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Eigenfunction Non-Orthogonality in Open Wave Chaotic Systems: Non-Perturbative RMT Results for Single-Channel Scattering

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We present explicit distributions of diagonal non-orthogonality overlaps (also known as the Petermann factors) characterizing eigenmodes in a wave-chaotic cavity open via a single decay channel. We show that such factors should determine the shape of deep dips in the wave reflection experiments. The explicit distributions of the Petermann factors valid for an arbitrary strength of coupling to the scattering channel are obtained within the random matrix theory framework. The regime of weak-coupling allows further simplifications.

topics: coherent perfect absorption, non-hermitian random matrices, eigenvector non-orthogonality

1. Introduction

Non-Hermitian wave physics/engineering has recently emerged as an exciting new field with both fundamental and application-oriented insights [1, 2]. Among other questions in this area, the proposal of constructing the so-called coherent perfect absorbers (CPA) [3] has attracted a lot of attention recently [4]. Along these lines, in recent experiments the microwave technology has been used to build the first random "anti-laser" (at its lasing threshold), by demonstrating its ability to absorb suitably engineered incoming radiation fields with close to perfect efficiency [5], paving the way for the construction of CPAs based on disordered cavities. In another recent experiment, a CPA has been realised with a two-port microwave graph system, both with and without time-reversal symmetry [6], and yet in another way with chaotic cavity with programmable meta-surface inclusions [7].

Although construction of CPA's is a new development, related phenomena have been discussed actually long ago. To this end one may invoke an early influential experiment by Doron, Smilansky and Frenkel [8], who studied the frequency-dependent reflection coefficient $R(\omega) = |S(\omega)|^2$ of an electromagnetic wave sent via a single-mode waveguide to a cavity shaped in the form of a chaotic billiard. Here $S(\omega)$ stands for the associated scattering matrix at frequency ω, which for the single-channel setup is a single complex number. If no gain or loss is possible in the system, the scattering matrix must be unimodular due to flux conservation $S(\omega) = \mathrm{e}^{\mathrm{i}\,\delta(\omega)}$, where real $\delta(\omega)$ is known as the scattering phase shift. If this were true under experimental conditions, the reflection coefficient would be trivial $R(\omega) = 1$. One of the main experimental observations made in [8] was that the reflection coefficient $R(\omega)$ showed considerable variation with frequency ω, with many pronounced dips to low values $R(E) \lesssim 0.1$ at some frequencies, reminiscent of an "imperfect" version of modern CPA. Such a behaviour has been attributed to the presence of uniform losses in resonator walls which can be taken into account phenomenologically by adding a small imaginary increment to the real frequency $\omega \to \omega + \mathrm{i}\epsilon$ with $\epsilon > 0$. Recall that the frequency derivative of the scattering phase shift is an important scattering characteristics τ_W called the Wigner time delay $\tau_W = \frac{\mathrm{d}\delta}{\mathrm{d}\omega}$ [9]. Assuming that absorption is weak, i.e., $\epsilon \ll \Delta$, with Δ standing for the mean spacing between eigenfrequencies in the closed cavity in a given frequency range, one may expand the scattering matrix in ϵ yielding a relation between $R(\omega)$ and the Wigner time-delay $|S(\omega+\mathrm{i}\epsilon)|^2 \approx \mathrm{e}^{-2\epsilon\tau_W(\omega)}$. Though such an approximation was found to be in overall good agreement between measured reflection and anticipated statistics of the Wigner time-delays, it is clear that it must break down in the vicinity of the deepest dips. This motivates to have a closer look at $R(\omega)$ in the above setting. Our aim is to shed some light on the relation between the shape of the deepest dips and the property of eigenfunction non-orthogonality known to occur in scattering systems. This relation enables us to provide a detailed statistical description of such shapes in systems with chaotic wave scattering. The analysis sketched below is a short outline of our recent paper [10], where necessary technical detail can be found.

2. Dips shape analysis and eigenfunction non-orthogonality

The proper framework for such an analysis is the so called effective Hamiltonian formalism for wave-chaotic scattering [11–15]. The single-channel scattering matrix $S(\omega + \mathrm{i}\epsilon)$ of a system with spatially-uniform losses ϵ in this approach can be represented as

$$S(\omega + \mathrm{i}\epsilon) = \prod_{n=1}^{N} \frac{\omega + \mathrm{i}\epsilon - z_n^*}{\omega + \mathrm{i}\epsilon - z_n}, \tag{1}$$

where the positions $z_n = E_n - \mathrm{i}\Gamma_n$, $n = 1, \ldots, N$, of the S-matrix poles in the lower half of the complex frequency plane are considered to be complex eigenvalues of the $N \times N$ non-selfadjoint matrix $\mathcal{H}_{\mathrm{eff}} = \mathbf{H}_N - \mathrm{i}\boldsymbol{w} \otimes \boldsymbol{w}^\dagger$ known as the effective Hamiltonian, with the self-adjoint part $\mathbf{H}_N = \mathbf{H}_N^\dagger$ modelling the Hamiltonian/wave operator of the closed cavity resonator decoupled from the input antenna. The coupling between the cavity and the antenna channel is then characterised by the vector $\boldsymbol{w} = (w_1, \ldots, w_N)$ of coupling amplitudes.

Using the representation (1) it is easy to see that deepest dips in the reflection coefficient $R(\omega) = |S_\epsilon(\omega)|^2$, like shown in Fig. 1, happen when absorption ϵ matches some imaginary parts Γ_n (also known as "widths") of the resonance poles, i.e., when $\delta_n = \epsilon - \Gamma_n \ll \epsilon$. For a dip occurring in the vicinity of the energy/frequency $\omega = E_n$, for the frequency satisfying $|\omega - E_n| \ll \Delta$, its shape is well described by the following profile

$$R(\omega) = K_n \frac{(\omega - E_n)^2 + \delta_n^2}{(\omega - E_n)^2 + 4\epsilon^2}, \tag{2}$$

where

$$K_n = \prod_{k \neq n}^{N} \frac{|z_k - z_n|^2}{|z_k^* - z_n|^2}. \tag{3}$$

In particular, the locally minimal value $R_{\min}$ of the reflection signal and the curvature at the minimum $C := \frac{\mathrm{d}^2}{\mathrm{d}\omega^2} R(\omega)|_{\omega = E_n}$ are given by

$$R_{\min} = K_n \frac{\delta_n^2}{4\epsilon^2}, \quad \text{and} \quad C = \frac{K_n}{2\epsilon^2}, \tag{4}$$

showing that the statistics of the dip shapes over different reflection minima are controlled by the statistics of the resonance widths Γ_n and that of the factors K_n. Whereas the information about Γ_n is available theoretically [16–19] and is in a good agreement with experiments [20, 21], the factor K_n looks less familiar, and its meaning is not immediately obvious.

To this end let us first recall that the effective non-Hermitian Hamiltonian $\mathcal{H}_{\mathrm{eff}} = \mathbf{H}_N - \mathrm{i}\boldsymbol{w} \otimes \boldsymbol{w}^\dagger$ is non-normal (i.e., does not commute with its adjoint $\mathcal{H}_{\mathrm{eff}}^\dagger = \mathbf{H}_N + \mathrm{i}\boldsymbol{w} \otimes \boldsymbol{w}^\dagger$) hence has a set

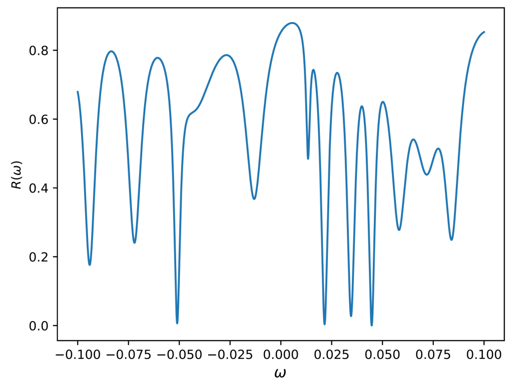

Fig. 1. The reflection coefficient generated according to (1) for $N = 200$, absorption $\epsilon = 0.001 \ll \Delta$ and effective coupling constant $g = 1.25$ after taking the positions of resonances $z_n = E_n - \mathrm{i}\Gamma_n$ numerically generated as eigenvalues of the corresponding RMT effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}$.

of "right" $\boldsymbol{r}_i$ and "left" $\boldsymbol{l}_i$ eigenvectors with standard bi-orthogonality properties $(\boldsymbol{l}_i^* \boldsymbol{r}_j) = \delta_{ij}$. The corresponding non-orthogonality overlap matrix $\mathcal{O}_{mn} = (\boldsymbol{l}_m^* \boldsymbol{l}_n)(\boldsymbol{r}_n^* \boldsymbol{r}_m)$ shows up in various physical observables of quantum chaotic systems, most notably in excess noise in open laser resonators, see [22] and references therein, in which context the diagonal overlaps $\mathcal{O}_{nn}$ are known as the Petermann factors. Other manifestations of the eigenvector/eigenfunction non-orthogonality in chaotic scattering can be found in sensitivity of the resonance widths to small perturbations [23, 24] and in transmission statistics [25, 26]. Our work [10] adds to this context by observing that the factors $K_n = \prod_{k \neq n}^{N} \frac{|z_k - z_n|^2}{|z_k^* - z_n|^2}$ introduced in (3) turn out to be simply *reciprocals* of the Petermann factors $K_n = \mathcal{O}_{nn}^{-1}$, hence the shapes of CPA-like minima in the single-channel case reflect the eigenfunction non-orthogonality.

Although statistics of the Petermann factors was addressed in a few papers on chaotic wave scattering [22, 27], non-perturbative results remain scarce, especially in the most interesting case of systems with preserved time reversal invariance. It turns out that in the case of the single-channel reflection the full distribution of diagonal overlaps/Pettermann factors can be found using the effective Hamiltonian framework combined with the random matrix theory (RMT) for modelling the properties of the associated self-adjoint part $\mathbf{H}_N$. We report these results below, referring to our paper [10] for a detailed derivation and further discussion.

3. Distribution of Petermann factors in single-channel chaotic reflection

Define the probability density of the non-orthogonality factor $t = \mathcal{O}_{nn} - 1$ corresponding to resonances in the vicinity of a point $z = X - \mathrm{i}Y$, $Y > 0$, as

$$\mathcal{P}(t;z) = \left\langle \frac{1}{N}\sum_{i=1}^{N} \delta(\mathcal{O}_{nn}-1-t)\delta(z-z_n) \right\rangle . \tag{5}$$

Then, aiming to describe the universal properties of wave-chaotic systems with preserved time-reversal invariance one can follow the standard route [28] and replace the self-adjoint part $\mathbf{H}_N$ in the effective Hamiltonian with a large random real symmetric matrix $H \in GOE$, where GOE stands for the Gaussian orthogonal ensemble normalized for $N \gg 1$ to have the semicircular eigenvalue density $\rho = \frac{1}{2\pi}\sqrt{4-E^2}$, ensuring the mean eigenvalue spacing of the order $\Delta \sim N^{-1}$. The coupling amplitudes $w_i, i = 1,\ldots,N$ are then normalized in such a way that $\gamma = \sum_i w_i^2$ tends to a finite limit as $N \to \infty$. As is well-known, see e.g. [11, 12, 28], as $N \gg 1$ statistics of all scattering characteristics varying at frequency/energy scales of the order of Δ turns out to be universal and depending only on the "renormalized" coupling strength $g = \frac{1}{2\pi\rho}\left(\gamma + \frac{1}{\gamma}\right)$ but not on the particular choice of w_i. One convenient choice is $\boldsymbol{w} = \sqrt{\gamma}\tilde{\boldsymbol{w}}$, with components $\tilde{w}_i$ being independent, mean zero Gaussian variables, with covariance $\langle \tilde{w}_i \tilde{w}_j \rangle = N^{-1}\delta_{ij}$ [11] where the angular brackets always denote averaging over relevant distributions. Equivalently, one may choose $\tilde{\boldsymbol{w}}$ to be any fixed of unit length [28], e.g. simply $\tilde{\boldsymbol{w}} = \boldsymbol{e}$ with $\boldsymbol{e}^{\mathrm{T}} = (1,0,\ldots,0)$. The final results in the limit $N \to \infty$ will be the same for these two choices.

One of the main findings of the paper [10] is that the limiting distribution $\mathcal{P}_y(t) := \lim_{N\to\infty} \frac{1}{\pi\rho N}\mathcal{P}(t; z = E - \mathrm{i}y/\pi\rho N)$ can be found explicitly and is given by

$$\mathcal{P}_y(t) = \frac{1}{2}\frac{\mathrm{e}^{-gy}}{\sqrt{t^5(1+t)}} \times \mathbb{L}_y\left(\mathrm{e}^{-gy\left(1+\frac{2}{t}\right)} I_0\left(\frac{2y\sqrt{(g^2-1)(1+t)}}{t}\right)\right), \tag{6}$$

where $\mathbb{L}_y$ is the following differential operator

$$\mathbb{L}_y = 2\sinh(2y) - \left(\cosh(2y) - \frac{\sinh(2y)}{2y}\right) \times \left(\frac{3}{y} + 2\frac{\mathrm{d}}{\mathrm{d}y}\right) \tag{7}$$

and $I_0(x)$ stands for the modified Bessel function.

Note that integration of the above expression over t gives the probability density of the scaled resonance widths $y = \pi\Gamma_n/\Delta$ for any value of the coupling parameter g in the following form

$$\rho(y) = \frac{1}{4\sqrt{2}}\mathrm{e}^{-gy}\mathbb{L}_y \int_1^\infty \mathrm{d}a\, \mathrm{e}^{-gya}\frac{(a-1)}{\sqrt{a+1}} \times I_0\left(y\sqrt{(g^2-1)(a^2-1)}\right), \tag{8}$$

which is much simpler expression than the threefold integral representation for $\rho(y)$ reported earlier in [18], though can be shown to be fully equivalent to it.

The parameter $g \geq 1$ determines the effective quality of coupling between the channel and the chaotic cavity, its minimal possible value $g = 1$ corresponds to the case of the so-called "perfect coupling" condition. The latter condition physically corresponds to absence of the so-called fast "direct reflection" processes, so that all the incoming flux penetrates the medium and participates in the formation of long-living resonances. Under these conditions there are many broad ("overlapping") resonances with $\Gamma \gg \Delta$. In contrast, the value $g \gg 1$ corresponds to the regime of weak coupling, when the typical widths of the resonance can be shown to satisfy $\Gamma_n \sim \Delta/g \ll \Delta$, or equivalently $y \sim g^{-1} \ll 1$. In this regime we can use the large-argument asymptotic form of the modified Bessel function $I_0(z) \approx \mathrm{e}^z/\sqrt{2\pi z}$ to notice that the integral in (8) is dominated by large values $a \sim g/y \gg 1$. We also can write, to the leading order, $\mathbb{L}_y \approx -\frac{8}{3}y^2\frac{\mathrm{d}}{\mathrm{d}y}$. After introducing the appropriate rescaling $\tilde{y} := gy$ and evaluating the resulting integral one arrives at the density

$$\rho_{PT}(\tilde{y}) = \frac{1}{\sqrt{\pi\tilde{y}}}\mathrm{e}^{-\tilde{y}} \tag{9}$$

which is the well-known Porter–Thomas distribution characterising the regime of weak coupling, see [19] and discussion therein. Similarly, one can study what the small coupling regime entails for the distribution of the non-orthogonality factor t. Close inspection shows that for $g \gg 1$ the typical values are $t \sim g^{-2} \ll 1$. We find it convenient to define the scaled variable $\tau = tg/y$, which is typically of the order of unity. Accurately implementing all rescalings and taking $g \to \infty$ in (6), one finds that the joint density of τ and $\tilde{y}$ in this limit factorizes, respectively, as

$$\mathcal{P}_{\tilde{y}}(\tau) = \rho_{PT}(\tilde{y})\mathcal{P}(\tau), \tag{10}$$

and

$$\mathcal{P}(\tau) = \frac{\mathrm{e}^{-1/\tau}}{3\tau^2}\left(1 + \frac{2}{\tau}\right). \tag{11}$$

A few remarks are appropriate here. First, as was already mentioned, the paper [22] addressed the distribution of the non-orthogonality factors in the limit of weak coupling by employing a perturbation theory in parameter $\gamma \ll 1$. So it would be natural to compare (10) and (11) to the results of that paper. However, the emphasis in [22] has been put on the non-orthogonality of the most narrow rather than typical resonances. For this reason the distributions presented there are somewhat different in form from (10) and (11). It is, however, straightforward to inspect what the perturbation theory developed in (10) and (11) entails for resonances of typical widths in the weak coupling regime. Not surprisingly, one then arrives exactly at expressions (10) and (11).

Second remark is that exactly the same expression $\mathcal{P}(\tau) = \frac{1}{3\tau^2}\mathrm{e}^{-\frac{1}{\tau}}\left(1 + \frac{2}{\tau}\right)$ recently appeared in studies of the so-called fidelity susceptibility in [29].

Employing the same perturbation theory as in [22] one indeed finds a direct relation between the two quantities. By the same perturbative reasoning one should expect that the same distribution must control perturbative non-orthogonality factors for real eigenvalues in small antisymmetric full-rank random Gaussian deformations of GOE matrices. In that latter model the non-orthogonality factor distribution has been studied recently for real Gaussian deformations, fully non-perturbatively [30]. One can easily check that the results in the weak non-Hermiticity regime, Eq. (17) of [30], indeed converge, after appropriate rescaling, to $\mathcal{P}(\tau)$, when the non-Hermiticity parameter $A \to 0$, keeping $t/A = \tau$ fixed.

Our nonperturbative methods can be also used to treat the systems with broken time-reversal symmetry, which in this framework is obtained by replacing the self-adjoint part $\mathbf{H}_N$ in the effective Hamiltonian with a random Hermitian matrix $H \in$ GUE, with GUE standing for the Gaussian unitary ensemble

$$\mathcal{P}_y(t) = \frac{16}{t^3}\, \mathrm{e}^{-2gy}\tilde{\mathbb{L}}_y \times \left[\mathrm{e}^{-2gy\left(1+\frac{2}{t}\right)} I_0\left(\frac{4y}{t}\sqrt{(g^2-1)(1+t)}\right)\right], \tag{12}$$

where now $\tilde{\mathbb{L}}_y$ is a differential operator acting on functions $f(y)$ as

$$\tilde{\mathbb{L}}_y f(y) = \left[1 + \left(\frac{\sinh(2y)}{2y}\right)^2 + \frac{1}{2y}\left(1 - \frac{\sinh(4y)}{4y}\right)\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{4}\left(\left(\frac{\sinh 2y}{2y}\right)^2 - 1\right)\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right] y^2 f(y). \tag{13}$$

The following remarks on the comparison between (12) and (6) are due. The most prominent feature in both cases is the same heavy-tail behaviour $\mathcal{P}_y^{(2)}(t) \sim t^{-3}$ for $t \gg 1$ rendering all moments $\langle O_{nn}^l \rangle$, $l \geq 2$ divergent. This tail behaviour is exactly the same as found earlier in other complex-valued non-normal random matrices [31–33] and seems to be the most universal feature of random diagonal overlaps/Petermann factors. Interestingly, in the weak-coupling regime $g \gg 1$ the difference between the cases of broken and preserved time-reversal invariance becomes much more pronounced. One can repeat the same calculation which lead to (10) and (11) starting from (12) and to obtain instead

$$\mathcal{P}(\tau) = \frac{\mathrm{e}^{-\frac{1}{\tau}}}{3\sqrt{\pi}\tau^{5/2}}\left(\frac{3}{4} + \frac{1}{\tau} + \frac{1}{\tau^2}\right), \tag{14}$$

showing that in the weak-coupling case the tail behaviour does reflect the presence or absence of the time-reversal symmetry.

4. Summary

We identified a direct relation between the shape of the deepest dips in the frequency-dependent reflection coefficient and the Petermann factors characterizing non-orthogonality of eigenmodes in the case of a single channel scattering. Using the random matrix theory we then found the full non-perturbative distribution of the Petermann factors for the case of chaotic scattering, both for preserved and broken time-reversal invariance. As a by-product, we arrived at a new and much simpler formula describing distribution of resonance widths in a single-channel scattering in time-reversal system.

With suitable adjustments, our method should be applicable to the rank-one family of *subunitary* deformations of the Haar-distributed unitary random CUE matrices. The corresponding model introduced in [34, 35] naturally appears in the context of time-periodic scattering. The statistics of non-orthogonality factors are then expected to be exactly the same as given by (12), with the perfect coupling case $g = 1$ corresponding in that context to the so-called *truncated* CUE [36].

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