W. Goethe University, Max-von-Laue-Str. 1, 60438 Frankfurt, Germany*

Doi: 10.12693/APhysPolA.142.436 [*]e-mail: fgiacosa@ujk.edu.pl

In the study of decays, it is quite common that an unstable quantum state/particle has multiple distinct decay channels. In this case, besides the survival probability $p(t)$, also the probability $w_i(t)$ that a decay occurs between $(0, t)$ in the i-th channel is a relevant object. The general form of the function $w_i(t)$ was recently presented in *PLB* **831**, 137200 (2022). Here, we provide a novel and detailed "joint" derivation of both $p(t)$ and $w_i(t)$. As it is well known, $p(t)$ is not an exponential function; similarly, $w_i(t)$ is not one either. In particular, the ratio w_i/w_j (for $i \neq j$) is not a simple constant as it would be in the exponential limit. The functions $w_i(t)$ and their mutual ratios may therefore represent a novel tool to study the non-exponential nature of the decay law.

topics: decay law, unstable particles, multichannel decay

1. Introduction

In the study of unstable states, both in quantum mechanics (QM) and in quantum field theory (QFT), the survival probability $p(t)$ (the probability that the state formed at $t = 0$ has not decayed yet at a later time $t > 0$) is of crucial importance [1–15]. Yet, usually unstable states can decay in more than a single decay channel [16]. Then, an equally useful and relevant object is the decay probability $w_i(t)$ that the decay has occurred between 0 and $t > 0$ in a certain i-th channel. Of course, the equality

$$p(t) + \sum_{i=1}^{N} w_i(t) = 1 \tag{1}$$

must hold for each t because at any given time the state has either decayed in one of the N possible channels or it is undecayed (*tertium non-datur*). As it is well established, the survival probability $p(t)$ can be well approximated with an exponential expression $p(t) \simeq \mathrm{e}^{-t/\tau}$, but the latter is not exact as shown by direct and indirect experimental analyses [17–21]. Since $p(t)$ is not an exponential, it follows that the functions $w_i(t)$ are also not such.

The explicit form for $w_i(t)$ was recently derived in [22]. The preliminary approximate expression was previously put forward in [11]. Here we present the novel joint determination of $p(t)$ and $w_i(t)$ that makes use of the Lippmann–Schwinger equation at the level of operators, see e.g. [23].

2. Evaluation of $p(t)$ and $w_i(t)$

Let H be the Hamiltonian of a physical system that contains an unstable state $|S\rangle$. We assume that H can be split into $H = H_0 + H_{int}$ with $H_{int} = \sum_{i=1}^{N} H_i$, where H_i is responsible for the i-th decay channel. The orthogonal–normalized–complete (ONC) eigenstates of the non-interacting Hamiltonian H_0 are $\{|S\rangle, |E,i\rangle\} : H_0 |S\rangle = M |S\rangle$, $H_0 |E,i\rangle = E |E,i\rangle$ with $E \geq E_{\mathrm{th},i}$, where $E_{\mathrm{th},i}$ is the energy threshold of the i-th channel; here, we assume as the definition that $E_{\mathrm{th},1} \leq E_{\mathrm{th},2} \leq \ldots \leq E_{\mathrm{th},N}$. The ONC conditions of the underlying Hilbert space read

$$\langle S|S\rangle = 1, \quad \langle S|E,i\rangle = 0,$$

$$\langle E,i|E',j\rangle = \delta_{ij}\, \delta(E - E'), \tag{2}$$

and

$$|S\rangle \langle S| + \sum_{i=1}^{N} \int_{E_{\mathrm{th},i}}^{\infty} \mathrm{d}E \; |E,i\rangle \langle E,i| = 1. \tag{3}$$

The decays $|S\rangle \to |E,i\rangle$ are encoded in the matrix elements

$$\langle S|H_j|E,j\rangle = \delta_{ij} \sqrt{\frac{\Gamma_i(E)}{2\pi}}, \tag{4}$$

where $\Gamma_i(E)$ is the i-th decay width, which generally is a function of energy (it reduces to a constant in the exponential limit or the Breit–Wigner (BW) limit [24–26]). (Note, in (4) the sum over other d.o.f. such as spin and momenta has been implicitly taken into account; the functions $\Gamma_i(E)$ are assumed to be known for a specific quantum system, even though usually this is not a simple task.) An explicit expression for H that fulfills the properties listed above can be written in the form of the Friedrichs–Lee Hamiltonian [27, 28] (for various applications, see [29–41] and refs. therein)

$$H = H_0 + H_{int}, \tag{5}$$

with

$$H_0 = M\,|S\rangle\,\langle S| + \sum_{i=1}^{N}\int_{E_{\mathrm{th},i}}^{\infty} \mathrm{d}E\,E\,\,|E,i\rangle\,\langle E,i|\,, \tag{6}$$

$$H_{int} = \sum_{i=1}^{N}\int_{E_{\mathrm{th},i}}^{\infty} \mathrm{d}E\,\sqrt{\frac{\Gamma_i(E)}{2\pi}}\Big(|E,i\rangle\langle S|+|S\rangle\langle E,i|\Big). \tag{7}$$

Note, H actually represents an infinite class of models, since it depends on the functions $\Gamma_i(E)$.

The quantity $U(t) = \mathrm{e}^{-\mathrm{i}Ht/\hbar}$ is a well-known time evolution operator. In our case, we are interested in the evaluation of the survival probability amplitude and the i-th channel decay amplitude

$$\langle S|U(t)|S\rangle\,, \quad \langle E,i|U(t)|S\rangle\,. \tag{8}$$

In order to accomplish it, let us introduce the operator $F(t)$ (F for "future") as

$$F(t) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E\;\mathrm{e}^{-\mathrm{i}Et/\hbar}}{E-H+\mathrm{i}\varepsilon} = \begin{cases} U(t), & \text{for } t>0,\\ 0, & \text{for } t<0. \end{cases} \tag{9}$$

The previous equation should be understood as an operatorial equation, i.e., for an arbitrary eigenstate $|\Psi_0\rangle$ with $H\,|\Psi_0\rangle = E_0\,|\Psi_0\rangle$, one has

$$F(t)\,|\Psi_0\rangle = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E\;\mathrm{e}^{-\mathrm{i}Et/\hbar}}{E-H+\mathrm{i}\varepsilon}\,|\Psi_0\rangle =$$

$$\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E\;\mathrm{e}^{-\mathrm{i}Et/\hbar}}{E-E_0+\mathrm{i}\varepsilon}\,|\Psi_0\rangle = \begin{cases} \mathrm{e}^{-\mathrm{i}E_0t/\hbar}\,|\Psi_0\rangle\,, \\ \quad \text{for } t>0,\\ 0, \quad \text{for } t<0 \end{cases} \tag{10}$$

where the last equation is obtained by integrating on the lower half-plane of the complex variable E for $t>0$ and on the upper half-plane for $t<0$. Formally, $F(t)$ is not defined for $t=0$ since the integral $\int_{-\infty}^{+\infty}\mathrm{d}E\,\frac{1}{E-E_0+\mathrm{i}\varepsilon}$ does not converge. Now, we summarize (10) by writing

$$F(t) = \theta(t)\,U(t) \tag{11}$$

together with the choice $\theta(0)=\frac{1}{2}$, thus $F(0)=\frac{1}{2}$. Similarly, let us introduce the operator $P(t)$ (P for "past")

$$P(t) = F^*(-t) = -\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E\;\mathrm{e}^{-\mathrm{i}Et/\hbar}}{E-H-\mathrm{i}\varepsilon} =$$

$$\begin{cases} 0, & \text{for } t>0,\\ U(t), & \text{for } t<0 \end{cases} \tag{12}$$

hence $P(t) = \theta(-t)U(t)$ and $P(0)=\frac{1}{2}$. For each time t (including $t=0$) we get a consistent result

$$U(t) = \mathrm{e}^{-\mathrm{i}Ht/\hbar} = F(t) + P(t) =$$

$$\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E\;\mathrm{e}^{-\mathrm{i}Et/\hbar}}{E-H+\mathrm{i}\varepsilon} - \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E\;\mathrm{e}^{-\mathrm{i}Et/\hbar}}{E-H-\mathrm{i}\varepsilon} =$$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E\,\varepsilon\;\mathrm{e}^{-\mathrm{i}Et/\hbar}}{(E-H)^2+\varepsilon^2} = \int_{-\infty}^{+\infty} \mathrm{d}E\,\delta(E-H)\,\mathrm{e}^{-\mathrm{i}Et/\hbar}. \tag{13}$$

Next, we return to the time evolution of the expectation values of (8). To evaluate them, we need to determine propagators defined as

$$G_S(E) = \left\langle S\left|\frac{1}{E-H+\mathrm{i}\varepsilon}\right|S\right\rangle,$$

$$T_i(E',E) = \left\langle E',i\left|\frac{1}{E-H+\mathrm{i}\varepsilon}\right|S\right\rangle. \tag{14}$$

Namely, once these quantities are known, the time evolution is obtained by using the "future" representation $F(t)$ of (9). For this, we write down the operatorial Lippmann–Schwinger equation

$$\frac{1}{E-H+\mathrm{i}\varepsilon} = \frac{1}{E-H_0+\mathrm{i}\varepsilon}\left[1 + H_{int}\frac{1}{E-H+\mathrm{i}\varepsilon}\right], \tag{15}$$

which can be proven considering the operator O defined as (note that when dealing with the operators, the order is important)

$$O = (E-H_0+\mathrm{i}\varepsilon)\left[\frac{1}{E-H+\mathrm{i}\varepsilon} - \frac{1}{E-H_0+\mathrm{i}\varepsilon}\right] =$$

$$(E-H_0+\mathrm{i}\varepsilon)\frac{1}{E-H+\mathrm{i}\varepsilon} - 1 =$$

$$(E-H_0+\mathrm{i}\varepsilon)\frac{1}{E-H+\mathrm{i}\varepsilon} - (E-H+\mathrm{i}\varepsilon)\frac{1}{E-H+\mathrm{i}\varepsilon} =$$

$$(H-H_0)\frac{1}{E-H+\mathrm{i}\varepsilon} = H_{int}\,\frac{1}{E-H+\mathrm{i}\varepsilon}. \tag{16}$$

Then, the propagator of the unstable state S reads

$$G_S(E) = \left\langle S\left|\frac{1}{E-H+\mathrm{i}\varepsilon}\right|S\right\rangle = \frac{1}{E-M+\mathrm{i}\varepsilon} + \frac{1}{E-M+\mathrm{i}\varepsilon}\left\langle S\left|H_{int}\frac{1}{E-H+\mathrm{i}\varepsilon}\right|S\right\rangle =$$

$$\frac{1}{E-M+\mathrm{i}\varepsilon} + \frac{1}{E-M+\mathrm{i}\varepsilon}\sum_{i=1}^{N}\int_{E_{\mathrm{th},i}}^{\infty}\mathrm{d}E'\,\sqrt{\frac{\Gamma_i(E')}{2\pi}}\,T_i(E',E), \tag{17}$$

while the propagators for the transitions $|S\rangle \to |E,i\rangle$ are given by

$$T_i(E',E) = \left\langle E',i \left| \frac{1}{E-H+\mathrm{i}\varepsilon} \right| S \right\rangle = \frac{1}{E-E'+i\varepsilon} \left\langle E',i \left| H_{int} \frac{1}{E-H+\mathrm{i}\varepsilon} \right| S \right\rangle = \sqrt{\frac{\Gamma_i(E')}{2\pi}} \frac{G_S(E)}{E-E'+\mathrm{i}\varepsilon}. \tag{18}$$

Plugging $T_i(E',E)$ into (17), we obtain the Dyson–Schwinger equation of the S propagator

$$G_S(E) = \frac{1}{E-M+\mathrm{i}\varepsilon} - \frac{\Pi(E)\, G_S(E)}{E-M+\mathrm{i}\varepsilon}, \tag{19}$$

where the total self-energy $\Pi(E)$ and the partial self-energies $\Pi_i(E)$ read, respectively,

$$\Pi(E) = \sum_{i=1}^{N} \Pi_i(E) \tag{20}$$

and

$$\Pi_i(E) = -\int_{E_{\mathrm{th},i}}^{\infty} \frac{\mathrm{d}E'}{2\pi} \frac{\Gamma_i(E')}{E-E'+\mathrm{i}\varepsilon}, \tag{21}$$

for which $\mathrm{Im}(\Pi_i(E)) = \Gamma_i(E)/2$ (optical theorem)†. Then,

$$G_S(E) = \frac{1}{E-M+\Pi(E)+\mathrm{i}\varepsilon} \tag{22}$$

is the state S propagator being searched. As it is well known, this expression can be also obtained by performing the standard Dyson resummation, see e.g. [39]. We thus have provided a simple alternative derivation of this object.

The propagator $G_S(E)$ can be also rewritten as

$$G_S(E) = \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}E' \, \frac{d_S(E')}{E-E'+\mathrm{i}\varepsilon} \tag{23}$$

with

$$d_S(E) = -\frac{1}{\pi}\, \mathrm{Im}(G_S(E)) = \frac{\Gamma(E)}{2\pi} \left|G_S(E)\right|^2 . \tag{24}$$

The function $d_S(E)$ is a correctly normalized energy distribution (or spectral function) of the unstable state ($\mathrm{d}E \; d_S(E)$ is the probability that the state S has an energy between $(E, E+\mathrm{d}E)$). Then one proceeds as usual to determine the survival probability amplitude

$$a(t) = \langle S|\, U(t)\, |S\rangle \overset{t\geq 0}{=} \langle S|\, F(t)\, |S\rangle = \int_{-\infty}^{+\infty} \frac{\mathrm{i}\,\mathrm{d}E\, G_S(E)\mathrm{e}^{-\mathrm{i}Et/\hbar}}{2\pi} = \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}E\, d_S(E)\mathrm{e}^{-\mathrm{i}Et/\hbar}. \tag{25}$$

This is indeed the amplitude that starting with $|S\rangle$, we still have $|S\rangle$ at the time $t > 0$. The survival probability

$$p(t) = \left| \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}E\, d_S(E)\mathrm{e}^{-\mathrm{i}Et/\hbar} \right|^2 \tag{26}$$

emerges. This is indeed the starting point of many studies on the decay law [1–15].

As a consequence of the adopted formalism, once $G_S(E)$ is fixed, also $T_i(E',E)$ in (18) is determined. We then calculate the probability that the decay takes place in the i-th channel between 0 and $t > 0$ as

$$w_i(t) = \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}E' \left| \langle E',i|\, U(t)\, |S\rangle \right|^2 \overset{t\geq 0}{=} \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}E' \left| \langle E',i|\, F(t)\, |S\rangle \right|^2 = \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}E' \left| \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}E\, T_i(E',E)\mathrm{e}^{-\mathrm{i}Et/\hbar} \right|^2$$
$$\times \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}E' \frac{\Gamma_i(E')}{2\pi} \left| \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}E \frac{G_S(E)}{E-E'+\mathrm{i}\varepsilon}\, \mathrm{e}^{-\mathrm{i}Et/\hbar} \right|^2 . \tag{27}$$

This is indeed the expression for the quantity $w_i(t)$ that we were looking for. However, it still involves the complex propagator $G_S(E)$, so it is better to recast it into a form that is simpler for practical applications. By introducing the spectral representation of (23) of the form

$$\frac{\mathrm{i}}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{d}E}{E-E'+\mathrm{i}\varepsilon}\, G_S(E)\mathrm{e}^{-\mathrm{i}Et/\hbar} = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}E \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}y\, \frac{d_S(y)\, \mathrm{e}^{-\mathrm{i}Et/\hbar}}{(E-E'+\mathrm{i}\varepsilon)(E-y+\mathrm{i}\varepsilon)} = \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}y\, \frac{d_S(y)}{E'-y} \left[\mathrm{e}^{-\mathrm{i}E't/\hbar} - \mathrm{e}^{-\mathrm{i}yt/\hbar} \right] \tag{28}$$

(note, the integrand contains no singularity), we obtain the expression [22]

$$w_i(t) = \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}E' \, \frac{\Gamma_i(E')}{2\pi} \times \left| \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}y\, \frac{d_S(y)}{E'-y} \left[\mathrm{e}^{-\mathrm{i}E't/\hbar} - \mathrm{e}^{-\mathrm{i}y't/\hbar} \right] \right|^2 . \tag{29}$$

[†1] It is often common to perform the replacements $\Pi_i(E) \to \Pi_i(E) + C_i$, where the latter are real subtraction constants such that $\mathrm{Re}(\Pi_i(M)) = 0$. In this way, the bare mass M of the unstable state is left unchanged by quantum the fluctuations.

This quantity can be calculated numerically when the functions $\Gamma_i(E)$ (and thus also $d_S(E)$) are known. Roughly speaking, it is ready to be used, just "plug in and calculate".

There is another useful way to express $w_i(t)$ mentioned in [22]. By introducing

$$I(t) = \frac{1}{\hbar}\int_0^t \mathrm{d}t'\, a(t')\,\mathrm{e}^{\mathrm{i}\,E't'/\hbar} = \int_0^t \mathrm{d}t' \left[\int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}y\, d_S(y)\,\mathrm{e}^{-\mathrm{i}\,yt'/\hbar}\right] \mathrm{e}^{\mathrm{i}\,E't'/\hbar} = \frac{1}{\hbar}\int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}y\, d_S(y) \int_0^t \mathrm{d}t'\ \mathrm{e}^{\mathrm{i}\,(E'-y)t'/\hbar} \int_{-\infty}^{+\infty} \frac{\mathrm{d}y\, d_S(y)}{\mathrm{i}\,(E'-y)} \left[\mathrm{e}^{\mathrm{i}\,(E'-y)t/\hbar} - 1\right] = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\,E't/\hbar} \int_{E_{\mathrm{th},1}}^{+\infty} \mathrm{d}y\, \frac{d_S(y)}{E'-y} \left[\mathrm{e}^{-\mathrm{i}\,E't/\hbar} - \mathrm{e}^{\mathrm{i}\,y/\hbar}\right], \tag{30}$$

we find

$$w_i(t) = \int_{E_{\mathrm{th},i}}^{+\infty} \mathrm{d}E'\, \frac{\Gamma_i(E')}{2\pi} \left| \int_0^t \mathrm{d}t'\, \frac{a(t')\,\mathrm{e}^{\mathrm{i}\,E't'/\hbar}}{\hbar} \right|^2. \tag{31}$$

Once $a(t)$ is calculated (a necessary step for getting the survival probability $p(t)$), $w_i(t)$ can be numerically evaluated from the previous expression.

Next, we recall some relevant properties and extensions.

- We can prove (1) by using the formal expression for the transitions $w_i(t)$ in (27) and the completeness relation of (3)

$$\sum_{i=1}^N w_i(t) = \sum_{i=1}^N \int_{E_{th,i}}^{+\infty} \mathrm{d}E'\ |\langle E', i|\, U(t)\, |S\rangle|^2 = \langle S|\, U^\dagger(t) \Big[\sum_{i=1}^N \int_{E_{th,i}}^{+\infty} \mathrm{d}E'\ |E', i\rangle \langle E', i| \Big] U(t)\, |S\rangle = \langle S|\, U^\dagger(t) \big[1 - |S\rangle \langle S| \big] U(t)\, |S\rangle = 1 - p(t). \tag{32}$$

It is an important consistency check for the correctness of the obtained results.

- The exponential (or Breit–Wigner) limit [24–26] is obtained for $\Gamma_i = \text{const}$ and $\Gamma = \sum_{i=1}^N \Gamma_i$ (no energy dependence). The survival probability $p(t)$ and the decay probabilities $w_i(t)$ reduce to [11, 22]

$$p(t) = \mathrm{e}^{-\Gamma/\hbar}, \quad w_i(t) = \frac{\Gamma_i}{\Gamma}\left(1 - \mathrm{e}^{-\Gamma t/\hbar}\right), \quad w_i(t) \to \frac{w_i(t)}{w_j(t)} = \frac{\Gamma_i}{\Gamma_j} = \text{const}. \tag{33}$$

- In the general case, the ratio $w_i/w_j \neq \text{const}$ (for $i \neq j$). This fact is shown in [22] with the widths $\Gamma_i(E) = 2g_i^2\sqrt{E - E_{th,i}}/(E^2 + \Lambda^2)$ inspired by the expressions derived in [42] in the case of hydrogen-like atoms. In [11], w_i/w_j was also shown to be not a simple constant (in the framework of an approximate solution) for various choices of $\Gamma_i(E)$.
- The related interesting quantity is $h_i(t) = w_i'(t)$, where $h_i(t)\,\mathrm{d}t$ is the probability that the decay takes place in the i-th channel in the interval $(t, t + \mathrm{d}t)$. In the BW limit, $h_i(t)/h_j(t) = \Gamma_i/\Gamma_j = \text{const}$, but this generally does not apply [11, 22].
- In [43], the two-channel decay was studied by in the framework of the asymmetric double-delta potential $V(x) = V_0\big(\delta(x-a) + k\delta(x+a)\big)$, where $k \neq 1$ means that two channels were represented by tunneling to "left" and to "right". The numerical accurate solutions of the Schrödinger equation clearly shown that $w_R(t)/w_L(t)$ as well as $h_R(t)/h_L(t)$ (where R stays for the right and L for the left) are not constant.
- The results can be extended to QFT. For this, the variable E must be replaced by $s = E^2$ (for the relativistic version of the Friedrichs–Lee approach, see e.g. [44–46]). The propagator reads $G_S(s) = [s - M^2 + \Pi(s) + \mathrm{i}\varepsilon]^{-1}$, where $\Pi(s) = \sum_{i=1}^N \Pi_i(s)$ (with $\mathrm{Im}(\Pi_i(s)) = \sqrt{s}\Gamma_i(s)$) is the sum of the self energies for the N distinct decay channels. The spectral function is $d_S(s) = -\frac{1}{\pi}\,\mathrm{Im}(G_S(s))$ (e.g. [47, 48]). The survival probability $p(t)$ takes an analogous form of (25) (e.g. [49, 50]

$$p^{\mathrm{QFT}}(t) = \left| \int_{s_{th,1}}^{+\infty} \mathrm{d}s\, d_S(s)\,\mathrm{e}^{-\mathrm{i}\sqrt{s}\,t/\hbar} \right|^2, \tag{34}$$

while the partial decay probability $w_i(t)$ reads

$$w_i^{\mathrm{QFT}}(t) = \int_{s_{th,i}}^{+\infty} \mathrm{d}s\, \frac{\sqrt{s}\Gamma_i(s)}{\pi} \left| \int_{s_{th,1}}^{+\infty} \mathrm{d}s'\, d_S(s') \left(\frac{\mathrm{e}^{-\mathrm{i}\sqrt{s}\,t/\hbar} - \mathrm{e}^{-\mathrm{i}\sqrt{s'}\,t/\hbar}}{s - s'} \right) \right|^2. \tag{35}$$

This expression can be calculated numerically once the functions $\Gamma_i(s)$ are known.

- In QFT, there is no BW limit and no exponential decay (the threshold is always present because $s \geq 0$). Setting $\Gamma_i(s)$ to a constant leads to some inconsistencies. An interesting model, discussed in [51], postulates $\Pi_i(s) = \mathrm{i}\tilde{\Gamma}_i\sqrt{s - s_{\mathrm{th},i}}$ for which $\Gamma_i(s) = \tilde{\Gamma}_i\sqrt{\frac{1}{s}(s - s_{\mathrm{th},i})}\,\theta(s - s_{\mathrm{th},i})$ (which reduces to a constant for large s). Despite its simplicity, it allows the spectral functions of various broad hadrons to be fitted quite well. The function $w_i(t)$ turns out to be, as expected, non-exponential, in agreement with the QM case.

3. Conclusions

In this work, we presented a novel and simple way to obtain the expressions of the survival probability $p(t)$ and the decay probability into the i-th channel $w_i(t)$ by using the Lippmann–Schwinger equation at the level of operators. The propagator for the state S and the transition propagator for S into any decay product are intertwined. In this way, $p(t)$ and $w_i(t)$ naturally emerge, and the results coincide with the ones shown in [22]. In the future, the study of $w_i(t)$ in various physical systems is planned.

Acknowledgments

The author thanks L. Tinti, G. Pagliara and S. Mrówczyński for stimulating and useful discussions. Financial support from the OPUS project 2019/33/B/ST2/00613 is acknowledged.

References

[1] L. Fonda, G.C. Ghirardi, A. Rimini, *Rep. Prog. Phys.* **41**, 587 (1978).

[2] L.A. Khalfin, *Zh. Eksp. Teor. Fiz.* **33**, 1371 (1957); *Sov. Phys. JETP* **6**, 1053 (1958)).

[3] R.G. Winter, *Phys. Rev.* **123**, 1503 (1961).

[4] J. Levitan, *Phys. Lett. A* **129**, 267 (1988).

[5] D.A. Dicus, W.W. Repko, R.F. Schwitters, T.M. Tinsley, *Phys. Rev. A* **65**, 032116 (2002).

[6] M. Peshkin, A. Volya, V. Zelevinsky, *EPL* **107**, 40001 (2014).

[7] G. García-Calderón, R. Romo, *Phys. Rev. A* **93**, 022118 (2016).

[8] T. Koide, F. M. Toyama, *Phys. Rev. A* **66**, 064102 (2002).

[9] F.V. Pepe, P. Facchi, Z. Kordi, S. Pascazio, *Phys. Rev. A* **101**, 013632 (2020).

[10] A. Kofman and G. Kurizki, *Nature* **405**, 546 (2000).

[11] F. Giacosa, *Found. Phys.* **42**, 1262 (2012).

[12] F. Giacosa, *Phys. Rev. A* **88**, 052131 (2013).

[13] K. Raczynska, K. Urbanowski, *Acta Phys. Pol. B* **49**, 1683 (2018).

[14] D.F. Ramírez Jiménez, N.G. Kelkar, *Phys. Rev. A* **104**, 022214 (2021).

[15] D.F. Ramírez Jiménez, N.G. Kelkar, *J. Phys. A* **52**, 055201 (2019).

[16] R.L. Workman, V.D. Burkert, V. Crede et al. (Particle Data Group), *Prog. Theor. Exp. Phys.* **2022**, 083C01 (2022).

[17] S.R. Wilkinson, C.F. Bharucha, M.C. Fischer, K.W. Madison, P.R. Morrow, Q. Niu, B. Sundaram, M.G. Raizen, *Nature* **387**, 575 (1997).

[18] M.C. Fischer, B. Gutierrez-Medina, M.G. Raizen, *Phys. Rev. Lett.* **87**, 040402 (2001).

[19] N.G. Kelkar, M. Nowakowski, K.P. Khemchandani, *Phys. Rev. C* **70**, 024601 (2004).

[20] C. Rothe, S.I. Hintschich, A.P. Monkman, *Phys. Rev. Lett.* **96**, 163601 (2006).

[21] A. Crespi, F.V. Pepe, P. Facchi, F. Sciarrino, P. Mataloni, H. Nakazato, S. Pascazio, R. Osellame, *Phys. Rev. Lett.* **122**, 130401 (2019).

[22] F. Giacosa, *Phys. Lett. B* **831**, 137200 (2022).

[23] H. Müther, O.A. Rubtsova, V.I. Kukulin, V.N. Pomerantsev, *Phys. Rev. C* **94**, 024328 (2016).

[24] V. Weisskopf, E.P. Wigner, *Z. Phys.* **63**, 54 (1930).

[25] V. Weisskopf, E. Wigner, *Z. Phys.* **65**, 18 (1930).

[26] G. Breit, in: *Handbuch der Physik*, vol. 8/41/1, Springer, Berlin 1959.

[27] K.O. Friedrichs, *Commun. Pure Appl. Math.* **1**, 361 (1948).

[28] T.D. Lee, *Phys. Rev.* **95**, 1329 (1954).

[29] C.B. Chiu, E.C.G. Sudarshan, G. Bhamathi, *Phys. Rev. D* **46**, 3508 (1992).

[30] E.T. Jaynes, F.W. Cummings, *Proc. IEEE.* **51**, 89 (1963).

[31] O. Civitarese, M. Gadella, *Phys. Rep.* **396**, 41 (2004).

[32] A.G. Kofman, G. Kurizki, B. Sherman, *J. Modern Opt.* **41**, 353 (1994).

[33] Z.W. Liu, W. Kamleh, D.B. Leinweber, F.M. Stokes, A.W. Thomas, J.J. Wu, *Phys. Rev. Lett.* **116**, 082004 (2016).

[34] M. Scully, M. Zubairy, *Quantum Optics*, Cambridge University Press, Cambridge 1997.

[35] G. Ordonez, T. Petrosky, I. Prigogine, *Phys. Rev. A* **63**, 052106 (2001).

[36] Z. Xiao, Z.Y. Zhou, *Phys. Rev. D* **94**, 076006 (2016).

[37] Z. Xiao, Z.Y. Zhou, *J. Math. Phys.* **58**, 062110 (2017).

[38] Z.Y. Zhou, Z. Xiao, *Phys. Rev. D* **96**, 054031 (2017); erratum: *Phys. Rev. D* **96**, 099905 (2017).

[39] F. Giacosa, *J. Phys. Conf. Ser.* **16129**, 012012 (2020).

[40] P.M. Lo, F. Giacosa, *Eur. Phys. J. C* **79**, 336 (2019).

[41] D. Lonigro, *Eur. Phys. J. Plus* **137**, 492 (2022).

[42] P. Facchi, S. Pascazio, *Phys. Lett. A* **241**, 139 (1998).

[43] F. Giacosa, P. Kościk, T. Sowiński, *Phys. Rev. A* **102**, 022204 (2020).

[44] I. Antoniou, M. Gadella, I. Prigogine, G.P. Pronko, *J. Math. Phys.* **39**, 2995 (1998).

[45] Z.Y. Zhou, Z. Xiao, *Eur. Phys. J. C* **80**, 1191 (2020).

[46] Z.Y. Zhou, Z. Xiao, *Eur. Phys. J. C* **81**, 551 (2021).

[47] P.T. Matthews, A. Salam, *Phys. Rev.* **112**, 283 (1958).

[48] F. Giacosa, G. Pagliara, *Phys. Rev. C* **76** 065204 (2007).

[49] F. Giacosa, G. Pagliara, *Mod. Phys. Lett. A* **26**, 2247 (2011).

[50] P. Facchi, S. Pascazio, *Chaos Solitons Fractals* **12**, 2777 (2001).

[51] F. Giacosa, A. Okopińska, V. Shastry, *Eur. Phys. J. A* **57**, 336 (2021).