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Intriguing Problems with Static Friction on Stage

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The phenomenon of static friction is demonstrated in some challenging situations. The work-energy theorem and the center-mass equation are elucidated and clearly discerned when used and interpreted in problems with static friction. Especially, it is demonstrated that it is indispensable to refer to thermodynamics to provide a complete and adequate description of some seemingly purely mechanical problems with static friction involved. The static frictional force, although being a zero-work force, is proved to influence energy acquired by a system. Unexpected values and directions of the static frictional forces are shown for some specific cases. Finally, it is explained how static friction plays a crucial role in the working of a store security tag, a device commonly used at clothes shops to prevent theft.

topics: static friction, work-energy theorem, security tag

1. Introduction

Static friction is a well-known and thoroughly elaborated phenomenon. Yet, there are some theoretical and practical situations where static friction enters in a surprising and thought-provoking manner. In this review paper, we want to demonstrate some examples of such challenging problems with static friction on stage. Especially, we point out how — in this context — to use correctly the center-of-mass and work-energy equations and why and how one must engage thermodynamics in analyzing problems with static friction. We also explain a paradox that although static friction does no work, it can modify energy acquired by a body. It is also surprising that static friction may enhance acceleration caused by another force. It can be even greater than the applied force and make the body move (accelerate) in the direction opposite to the direction of the exerted force. Finally, we describe how static friction, thanks to a specific positive feedback loop, is harnessed in a store security tag, a well-known device used at clothes shops to prevent potential thefts.

2. Mechanics meets thermodynamics

When a car accelerates, it is possible owing to the static frictional force acting on the car. It appears that the acquired kinetic energy is equal to the work done on the car by the frictional force exerted by the road, acting through the displacement of the car. Yet, we know that the static frictional force does no work and the kinetic energy comes from burning of gasoline, not from the road [1].

Actually, Newton's second law of motion for a system of particles,

$$\sum_i \boldsymbol{F}_{\mathrm{ext},i} = M\frac{\mathrm{d}\boldsymbol{v}_{\mathrm{CM}}}{\mathrm{d}t}, \tag{1}$$

integrated through a displacement of the center-of-mass point, leads to the equation

$$\sum_i \int \boldsymbol{F}_{\mathrm{ext},i}\cdot \mathrm{d}\boldsymbol{r}_{\mathrm{CM}} = \Delta\left(\frac{1}{2}M\boldsymbol{v}_{\mathrm{CM}}^2\right). \tag{2}$$

It is called the CM (center-of-mass) equation [1, 2]. For the total static frictional force f_1+f_2 acting on the car through a displacement d_{CM}, where f_1 and f_2 represent the total forces on the rear and front wheels (see Fig. 1), we get from (2):

$$(f_1+f_2)d_{\mathrm{CM}} = \Delta\left(\frac{1}{2}M\boldsymbol{v}_{\mathrm{CM}}^2\right). \tag{3}$$

The left-hand side of this equation seems to be the work and the right-hand side looks like the change of kinetic energy of the car. This, however, is not true. It is important to notice that the left-hand side of the CM equation (2) is *not* the work (some authors call it "pseudowork" [2, 3]) because in general the displacement $\mathrm{d}\boldsymbol{r}_i$ of the point of application of the i-th force is not equal to the displacement of the center-of-mass of the body, $\mathrm{d}\boldsymbol{r}_i \neq \mathrm{d}\boldsymbol{r}_{\mathrm{CM}}$. In effect,

$$\int \boldsymbol{F}_{\mathrm{ext},i}\cdot \mathrm{d}\boldsymbol{r}_{\mathrm{CM}} \neq \int \boldsymbol{F}_{\mathrm{ext},i}\cdot \mathrm{d}\boldsymbol{r}_i \equiv \text{work}. \tag{4}$$

Similarly, the right-hand side of the CM equation does *not* represent the change of the kinetic energy of the system because it involves only the center-of-mass speed.

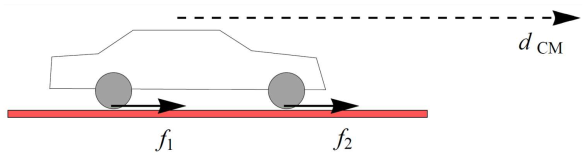

Fig. 1. A car accelerating thanks to static frictional forces.

In the literature [2, 4], it is emphasized that one has to precisely distinguish between the CM equation and the work-energy (WE) theorem [5–7]. The WE theorem can be obtained by integrating and summing up Newton's law equations for each particle of the system, i.e.,

$$\boldsymbol{F}_i = m_i \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t}, \tag{5}$$

which leads to

$$\sum_i \int \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i = \Delta K, \tag{6}$$

where the left-hand side represents the total work done in the system by both external and internal forces and ΔK is the change of the total kinetic energy of the system. The work done by external forces in general consists of the work done by macroscopic forces, W_{ext}, and microscopic forces that are responsible for the heat transfer Q. In turn, the work done by internal forces is equivalent to the minus change of potential (or/and chemical) energy $-\Delta E_{\mathrm{pot(chem)}}$ of the system. The right-hand side of (6) can be decomposed into two terms: the change of the kinetic energy of the center-of-mass ΔK_{CM} and the change of internal kinetic energy ΔK_{int} (i.e., rotational energy or energy connected with other forms of macroscopic internal motions of the macroscopic parts of the system and the thermal energy connected with the chaotic motion of the system microscopic constituents). In effect, (6) may be expressed as

$$W_{\mathrm{ext}} + Q = \Delta K_{\mathrm{CM}} + \Delta K_{\mathrm{int}} + \Delta E_{\mathrm{pot,chem}}. \tag{7}$$

Typically, the above equation is presented in a limited form (without the term $\Delta E_{\mathrm{pot,chem}}$) and has different names: it is called the energy equation [1] or the first law of thermodynamics [3, 8, 9] or just the extended work energy theorem [10]. The left-hand side of (7) represents the net external inputs to the system and the right-hand side is the change in the system energy. However, no author explains that (7) can be derived from the basic form of the work-energy theorem (6) obtained directly from Newton's law of motion — as it is shown in this paper. The derivation shows, in fact, how mechanical problems are in a natural way interconnected with the thermodynamical aspects of physical processes. This fact manifests itself particularly clearly in problems with static friction.

Let us come back to the example with the accelerating car, where there is no work done by external forces. The WE equation given by (7) can be brought to the following form [1]:

$$Q = \Delta\left(\frac{1}{2} M \boldsymbol{v}_{\mathrm{CM}}^2\right) + \Delta K_{\mathrm{int,macro}}$$
$$+\Delta K_{\mathrm{int,thermal}} + \Delta E_{\mathrm{chem}}, \tag{8}$$

where Q is the net heat transfer into the car from the surroundings (it is mainly the negative heat transfer from the hot engine to the air). Next, $\Delta K_{\mathrm{int,macro}}$ represents the increased energy of motion of the internal parts of the car (engine, wheels), $\Delta K_{\mathrm{int,thermal}}$ is associated mainly with the temperature rise of the engine and ΔE_{chem} is the (negative) change in chemical energy that pays for all the other terms.

Another instructive example of the correct use of the CM equation and WE theorem is the case of a climber who slowly inches up a vertical cliff, with a constant velocity (see Fig. 2) [1]. Let us assume that *the climber* is the system of interest. The CM equation then takes the form of

$$(f_1 + f_2 - Mg) d_{\mathrm{CM}} = 0 \tag{9}$$

and the WE theorem is given by

$$-Mg d_{\mathrm{CM}} - Q_{\mathrm{loss}} = \Delta E_{\mathrm{chem}}. \tag{10}$$

The static frictional forces f_1 and f_2 do no work since there is no slippage at the point of contact between the climber and the cliff. The nonzero inputs to (10) come from the negative work done by the external force Mg and the negative heat transfer to the air Q_{loss}. The change in the energy of the system is the (negative) change of chemical energy ΔE_{chem} burnt by the climber.

Notice that there is *no* change of the gravitational potential energy ΔE_{pot} of the climber on the right-hand side of (10). Considering the introduction of ΔE_{pot} into (10), we would make a double-entry mistake. Notice that the energetic contribution of the gravitational force Mg is already taken into account by the term $-Mgd_{\mathrm{CM}}$ on the left-hand side of (10). This term indeed represents the work done by the force Mg. Therefore, the frequently heard statement in this context that "the climber does the work to increase his gravitational potential energy" is wrong. No work is done by the climber and no increase of potential energy

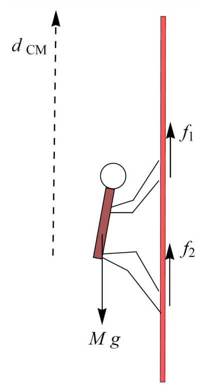

Fig. 2. A climber moves up a vertical cliff with a constant velocity.

appears in (10). The term Mg should be then treated as an *external* force and not as a force of a field with a potential energy defined.

It is instructive to notice that if we take as the system of interest both *the climber and the Earth*, then there are no inputs in the WE theorem for such a system, i.e., the left-hand side of (7) is zero. Especially, the gravitational force cannot be treated now as an external one. All the terms appear on the right-hand side of (7) as the change of the energy of the system: $+Mgd_{\mathrm{CM}}$ is the increase of the potential energy ΔE_{pot} connected with the enlarged separation between the climber and the Earth and $+Q_{\mathrm{loss}}$ represents the increased thermal energy in the atmosphere. To avoid mistakes in using the WE theorem, one must then be clear about the choice of the system [1].

3. Paradox of zero-work forces

Some forces do no work because they are perpendicular to the displacements $\mathrm{d}\boldsymbol{r}_i$ or their displacements $\mathrm{d}\boldsymbol{r}_i$ are zero, as it is for the static frictional forces. In the WE equation (6), the terms with forces that do no work (zero-work forces) can be omitted. In effect, we have

$$\sum_i \int \mathrm{d}\boldsymbol{r}_i \cdot {}^{w}\boldsymbol{F}_i = \Delta K, \tag{11}$$

where ${}^{w}\boldsymbol{F}_i$ denotes forces doing nonzero work. The result might suggest that the zero-work forces do not contribute to the change of the system energy. On the other hand, however, all forces in (2) (the CM equation), including the zero-work ones ${}^{0}\boldsymbol{F}_i$, are important and cannot be neglected. Thus, the CM equation shows directly that the zero-work forces *do* influence the change of the system velocity $\boldsymbol{v}_{\mathrm{CM}}$, so consequently also the change of the kinetic energy acquired by the system [11].

To reconcile (2) and (11), one has to notice that the zero-work forces, contrary to appearances, actually are in general present in a veiled manner in (11) and determine the energy achieved by the body. Namely, through Newton's law of motion the zero-work forces influence velocities $\boldsymbol{v}_i$ acquired by the system points to which the working forces ${}^{w}\boldsymbol{F}_i$ are applied. In this way, they modify the displacements $\mathrm{d}\boldsymbol{r}_i = \boldsymbol{v}_i\,\mathrm{d}t$ and affect the work $\int^{w} \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i$ done by the "working" forces.

To be more formal, we can write

$$\boldsymbol{v}_i = \int_0^t \mathrm{d}t' \boldsymbol{a}_i(t') = \frac{1}{m_i} \int_0^t \mathrm{d}t' \left[{}^{w}\boldsymbol{F}_i(t') + {}^{0}\boldsymbol{F}_i(t') \right], \tag{12}$$

where we have assumed that the velocity equals zero at $t = 0$. Then, the work $W_i = \int \mathrm{d}t({}^{w}\boldsymbol{F}_{\mathrm{ext},i} \cdot \boldsymbol{v}_i)$ done by the force acting on the i-th point is

$$W_i = \frac{1}{m_i} \int \mathrm{d}t\ {}^{w}\boldsymbol{F}_i \cdot \left[\int_0^t \mathrm{d}t' \left({}^{w}\boldsymbol{F}_i(t') + {}^{0}\boldsymbol{F}_i(t') \right) \right]. \tag{13}$$

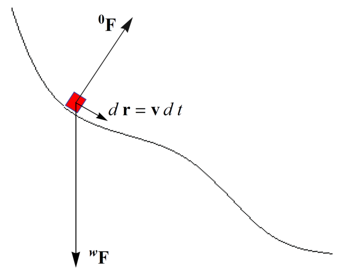

Fig. 3. A body on a slope is pulled by a force ${}^{w}\boldsymbol{F}$ (e.g., its weight). The zero-work slope reaction force ${}^{0}\boldsymbol{F}$ influences the orientation of the vector ($\boldsymbol{v}$) and therefore the orientation of $\mathrm{d}\boldsymbol{r}$ at the consecutive moments which has an impact on the work done by the force ${}^{w}\boldsymbol{F}$, according to (13).

This result explicitly shows that the zero-work forces are actually involved in the process of work. Furthermore, this conclusion refers even to a system consisting of a single particle (see Fig. 3).

In some cases, the zero-work forces are explicitly present in the WE theorem (6) (not only implicitly through the displacements $\mathrm{d}\boldsymbol{r}_i$) and cannot be removed from the description of the physical situation. However, the status of their "explicit" presence differs from (13). For example, for a rigid body, it is convenient to write the displacements $\mathrm{d}\boldsymbol{r}_i$ as

$$\mathrm{d}\boldsymbol{r}_i = \mathrm{d}\boldsymbol{r}_{\mathrm{CM}} + \mathrm{d}\boldsymbol{r}'_i, \tag{14}$$

where $\mathrm{d}\boldsymbol{r}_{\mathrm{CM}}$ is the displacement of the center of mass of the body and $\mathrm{d}\boldsymbol{r}'_i$ is the displacement with respect to the center of mass. Although ${}^{0}\boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i$ is zero, the products ${}^{0}\boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_{\mathrm{CM}}$ and ${}^{0}\boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}'_i$ in general are nonzero, thus the terms with the zero-work forces cannot be omitted

$$W = \sum_i \int \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_i \int \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_{\mathrm{CM}} + \sum_i \int \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}'_i. \tag{15}$$

Then, the WE theorem for a rigid body is [11]:

$$\int \mathrm{d}t \boldsymbol{F} \cdot \boldsymbol{v}_{\mathrm{CM}} + \int \mathrm{d}t \boldsymbol{M}' \cdot \boldsymbol{\omega} = \Delta\left(\frac{1}{2}M\boldsymbol{v}_{\mathrm{CM}}^2\right) + \Delta\left(\frac{1}{2}I\omega^2\right), \tag{16}$$

where the total external force $\boldsymbol{F}$ and the total torque $\boldsymbol{M}'$ include *both* the "working" and the zero-work forces. For example, for a cylinder rolling without slippage when pulled by a force F (see Fig. 4), assuming it was initially at rest, we have from (16) that

$$(\boldsymbol{F} - \boldsymbol{F}_s)\boldsymbol{s} + (\boldsymbol{F} + \boldsymbol{F}_s)\boldsymbol{R}\theta = \frac{1}{2}M\boldsymbol{v}_{\mathrm{CM}}^2 + \frac{1}{2}I\omega^2, \tag{17}$$

where $\boldsymbol{s} = \int \mathrm{d}t \boldsymbol{v}_{\mathrm{CM}}$ is a displacement of the center of mass and $\theta = \int \mathrm{d}t\omega$ is the angle of rotation of the cylinder. Because the rolling proceeds without slipping, we have $\boldsymbol{R}\theta = \boldsymbol{s}$ and — as expected — the terms with the static friction F_s cancel out.

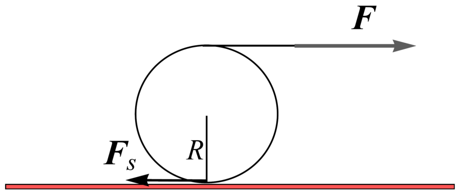

Fig. 4. A cylinder moving on a rough surface without slippage in the presence of a static frictional force F_s.

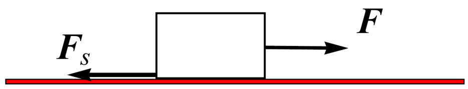

Fig. 5. A block pulled by a force F remains at rest due to a static frictional force F_s precisely compensating the force F.

In fact, the zero-work forces influence the work only implicitly by modifying the displacements $\mathrm{d}\boldsymbol{r}_i$, as shown in (13). In our example, the total displacement of the force application point is equal to (i) $2\boldsymbol{s}$ in the presence of static friction and (ii) $3\boldsymbol{s}$ for the case of motion on a perfectly smooth surface (no friction).

4. Direction and value of static frictional force

Based on typical physical situations, we are already accustomed that the static friction is opposite to the applied force and, in a sense, it hinders its action. For example, if a block which lays on a rough surface is pulled by a relatively small force F, it remains at rest as long as a static frictional force F_s, acting in the opposite direction, balances the force F (see Fig. 5). Or, if a cylinder or a ball moves down an incline without slippage, then the static frictional force is opposite to the force $mg\sin(\alpha)$ acting along the slope (see Fig. 6).

One would expect, in analogy to the case presented in Fig. 4, that as the force F is pointing to the right, the static frictional force is oriented to the left. Surprisingly, however, the direction of F_s is wrongly marked in Fig. 4. Solving the standard equations of motion for a cylinder pulled at a height h by a force F (see Fig. 7), we find that F_s acts in the same direction as the force F when $h > \frac{3}{2}R$ [12]. In Fig. 4, F_s should then be pointed to the right. One could say that in this case the static friction *helps* the force F in accelerating the cylinder. It would only mean that the acceleration is *greater* in the presence of static friction, as compared to a motion on a perfectly smooth surface. For $h = 2R$, the acceleration of the rolling cylinder is $a = \frac{4}{3}F/M$, while for the cylinder sliding on a smooth surface it is simply $a = F/M$.

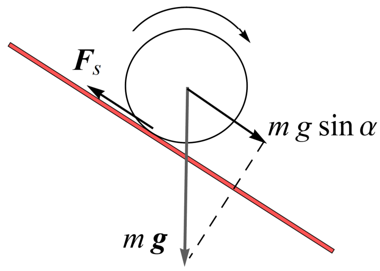

Fig. 6. A cylinder moves down a slope. Static friction F_s is opposite to the pulling force $mg\sin(\alpha)$.

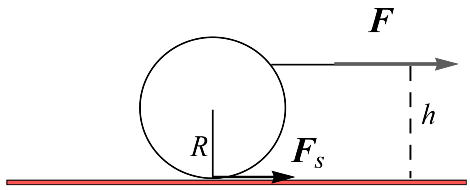

Fig. 7. The correctly indicated static frictional force F_s for a cylinder pulled by a force F applied at height $h > (3/2)R$.

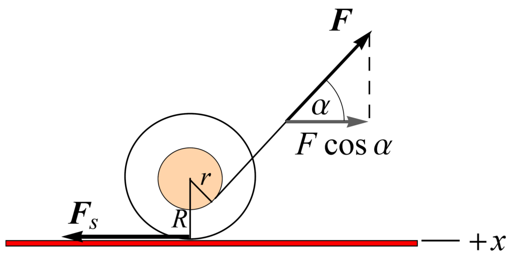

Fig. 8. A spool pulled by a force F at an angle α. For α satisfying the relation $\cos(\alpha) < r/R$, the static frictional force F_s is greater than the horizontal component $F\cos(\alpha)$ and the spool starts to move to the left.

One can encounter another surprise by analyzing the motion without slippage of a spool pulled by a force F through a thread inclined with an angle α (see Fig. 8). From the equations of motion we find that the acceleration along the x direction is

$$a = F\frac{\cos(\alpha) - \frac{r}{R}}{m + \frac{I}{R^2}}, \tag{18}$$

where m is the mass and I denotes the moment of inertia of the spool. For α fulfilling the inequality $\cos(\alpha) < \frac{r}{R}$, the acceleration is negative which means that the spool, being initially at rest, starts to move in the direction of the negative values on the x-axis. This, in turn, reflects the fact that the static frictional force F_s unexpectedly appears to be *greater* than the horizontal component $F\cos(\alpha)$ pulling the spool to the right. In other words, in this case, F_s not only hinders the action of the applied force but is able to dominate it and significantly influence the direction of the spool motion.

5. Store security tag

In this section, we want to demonstrate and explain a surprising behavior of a store security tag. Its mechanical operating principle is based on a specific positive feedback loop referring to static friction [13]. A security tag, widely used in shops for preventing theft of clothes, is a very simple device (see Fig. 9). Such a tag is attached to each garment with a pin. The pin passes through a bucket-like steel container and is placed between steel balls that press the pin thanks to a spring pushing the balls through a guide. All the parts are kept in place in a hard plastic container whose shape resembles UFO. While there is no problem with introducing the pin into the tag, it is practically impossible to remove it with your hands even if you use enormous force. Since each tag is equipped with an antenna, the alarm is triggered when one tries to leave the shop carrying a garment with the tag attached to it. The electronic part of the device, however, is out of scope of our presentation.

To understand the asymmetric behavior of the pin, one has to identify forces acting on the parts placed inside the tag. On the basis of Fig. 10, one can find the expression for the maximum static frictional force acting on the pin when one tries to remove it from the tag

$$T_{1\ \mathrm{max}} = \mu N_1 = \mu \left[F \cot(\alpha) + \frac{1+\cos(\alpha)}{\sin(\alpha)} T_1 \right], \tag{19}$$

where μ is the coefficient of static friction, N_1 is the normal force exerted upon the ball by the pin, F is the force produced by the guide connected to the spring and α is the angle at which the walls of the bucket are deviated from the right angle. When an external force F_{ext} is applied to the pin, a static frictional force T_1 starts to act on the pin due to its contact with the balls. In order to remove the pin from the tag, i.e., to cause its slipping on the balls, the force F_{ext} must exceed the maximum static friction given in (19). Note that F_{ext} is always equal to T_1. The problem, however, is such that $T_{1\ \mathrm{max}}$ depends on T_1 and in practical realizations of the tags the angle α satisfies the following relation:

$$\tan\left(\frac{\alpha}{2}\right) \leq \mu. \tag{20}$$

Therefore, F_{ext} $(= T_1)$ can never be greater than $T_{1\ \mathrm{max}}$. The greater the force F_{ext}, and respectively the static frictional force T_1, the greater $T_{1\ \mathrm{max}}$ (positive feedback loop) and no slippage is possible. In such a situation, the pin is firmly stuck in the tag.

In general, to remove the pin from the tag, shop assistants use strong external magnets. The magnet acts on the steel balls with forces greater than F and in this way removes, or makes negligible, the normal force N_1 exerted by the balls on the pin. In effect, no friction is possible and the pin is released.

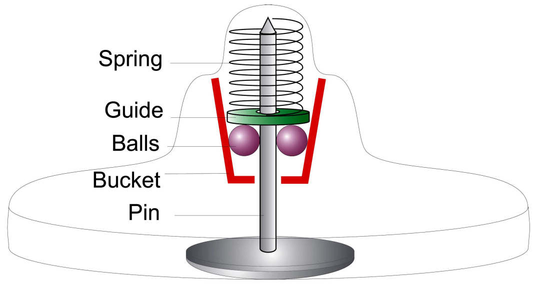

Fig. 9. The internal structure of a security tag.

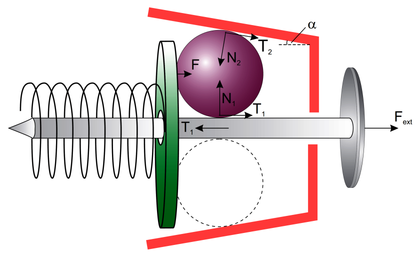

Fig. 10. Forces acting on a pin and a ball while one tries to pull the pin out of a tag.

For the process of inserting the pin into the tag, the second term in (19) is negative and there is always such a value of F_{ext} $(= T_1)$ that it becomes equal to and further greater than $T_{1\,max}$. Such a case is equivalent to the sliding of the pin on the ball surface. Accordingly, there is no problem with introducing the pin into the device.

It appears then that the inconspicuous static friction phenomenon may play an essential role in such, after all, intelligent devices as security tags.

6. Conclusions

Static friction appears to be a demanding phenomenon when one wants to use it correctly in the context of the WE theorem and the similarly looking CM equation. Unexpectedly, thermodynamic aspects of physical processes must be taken into account. Another feature of static friction is that although it does no work, it actively influences energy acquired by a system. The direction and value of static friction for rolling objects is also nontrivial and sometimes counterintuitive. Finally, we have shown how static friction is the basis on which store security tags work. We believe that physical problems with static friction involve interesting physics related to a fairly wide range of our everyday life experience. By analyzing these problems, the physical reality can be found more intriguing and — at the same time – it is better clarified and understood.

Acknowledgments

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