Papers submitted to the Conference "Applications of Physics in Mechanical and Material Engineering"

## The Physical Properties of Muon Cation: The *Ab Initio* Results

J.M. TRUSZKOWSKI<sup>a</sup>, M.W. JAROSIK<sup>b,\*</sup>, A.P. DURAJSKI<sup>b</sup> AND R. SZCZĘŚNIAK<sup>a,b</sup>

 <sup>a</sup> Division of Theoretical Physics, Jan Długosz University in Częstochowa, al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
 <sup>b</sup> Division of Physics, Częstochowa University of Technology, al. Armii Krajowej 19, 42-200 Częstochowa, Poland

Doi: 10.12693/APhysPolA.142.39

\*e-mail: marcin.jarosik@pcz.pl

We determined the physical properties of the muon cation  $(M_2^+)$ . The energy parameters of the Hamiltonian (the molecular orbital energy  $\varepsilon$  and the hopping integral t) were calculated by the variational method taking into account the Wannier's functions consisting of 1s Slater-type orbitals. We calculated the equilibrium distance  $R_0$  between the cores, the ground state energy  $E_0$ , and the dissociation energy  $E_D$ . Then we estimated the harmonic oscillation energy  $\omega_0^H$  and the anharmonic Morse energy  $\omega_0^M$ . We also determined the rotational constant  $B_0$  and the values of the muon-phonon coupling functions,  $g_{\varepsilon}$  and  $g_t$ . In the paper, we have demonstrated the influence of the environment on the properties of the muon cation in a balanced gain and loss energy scheme, where  $\gamma$  as energy, associated with the cation-environment interaction. We have shown that the cation may be in a metastable state. In the system, there is no the  $\mathcal{PT}$ -symmetry breaking of the Hamiltonian, because  $t > \gamma$ .

topics: muon cation, physical properties, stable and metastable state,  $\mathcal{PT}$ -symmetry

#### 1. Introduction

The physical properties of molecular systems can be theoretically determined with amazing accuracy. For this purpose, one can use the Schrödinger equation [1–4], its relativistic version — the Dirac equation [5, 6], and quantum electrodynamics [7, 8]. In most cases, the literature describes stable molecules in which chemical bonds are formed by electrons (also stable particles). In fact, such systems are stable and disintegrate under certain conditions as a result of interaction with the environment.

In the case of *exotic* molecules where chemical bonds are formed by negatively charged muons, the situation is different because muon has the finite lifetime  $\tau_{\mu} = 2.197034 \times 10^{-6} \text{ s} = 1.00427 \times 10^{11} \tau_0$ , and  $\tau_0 \simeq 2.418 \times 10^{-17}$  s is the unit of time in the atomic system of units [9]. It should be noted that muon differs from an electron by the finite lifetime and the mass, as  $m_{\mu}$  is about 207 times greater than the mass of the electron  $(m_e)$  [10]. So a muon is often called a heavy electron.

Note that for several decades, research on exotic systems containing negative muons has attracted the attention of the physics community due to the potential of muons to catalyze nuclear fusion processes (the first experimental observation was done in 1956 by Alvarez et al. [11]). Muon-catalyzed fusion is a process, which enables the release of nuclear energy at low temperatures (e.g. room temperature), and could in the future be practically used as a direct energy source or as a source of neutrons in the hybrid reactors. Muon entering a mixture of light elements exposed to a negative muons beam initiates a complicated chain of atomic and molecular processes. These processes occur in  $0.45 \times 10^{-6}$  s (faster than the time of the muon decay) [12]. The muon becomes free after a fusion event, so once the muon is released, it can initiate the next fusion reaction cycle (acts as a catalyst) [13]. Each muon catalyzed fusion releases the energy of 17.6 MeV [9]. Various theoretical methods of atomic and molecular physics are used in the Muon-Catalyzed Fusion (MCF) studies, e.g. hyperspherical expansion method [14] or variational methods [15, 16]. It is worth noting that due to the existence of muon molecules, it is possible to study nuclear reactions occurring at extremely low energies [17], which cannot be achieved in collision experiments [18].

In the presented paper, we calculated all important physical quantities characterizing the state of the muon cation  $M_2^+$ . We take into account the  $Cu_2^+$  cation [19], in which an electron from the valence shell was replaced by a muon. We treat atomic nuclei together with all the remaining electrons as the cores of the muon cation. In this case, the muon mass is significantly smaller than the core mass, which allows the Born–Oppenheimer approximation to be used [20]. We characterized the interaction of the muon cation with the environment in the framework of non-Hermitian quantum mechanics [21–25]. To describe the physical system, we used the Hamiltonian which is invariant with respect to the  $\mathcal{PT}$ -symmetry. It is the new formalism that allows to study of the interaction of the physical system with the environment in a balanced gain and loss energy scheme. Numerical calculations were carried out at the level required in quantum chemistry. This means that it will be possible to experimentally verify the obtained results.

#### 2. Formalism

The total energy of the muon cation is given by the formula  $E_T = E_p + E_{\mu}$ , where  $E_p = \frac{2}{R}$ , represents energy of core-core repulsion (*R* is the distance between the cores  $R = |\mathbf{R}|$ ). The energy of the muon ground state is denoted by  $E_{\mu}$  and it should be calculated by using Hamiltonian [26–32]

$$\hat{\mathcal{H}}_{\mu} = (\varepsilon + i\gamma)\,\hat{n}_1 + (\varepsilon - i\gamma)\,\hat{n}_2 + t\sum_{\sigma} \left(\hat{n}_{12\sigma} + \hat{n}_{21\sigma}\right),\tag{1}$$

where  $\hat{n}_j = \sum_{\sigma} \hat{n}_{j\sigma}$ ,  $\hat{n}_{j\sigma} = \hat{c}^{\dagger}_{j\sigma} \hat{c}_{j\sigma}$ , and  $\hat{n}_{ij\sigma} = \hat{c}^{\dagger}_{i\sigma} \hat{c}_{j\sigma}$ . The symbol  $\hat{c}^{\dagger}_{j\sigma}$  ( $\hat{c}_{j\sigma}$ ) represents the creation (annihilation) operator of the muon state on *j*-th core and the spin  $\sigma \in \{\uparrow, \downarrow\}$ . The Hamiltonian parameters are

$$\varepsilon = \int d^3 \boldsymbol{r} \, \Phi_i(\boldsymbol{r}) \left( -\frac{\nabla^2}{m_{\mu}} - \frac{2}{|\boldsymbol{r} - \boldsymbol{R}|} \right) \Phi_i(\boldsymbol{r}), \quad (2)$$

$$t = \int d^{3}\boldsymbol{r} \, \boldsymbol{\Phi}_{i}\left(\boldsymbol{r}\right) \left(-\frac{\nabla^{2}}{m_{\mu}} - \frac{2}{|\boldsymbol{r} - \boldsymbol{R}|}\right) \, \boldsymbol{\Phi}_{j}\left(\boldsymbol{r}\right), \quad (3)$$

where  $m_{\mu} = 206.768268 m_e$  [10]. The symbol  $\Phi_j(\mathbf{r})$  stands for the Wannier function. It can be constructed using the Slater orbitals as [29, 33, 34]

$$\Phi_{j}^{s}(\boldsymbol{r}) = A\left(\phi_{j}^{s}(\boldsymbol{r}) - B\phi_{l}^{s}(\boldsymbol{r})\right), \qquad (4)$$

where the parameters A and B ensuring function normalization are expressed by the following formulas

$$A = \frac{1}{\sqrt{2}} \sqrt{\frac{1 + \sqrt{1 - S_s^2}}{1 - S_s^2}}, \qquad B = \frac{S_s}{1 + \sqrt{1 - S_s^2}}.$$
 (5)

Additionally  $S_s = \int d^3 \boldsymbol{r} \ \phi_1^s(\boldsymbol{r})\phi_2^s(\boldsymbol{r})$ , where the 1s Slater-type orbital has the form  $\phi_j^s(\boldsymbol{r}) = \sqrt{\alpha_s^3/\pi} \exp(-\alpha_s |\boldsymbol{r} - \boldsymbol{R}_j|)$ , and  $\alpha_s$  is the variational parameter (the inverse size of orbital). The symbol  $\gamma$  in (1) represents the cation–environment interaction energy.

It is worth noting that the integrals defined in (2) can be calculated analytically. In particular, we have

$$\varepsilon = A^2 \left( 1 + B^2 \right) \varepsilon' - 2A^2 B t', \tag{6}$$

$$t = A^{2} \left( 1 + B^{2} \right) t' - 2A^{2}B \varepsilon', \tag{7}$$

where

$$\varepsilon' = \frac{1}{m_{\mu}} \alpha_s^2 - 2\alpha_s - \frac{2}{R} + 2\left(\alpha_s + \frac{1}{R}\right) e^{-2\alpha_s R},$$
(8)
$$t' = \frac{1}{m_{\mu}} \alpha_s^2 e^{-\alpha_s R} \left(1 + \alpha_s R - \frac{1}{3} \left(\alpha_s R\right)^2\right)$$

$$-4\alpha_s e^{-\alpha_s R} \left(1 + \alpha_s R\right).$$
(9)

 $-4\alpha_s e^{-\alpha_s R} (1 + \alpha_s R).$ The atomic overlap is given by

$$S_s = e^{-\alpha_s R} \left( 1 + \alpha_s R + \frac{1}{3} \left( \alpha_s R \right)^2 \right).$$
 (10)

The eigenvalues of the Hamiltonian  $\hat{\mathcal{H}}_{\mu}$  have the form

$$E_1 = E_2 = \varepsilon - \sqrt{t^2 - \gamma^2},\tag{11}$$

$$E_3 = E_4 = \varepsilon + \sqrt{t^2 - \gamma^2}.$$
(12)

The lowest energy value is  $E_1$ . It can be easily seen, that the ground state is degenerated, this being related to the existence of the muon spin. The degeneration can be removed by the constant external magnetic field applied to the cation.

#### 3. Results

#### 3.1. Physical state of the isolated muon cation

In Fig. 1a, we have plotted the total energy of the muon cation  $E_T = E_1 + \frac{2}{R}$  as a function of the distance R between the cores. The ground state energy  $E_0 = E_1 + \frac{2}{R_0}$  is equal to -242.541867 Ry, where the value of the equilibrium distance is  $0.009689 a_0$   $(a_0 \simeq 0.529 \times 10^{-10} \text{ m})$ . The results have been obtained for  $\alpha_{s0} = 255.981718 a_0^{-1}$ . Figure 1b presents the full form of the dependency of the variational parameter on the distance between the cores. In Fig. 1c, we presented the charge density distribution of muon cation in the equilibrium state calculated from the formula  $\rho(\mathbf{r}) = \sum_j \Phi_j^{s\star}(\mathbf{r}) \Phi_j^s(\mathbf{r})$ . In comparison with the results obtained for the

In comparison with the results obtained for the hydrogen molecule  $H_2^+$ , the muon cation is characterized by three orders lower value of the equilibrium distance and two orders lower value of the ground state energy. In particular, for  $H_2^+$ , we have  $R_0 = 2.003296 a_0$ ,  $E_0 = -1.173013$  Ry, where  $\alpha_{s0} = 1.238029 a_0^{-1}$ . It is worth noting that these results are consistent with the results obtained for  $H_2^+$  by Schaad and Hicks, i.e.,  $R_0 = 1.9972 a_0$  and  $E_0 = -1.20527$  Ry [35]. Differences in the values of the calculated physical quantities for the muon cation and the  $H_2^+$  cation cause the difference in dissociation energy  $E_D = \lim_{R \to +\infty} (E_T(R) - E_0)$ . In the case of the muon cation,  $E_D$  is equal to 35.682690 Ry, and for  $H_2^+$  we have 0.172417 Ry.

In Fig. 2, we have plotted the values of the molecular orbital energy and the hopping integral as a function of the core distance R. The equilibrium values of the molecular orbital energy and the hopping integral for the muon cation are equal to



Fig. 1. (a) The total energy  $E_T$  as a function of the distance R for the ion  $M_2^+$ , (b) the inverse size of the Slater orbital  $\alpha_s$  as a function of R, and (c) the distribution of muon charge for the equilibrium distance.



Fig. 2. (a) The molecular orbital energy  $\varepsilon$  and (b) the hopping integral t depending on the distance between the cores.

 $\varepsilon_0 = -351.142324$  Ry and  $t_0 = -97.819195$  Ry, respectively. In the case of the H<sub>2</sub><sup>+</sup> cation, we have  $\varepsilon_0 = -1.698245$  Ry and  $t_0 = -0.473122$  Ry, respectively.

In the next step, we characterized the phonon properties of the muon cation. As part of the harmonic approximation, the potential should be determined using the formula

$$V_{\rm H}(R) = E_0 + \frac{1}{2} k_{\rm H} \left( R - R_0 \right)^2, \qquad (13)$$

where  $k_{\rm H} = ({\rm d}^2 E_T(R)/{\rm d}R^2)|_{R=R_0}$ . The energy distribution of phonon states has the form  $E_n^{\rm H} = \omega_0^{\rm H} (n+1/2)$ , where  $\omega_0^{\rm H} = \sqrt{k_{\rm H}/m'}$ , and  $n=0, 1, 2, \ldots$  The reduced mass of the core system is  $m'=m_c$ , and  $m_c$  is the mass of the lightest core.

In Fig. 3, we presented the energy-harmonic potential curve. It can be seen that the values of the total energy  $E_T$  are correctly reproduced only in close vicinity of the equilibrium distance  $R_0$ . In a wider range of distances, the potential  $V_{\rm H}(R)$ is non-physical. For this reason the dissociation energy cannot be determined from  $V_{\rm H}(R)$ . It is worth



Fig. 3. The total energy  $E_T$  and the potentials  $V_{\rm H}$  and  $V_{\rm M}$  as a function of the inter-core distance R.



Fig. 4. (a) The muon-phonon coupling functions  $g_{\varepsilon}$  and (b)  $g_t$  versus the inter-core distance R.

noting that the above inconvenience can eliminated considering the Morse potential

$$V_{\rm M}(R) = E_0 + E_{\rm D} \left(1 - e^{-\alpha_{\rm M}(R-R_0)}\right)^2.$$
 (14)

The parameter  $\alpha_{\rm M}$  determines the curvature of the potential near its minimum. The constant  $k_{\rm M}$  should be estimated from the formula  $k_{\rm M} = (d^2 V_{\rm M}(R) / dR^2)_{R=R_0}$ , and the Morse energy is given by  $\omega_0^{\rm M} = \sqrt{k_{\rm M}/m'}$ . The full expression for the energy levels is  $E_n^{\rm M} = \omega_0^{\rm M}(n + \frac{1}{2}) + (\omega_0^{\rm M})^2/(4E_D)(n + \frac{1}{2})^2$ .

In Fig. 3, we have drawn the course of the curve  $V_{\rm M}(R)$ . It reproduces numerical results much better than  $V_{\rm H}(R)$ . The calculation results suggest that the vibrational energy of the muon cation is three orders of magnitude greater than the vibrational energy of the hydrogen cation. In particular, we have  $\left[\omega_0^{\rm H}\right]_{\rm M_2^+}=42.475033$  Ry,  $\left[\omega_0^{\rm M}\right]_{\rm M_2^+}=59.302329$  Ry, and  $\left[\omega_0^{\rm H}\right]_{\rm H_2^+}=0.014286$  Ry,  $\left[\omega_0^{\rm M}\right]_{\rm H_2^+}=0.021713$  Ry.

Then we determine the rotational energy  $E_r = B_0 l(l+1)$ , where  $B_0 = 1/(m'R_0^2)$ , and l = 0, 1, 2, .... For  $M_2^+$ , the value of  $B_0$  is 11.602813 Ry, while for  $H_2^+$ , we have  $B_0 = 0.000271$  Ry.

The muon-phonon coupling functions are defined by  $g_x = dx/dR$ , where  $x \in \{\varepsilon, t\}$ . They can be explicitly written as

$$g_{\varepsilon} = 6 \Biggl\{ 27 e^{4R\alpha_s} m_{\mu} - 3 e^{2R\alpha_s} \Biggl[ R^3 \alpha_s^4 \Bigl( -6 + R\alpha_s \bigl( -3 + 2R\alpha_s (1 + R\alpha_s) \bigr) \Bigr) \\ + m_{\mu} \Biggl( 18 + R\alpha_s \Bigl( 36 + R\alpha_s \Bigl( 27 + R\alpha_s \bigl( 12 + R\alpha_s (11 - 2R\alpha_s (-1 + R\alpha_s)) ) \Bigr) \Bigr) \Bigr) \Biggr) \Biggr] \Biggr] \\ + \Bigl( 3 + R\alpha_s (3 + R\alpha_s) \Bigr) \Biggl[ - R^3 \alpha_s^4 (2 + R\alpha_s) \bigl( 3 + R\alpha_s (3 + R\alpha_s) \bigr) \\ + m_{\mu} \Biggl( 9 + R\alpha_s \Bigl( 27 + R\alpha_s \Bigl( 33 + 2R\alpha_s \bigl( 12 + R\alpha_s (9 + R\alpha_s (5 + R\alpha_s)) ) \Bigr) \Bigr) \Bigr) \Biggr) \Biggr) \Biggr] \Biggr\} \\ \times \Biggl( m_{\mu} R^2 \Bigl( -9 e^{2R\alpha_s} + \bigl( 3 + R\alpha_s (3 + R\alpha_s) \Bigr)^2 \Bigr)^2 \Biggr)^{-1}, \tag{15}$$

$$d_{g_{t}} = \left\{ e^{-R\alpha_{s}} \left\{ -6M \left( 3 + R\alpha_{s} (3 + R\alpha_{s}) \right)^{2} \left( 3 + R\alpha_{s} \left( 3 + R\alpha_{s} \right) \left( 3 + R\alpha_{s} (3 + R\alpha_{s}) \right) \right) -54e^{4R\alpha_{s}} \left( R^{3} \alpha_{s}^{4} (2 - R\alpha_{s}) + m_{\mu} \left( 3 + R\alpha_{s} (-3 + R\alpha_{s}) \left( -1 + R\alpha_{s} (-1 + R\alpha_{s}) \right) \right) \right) + 6e^{2R\alpha_{s}} \left[ R^{3} \alpha_{s}^{4} \left( 3 + R\alpha_{s} (3 + R\alpha_{s}) \right) \left( 6 + R\alpha_{s} \left( 3 + R\alpha_{s} (1 + R\alpha_{s}) \right) \right) + m_{\mu} \left[ 54 + R\alpha_{s} \left( 162 - R\alpha_{s} \left( -225 + R\alpha_{s} \left( -117 + R\alpha_{s} \left( -3 + R\alpha_{s} (15 + R\alpha_{s} (3 + R\alpha_{s} (2 + R\alpha_{s}))) \right) \right) \right) \right) \right] \right] \right\} \right\}$$

$$\times \left( m_{\mu} R^{2} \left( -9e^{2R\alpha_{s}} + \left( 3 + R\alpha_{s} (3 + R\alpha_{s}) \right)^{2} \right)^{2} \right)^{-1}. \tag{16}$$



Fig. 5. (a) The ground state energy  $E_0$  and (b) the equilibrium distance  $R_0$  depending on the value of  $\gamma$  parameter.

Their equilibrium values compared to the values of electron-phonon coupling functions for the hydrogen cation are enormous, namely  $[g_{\varepsilon 0}]_{M_2^+}=4990.03/a_0 \text{ Ry}, [g_{t0}]_{M_2^+}=16315.15/a_0 \text{ Ry},$  and  $[g_{\varepsilon 0}]_{H_2^+}=0.116710/a_0 \text{ Ry}, [g_{t0}]_{H_2^+}=0.381646/a_0 \text{ Ry}.$  The full form of functions  $g_{\varepsilon}(R)$  and  $g_t(R)$  for the cation  $M_2^+$  has been presented in Fig. 4.

# 3.2. Influence of the environment on the physical state of the muon cation

The physical properties of the muon cation, which interact witch the environment have been determined in the balanced gain and loss energy scheme  $(\gamma \neq 0)$ .

In Fig. 5a and b, we presented the values of the ground state energy  $E_0$  and the equilibrium distance  $R_0$  as a function of the  $\gamma$  parameter. It can be noticed that as the interaction between the cation and the environment increases, the energy of ground state also increases, while the distance between the cores decreases. For  $\gamma = \gamma_{MS} = 87.527434$  Ry, the difference  $\lim_{R\to+\infty} (E_T(R) - E_0)$  is equal to zero. This means that for  $\gamma > \gamma_{MS}$ , the cation  $M_2^+$  is in the metastable state.

On the basis of (11), we find that the  $\mathcal{PT}$ -symmetry of the operator  $\hat{\mathcal{H}}_{\mu}$  is not broken, since  $t > \gamma$ . Note that in the case of the hydrogen molecule, for the value of  $\gamma = 0.520873$  Ry, there occurs the  $\mathcal{PT}$ -symmetry breaking of the electron Hamiltonian [36]. This result is related to two-body interactions between electrons. In the muon cation  $M_2^+$  such interactions are absent. However, as shown in the paper [36], the  $\mathcal{PT}$ -symmetry breaking of the electron Hamiltonian does not affect the ground state properties of the hydrogen molecule.

#### 4. Conclusions

In conclusion, we calculated all the relevant physical parameters of muon cation  $M_2^+$ . In particular, the ground state energy  $E_0$  is equal to -242.541867 Ry, the equilibrium distance  $R_0$  is equal to  $0.009689 a_0$ , and the dissociation energy  $E_D$  is equal to 35.682690 Ry. Moreover, for the harmonic oscillation energy  $\omega_0^{\rm H}$  and the anharmonic Morse energy  $\omega_0^{\rm M}$ , we obtained 42.475033 Ry and 59.302329 Ry, respectively. The rotational constant  $B_0$  equals to 11.602813 Ry. We also showed that the equilibrium values of the muon-phonon interaction functions  $g_{\varepsilon 0}$  and  $g_{t0}$  are 4990.03 Ry and 16315.15 Ry, respectively.

We performed numerical calculations at the level of the Schrödinger equation by using the variational method and the second quantization formalism. As shown before [33], these methods perfectly reproduce the results for the hydrogen molecule and its ions obtained by other researchers. This means that the results obtained by us for  $M_2^+$  are very accurate and can be directly compared with the experimental data.

In the paper, we characterized the influence of the cation's interaction with the environment on the physical properties of the cation state. We shown that the increase of  $\gamma$  interaction energy above the value of 87.527434 Ry causes the cation to be in a metastable state.

In the system under consideration, the  $\mathcal{PT}$ -symmetry of  $\hat{\mathcal{H}}_{\mu}$  is not broken due to the fact that the value of the hopping integral t is greater than the corresponding value of cation–environment interaction energy  $\gamma$ .

### References

- [1] E. Schrödinger, Ann. Phys. **79**, 361 (1926).
- [2] E. Schrödinger, Ann. Phys. 79, 489 (1926).
- [3] E. Schrödinger, Ann. Phys. 80, 734 (1926).
- [4] E. Schrödinger, Ann. Phys. 81, 109 (1926).
- [5] P.A.M. Dirac, Proc. R. Soc. Lond. 117, 610 (1928).
- [6] W. Greiner, *Relativistic Quantum Mechanics: Wave Equations*, Springer, 2000.
- [7] P.A.M. Dirac, Proc. R. Soc. Lond. 114, 243 (1927).
- [8] E. Fermi, *Rev. Mod. Phys.* 4, 87 (1932).
- [9] P. Froelich, Adv. Phys. 41, 405 (1992).
- [10] F. Moncadaa, D. Cruza, A. Reyesa, *Chem. Phys. Lett.* 570, 16 (2013).
- [11] L.W. Alvarez, H. Bradner, F.S. Crawford et al., *Phys. Rev.* **105**, 1127 (1956).
- [12] W. Czapliński, M. Rybski, *Phys. Lett. A* 380, 869 (2016).
- [13] E.J. Bleser, E.W. Anderson, L.M. Lederman et al., *Phys. Rev.* 132, 2679 (1963).

- [14] M. Decker, W. Sandhas, V.B. Belyaev, *Phys. Rev. A* 53, 726 (1996).
- [15] N.T. Padial, J.S. Cohen, R.B. Walker, *Phys. Rev. A* 37, 329 (1988).
- [16] A.M. Frolov, D. Wardlaw, *Eur. Phys. J. D* 63, 339 (2011).
- [17] W. Czapliński, A. Kravtsov, A. Mikhailov, N. Popov, *Eur. Phys. J. D* 3, 223 (1998).
- [18] V.B. Belyaev, S.A. Rakityansky,
   H. Fiedeldey, S. Sofianos, *Phys. Rev.* A **50**, 305 (1994).
- [19] F. Fantuzzi, D.O. de Sousa, M.C. Nascimento, *Chem. Select* 2, 604 (2017).
- [20] M. Born, R. Oppenheimer, Ann. Phys. 389, 457 (1927).
- [21] C.M. Bender, S. Boettcher, *Phys. Rev. Lett.* 80, 5243 (1998).
- [22] C.M. Bender, P.D. Mannheim, *Phys. Rev.* D 78, 025022 (2008).
- [23] G.A. Mezincescu, J. Phys. A: Math. Gen. 34, 3329 (2001).
- [24] M. Znojil, F. Cannata, B. Bagchi, R. Roychoudhury, *Phys. Lett. B* 483, 284 (2000).
- [25] A. Mostafazadeh, J. Phys. A: Math. Gen. 36, 7081 (2003).
- [26] J. Hubbard, Proc. R. Soc. Lond. A 276, 238 (1963).
- [27] M.C. Gutzwiller, *Phys. Rev. Lett.* 10, 159 (1963).
- [28] J. Kanamori, Prog. Theor. Phys. 30, 275 (1963).
- [29] A.P. Kądzielawa, A. Bielas, M. Acquarone, A. Biborski, M.M. Maśka, J. Spałek, *New J. Phys.* 16, 123022 (2014).
- [30] J. Spałek, Wstęp do Fizyki Materii Skondensowanej, PWN, 2015.
- [31] M. Klett, H. Cartarius, D. Dast, J. Main, G. Wunner, *Phys. Rev. A* **95**, 053626 (2017).
- [32] D. Dast, D. Haag, H. Cartarius, J. Main,
   G. Wunner, *Phys. Rev. A* 94, 053601 (2016).
- [33] M.W. Jarosik, R. Szczęśniak, A.P. Durajski, J.K. Kalaga, W. Leoński, *Chaos* 28, 013126 (2018).
- [34] E.A. Drzazga, R. Szczęśniak, I.A. Domagalska, A.P. Durajski, M. Kostrzewa, *Physica A* 518, 1 (2019).
- [35] L.J. Schaad, W.V. Hicks, J. Chem. Phys. 53, 851 (1970).
- [36] I.A. Wrona, M.W. Jarosik, R. Szczęśniak, K.A. Szewczyk, M.K. Stala, W. Leoński, *Sci. Rep.* 10, 215 (2020).