Solitons and Other Solutions for Two Higher-Order Nonlinear Wave Equations of KdV Type Using the Unified Auxiliary Equation Method

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Many new types of Jacobi-elliptic function solutions, solitons, and other solutions for two nonlinear partial differential equations, namely, the higher-order wave equation of KdV type (III) and the higher-order wave equation of KdV type (II) have been found using the unified auxiliary equation method combined with the conformable space-time fractional derivatives. The solitary wave solutions and the periodic wave solutions are obtained from the Jacobi elliptic function solutions when its moduli $m = 1$ or $m = 0$, respectively. Comparison of our new results with the well-known results are given.

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1. Introduction

Nonlinear evolution equations arise in a vast assortment of fields, like the physical sciences including thermodynamics, biology, soil mechanics, civil engineering, signal processing, control theory, finance, and non-Newtonian fluids, to the natural sciences including population ecology, infectious disease epidemiology, and natural networks. In the past few decades, particular attention has been given to the problem of finding the exact solutions of the nonlinear evolution equations. By virtue of these solutions, one may get better insight into the physical aspects of the nonlinear models studies. In recent years, quite a few methods for constructing explicit and solitary wave solutions of these nonlinear evolution equations have been presented. A variety of powerful methods, such as the semi-inverse variational method [1–5], the exp-function method [6–8], the Jacobi elliptic function method [9–12], the (G'/G)-expansion method [13, 14], the Kudryashov method [15–17], the multiple exp-function method [18, 19], the modified simple equation method [20–22], the auxiliary equation method [23, 24], the extended auxiliary equation method [25–28], the soliton ansatz method [29–35], the traveling wave hypothesis [36], the unified auxiliary equation method [37, 38], the conformable fractional derivatives [39-41], the Collocation finite element method [42] and so on.

The objective of this article is to apply the unified auxiliary equation method combined with the conformable space-time fractional derivatives for constructing the Jacobi elliptic function solutions, solitons, and other solutions of the following two nonlinear space-time fractional partial differential equations (PDEs):

(i) The higher-order wave equation of KdV type (III):

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^\beta u}{\partial x^\beta}+\alpha_1 u\frac{\partial^\beta u}{\partial x^\beta}+\beta_1\frac{\partial^{3\beta} u}{\partial x^{3\beta}}+\alpha_1^2\rho_1^2u^2\frac{\partial^\beta u}{\partial x^\beta}$$
$$+\alpha_1\beta_1\left(\rho_2 u\frac{\partial^{3\beta} u}{\partial x^{3\beta}}+\rho_3\frac{\partial^\beta u}{\partial x^\beta}\frac{\partial^{2\beta} u}{\partial x^{2\beta}}\right)=0, \tag{1.1}$$

(ii) The higher-order wave equation of KdV type (II):

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^\beta u}{\partial x^\beta}+\alpha_1 u\frac{\partial^\beta u}{\partial x^\beta}+\beta_1\frac{\partial^{3\beta} u}{\partial x^{3\beta}}+\alpha_1^2\rho_1^2u^2\frac{\partial^\beta u}{\partial x^\beta}$$
$$+\alpha_1\beta_1\left(\rho_2 u\frac{\partial^{3\beta} u}{\partial x^{3\beta}}+\rho_3\frac{\partial^\beta u}{\partial x^\beta}\frac{\partial^{2\beta} u}{\partial x^{2\beta}}\right)+\alpha_1\rho_4u^3\frac{\partial^\beta u}{\partial x^\beta}$$
$$+\alpha_1^2\beta_1\left(\rho_5u^2\frac{\partial^{3\beta} u}{\partial x^{3\beta}}+\rho_6u\frac{\partial^\beta u}{\partial x^\beta}\frac{\partial^{2\beta} u}{\partial x^{2\beta}}\right.$$
$$\left.+\rho_7\left(\frac{\partial^\beta u}{\partial x^\beta}\right)^3\right)=0, \tag{1.2}$$

where $0<\alpha,\beta\le 1$, and $u(x,t)$ is a real function, while $\rho_i\,(i=1,2,\ldots,7)$ are free parameters and α_1, β_1 are positive real constants. The two equations characterize the long wavelength and short amplitude of the waves, respectively. When $\alpha=\beta=1$, Tzirtzilakis et al. [43] have obtained some soliton-like solutions of Eqs. (1.1) and (1.2) and they said that these two equations are all water wave equations of KdV type which are more physically and practically meaningful. These two equations arise as models in the theory of water wave which were first proposed by Fokas [44]. Also these equations have been discussed in [45] when $\alpha=\beta=1$ using the integral bifurcation theory, where some traveling wave solutions with singular or non singular characters are ob-

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tained. The authors [46] have obtained some exact explicit parametric representations of solitary wave, kink and anti-kink wave solutions and breaking wave solutions of Eq. (1.1) when $\alpha=\beta=1$ under certain conditions. Also the authors [47] have considered the dynamic characters of travelling wave solutions of Eq. (1.1) when $\alpha=\beta=1$ under certain conditions.

This article is organized as follows: in Sect. 2, the description of conformable fractional derivative is given. In Sect. 3, the description of the unified auxiliary equation method combined with the conformable space-time fractional derivatives is obtained. In Sect. 4, we apply this method to the higher-order wave equation of KdV type (III) and the higher-order wave equation of KdV type (II). In Sect. 5, we present the graphical representations for some solutions of Eqs. (1.1) and (1.2). In Sect. 6, conclusions are obtained. To our best knowledge, the two Eqs. (1.1) and (1.2) are not discussed elsewhere using the unified auxiliary equation method combined with the conformable space-time fractional derivatives.

2. Description of the conformable fractional derivative

Khalil et al. [39] introduced a novel definition of fractional derivative named the conformable fractional derivative, which can rectify the deficiencies of the other definitions.

Definition 1. Suppose $f:[0,\infty)\to R$ is a function. Then, the conformable fractional derivative of f of order α is defined as:

$$T_\alpha\left(f\right)\left(t\right)=\lim_{\tau\to 0}\frac{f\left(t+\tau t^{1-\alpha}\right)-f\left(t\right)}{\tau}, \tag{2.1}$$

for all $t>0$ and $\alpha\in(0,1]$. Several properties of the conformable fractional derivative are given below as in [39–41].

Theorem 1. Suppose $\alpha\in(0,1]$, and f and g are $\alpha-$differentiable at $t>0$. Then

$$T_\alpha\left(af+bg\right)=aT_\alpha\left(f\right)+bT_\alpha\left(g\right),\quad \forall a,b\in R. \tag{2.2}$$

$$T_\alpha\left(t^\mu\right)=\mu t^{\mu-\alpha},\quad \forall\mu\in R. \tag{2.3}$$

$$T_\alpha\left(fg\right)=fT_\alpha\left(g\right)+gT_\alpha\left(f\right), \tag{2.4}$$

$$T_\alpha\left(\frac{f}{g}\right)=\left(\frac{gT_\alpha\left(f\right)-fT_\alpha\left(g\right)}{g^2}\right). \tag{2.5}$$

Furthermore, if f is differentiable, then

$$T_\alpha\left(f\right)\left(t\right)=t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}\left(t\right). \tag{2.6}$$

Theorem 2. Suppose $f:[0,\infty)\to R$ is a differentiable function and also $\alpha-$ differentiable. Let g be a function defined in the range of f and also differentiable. Then

$$T_\alpha\left(f\circ g\right)\left(t\right)=t^{1-\alpha}g'\left(t\right)f'\left(g\left(t\right)\right). \tag{2.7}$$

3. Description of the unified auxiliary equation method combined with the conformable space-time fractional derivatives

Consider the following nonlinear PDE:

$$F\left(u,\frac{\partial^\alpha u}{\partial t^\alpha},\frac{\partial^\beta u}{\partial x^\beta},\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}},\frac{\partial^{2\beta} u}{\partial x^{2\beta}},\ldots\right)=0,$$

$$0<\alpha,\beta\leq 1, \tag{3.1}$$

where F is a polynomial in u(x,t) and its partial derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1. We use the conformable space-time wave transformation

$$u(x,t)=u(\xi),\quad \xi=\frac{x^\beta}{\beta}-c_1\frac{t^\alpha}{\alpha}, \tag{3.2}$$

where c_1 is a constant, to reduce Eq. (3.1) to the following ODE:

$$P(u,u',u'',\ldots)=0, \tag{3.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, such that $'=\frac{\mathrm{d}}{\mathrm{d}\xi}$.

Step 2. We assume that Eq. (3.3) has the formal solution

$$u(\xi)=a_0+\sum_{i=1}^{N}f^{i-1}(\xi)\left[a_if(\xi)+b_ig(\xi)\right], \tag{3.4}$$

where a_0, a_i, b_i $(i=1,\ldots,N)$ are constants to be determined later,such that $a_N\neq 0$ or $b_N\neq 0$, while $f(\xi)$ and $g(\xi)$ satisfy the auxiliary ODEs

$$f'(\xi)=f(\xi)g(\xi), \tag{3.5}$$

$$g'(\xi)=q+g^2(\xi)+rf^{-2}(\xi), \tag{3.6}$$

$$g^2(\xi)=-\left[q+\frac{r}{2}f^{-2}(\xi)+cf^2(\xi)\right] \tag{3.7}$$

where q, r, c are constants.

Step 3. We determine the positive integer N in (3.4) by using the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (3.3).

Step 4. We substitute (3.4) along with (3.5)–(3.7) into Eq. (3.3) and collect all terms of the same order of $f^i(\xi)$ $g^j(\xi)$ $(i=0,\pm1,\pm2,..,j=0,1)$ and set them to zero, yield a set of algebraic equations which can be solved by using the Maple or Mathematica to find a_0, a_i, b_i, c_1, q, r, c.

Step 5. It is well-known [38] that the auxiliary ODEs (3.5) and (3.6) have the following Jacobi elliptic function solutions:

(1) If $q=1+m^2$, $r=-2m^2$, $c=-1$, then

$$f(\xi)=\frac{1}{\mathrm{sn}(\xi,m)},\quad g(\xi)=\frac{-\mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m)}{\mathrm{sn}(\xi,m)}.$$

(2) If $q=\left(1-2m^2\right)$, $r=2m^2$, $c=\left(m^2-1\right)$, then

$$f(\xi)=\frac{1}{\mathrm{cn}(\xi,m)},\quad g(\xi)=\frac{\mathrm{sn}(\xi,m)\mathrm{dn}(\xi,m)}{\mathrm{cn}(\xi,m)}.$$

(3) If $q = \left(-2+m^2\right)$, $r = 2$, $c = \left(1-m^2\right)$, then

$$f(\xi) = \frac{1}{\mathrm{dn}(\xi,m)}, \quad g(\xi) = \frac{m^2\mathrm{sn}(\xi,m)\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)}.$$

(4) If $q = \left(1+m^2\right)$, $r = -2$, $c = -m^2$, then

$$f(\xi) = \mathrm{sn}(\xi,m), \quad g(\xi) = \frac{\mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m)}{\mathrm{sn}(\xi,m)} \quad .$$

(5) If $q = \left(1-2m^2\right)$, $r = \left(-2+2m^2\right)$, $c = m^2$, then

$$f(\xi) = \mathrm{cn}(\xi,m), \quad g(\xi) = -\frac{\mathrm{sn}(\xi,m)\mathrm{dn}(\xi,m)}{\mathrm{cn}(\xi,m)}.$$

(6) If $q = \left(-2+m^2\right)$, $r = \left(2-2m^2\right)$, $c = 1$, then

$$f(\xi) = \mathrm{dn}(\xi,m), \quad g(\xi) = -\frac{m^2\mathrm{sn}(\xi,m)\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)}.$$

(7) If $q = \left(-2+m^2\right)$, $r = \left(-2+2m^2\right)$, $c = -1$, then

$$f(\xi) = \frac{\mathrm{cn}(\xi,m)}{\mathrm{sn}(\xi,m)}, \quad g(\xi) = -\frac{\mathrm{dn}(\xi,m)}{\mathrm{sn}(\xi,m)\mathrm{cn}(\xi,m)}.$$

(8) If $q = \left(1-2m^2\right)$, $r = \left(2m^2-2m^4\right)$, $c = -1$, then

$$f(\xi) = \frac{\mathrm{dn}(\xi,m)}{\mathrm{sn}(\xi,m)}, \quad g(\xi) = -\frac{\mathrm{cn}(\xi,m)}{\mathrm{sn}(\xi,m)\mathrm{dn}(\xi,m)}.$$

(9) If $q = \left(-2+m^2\right)$, $r = -2$, $c = \left(-1+m^2\right)$, then

$$f(\xi) = \frac{\mathrm{sn}(\xi,m)}{\mathrm{cn}(\xi,m)}, \quad g(\xi) = \frac{\mathrm{dn}(\xi,m)}{\mathrm{sn}(\xi,m)\mathrm{cn}(\xi,m)}.$$

(10) If $q = \left(1+m^2\right)$, $r = -2m^2$, $c = -1$, then

$$f(\xi) = \frac{\mathrm{dn}(\xi,m)}{\mathrm{cn}(\xi,m)}, \quad g(\xi) = \frac{\left(1-m^2\right)\mathrm{sn}(\xi,m)}{\mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m)}.$$

(11) If $q = \left(1-2m^2\right)$, $r = -2$, $c = \left(m^2-m^4\right)$, then

$$f(\xi) = \frac{\mathrm{sn}(\xi,m)}{\mathrm{dn}(\xi,m)}, \quad g(\xi) = \frac{\mathrm{cn}(\xi,m)}{\mathrm{sn}(\xi,m)\mathrm{dn}(\xi,m)}.$$

(12) If $q = \frac{1}{2}\left(-1+2m^2\right)$, $r = -\frac{1}{2}$, $c = -\frac{1}{4}$, then

$$f(\xi) = \frac{\mathrm{cn}(\xi,m) \pm 1}{\mathrm{sn}(\xi,m)}, \quad g(\xi) = \mp\mathrm{ds}(\xi,m).$$

There are many other Jacobi elliptic solutions of the auxiliary ODEs (3.5) and (3.6) which are omitted here for simplicity (see [38]). In Eqs. (1)–(12) $\mathrm{sn}(\xi,m)$, $\mathrm{cn}(\xi,m)$, $\mathrm{dn}(\xi,m)$ and so on are the Jacobi elliptic sine function, the Jacobi elliptic cosine function, the Jacobi elliptic function of the third kind respectively, and m denotes the modulus of the Jacobi elliptic functions, where $0 < m < 1$. It is well known [47, 48] that the Jacobi elliptic functions degenerate into hyperbolic functions when $m \to 1$ as follows:

$$\mathrm{sn}(\xi,1) \to \tanh(\xi), \quad \mathrm{cn}(\xi,1) \to \mathrm{sech}(\xi),$$

$$\mathrm{dn}(\xi,1) \to \mathrm{sech}(\xi), \quad \mathrm{ns}(\xi,1) \to \coth(\xi), \quad \mathrm{dc}(\xi,1) \to 1,$$

$$\mathrm{ds}(\xi,1) \to \mathrm{cosech}(\xi), \quad \mathrm{sc}(\xi,1) \to \sinh(\xi),$$

$$\mathrm{sd}(\xi,1) \to \sinh(\xi), \quad \mathrm{cs}(\xi,1) \to \mathrm{cosech}(\xi),$$

and into trigonometric functions when $m \to 0$ as follows:

$$\mathrm{sn}(\xi,0) \to \sin(\xi), \quad \mathrm{cn}(\xi,0) \to \cos(\xi), \quad \mathrm{dn}(\xi,0) \to 1,$$

$$\mathrm{ns}(\xi,0) \to \mathrm{cosec}(\xi), \quad \mathrm{cs}(\xi,0) \to \cot(\xi),$$

$$\mathrm{ds}(\xi,0) \to \mathrm{cosec}(\xi), \quad \mathrm{sc}(\xi,0) \to \tan(\xi),$$

$$\mathrm{sd}(\xi,0) \to \sin(\xi), \quad \mathrm{dc}(\xi,0) \to \sec(\xi).$$

Step 6. We substitute the values a_0, a_i, b_i, and the solutions (1)–(12) given in step 5 into (3.4), to get the exact solutions of Eq. (3.1).

4. Applications

In this section, we apply the method described in Sect. 3, to construct the exact solutions including the Jacobi elliptic function solutions, solitons and other wave solutions of the two nonlinear PDEs (1.1) and (1.2) as follows.

4.1. On solving the higher-order wave equation of KdV type (1.1)

To solve Eq. (1.1), we first use the conformable space-time wave transformation

$$u(x,t) = \phi(\xi), \quad \xi = \frac{x^\beta}{\beta} - c_1\frac{t^\alpha}{\alpha}, \tag{4.1}$$

where c_1 is a non zero constant and $0 < \alpha, \beta \leq 1$, to reduce Eq. (1.1) into the following nonlinear ordinary differential equation(ODE):

$$\begin{aligned}&(1-c_1)\phi' + \alpha_1\phi\phi' + \beta_1\phi''' + \alpha_1^2\rho_1\phi^2\phi'\\&+\alpha_1\beta_1\left(\rho_2\phi\phi''' + \rho_3\phi'\phi''\right) = 0.\end{aligned} \tag{4.2}$$

Integrating Eq. (4.2) with respect to ξ and the vanishing constant of integration, we get

$$\begin{aligned}&(1-c_1)\phi + \frac{1}{2}\alpha_1\phi^2 + \beta_1\left(1+\alpha_1\rho_2\phi\right)\phi''\\&+\frac{1}{3}\alpha_1^2\rho_1\phi^3 + \frac{1}{2}\alpha_1\beta_1\left(\rho_3-\rho_2\right)\phi'^2 = 0.\end{aligned} \tag{4.3}$$

Balancing $\phi\phi''$ with ϕ^3 in Eq. (4.3), we get $N = 2$. According to the form (3.4), Eq. (4.3) has the formal solution

$$\begin{aligned}\phi(\xi) &= a_0 + a_1 f(\xi) + b_1 g(\xi) + a_2 f^2(\xi)\\&+b_2 f(\xi)g(\xi),\end{aligned} \tag{4.4}$$

where a_0, a_1, a_2, b_1, and b_2 are constants to be determined, such that $a_2 \neq 0$ or $b_2 \neq 0$. Substituting (4.4) along with (3.5)–(3.7) into Eq. (4.3) and collecting all terms of the same order of $f^i(\xi)\,g^j(\xi)\,,(i = 0, \pm1, \pm2, \ldots, j = 0, 1)$, and setting them to zero, we get a system of algebraic equations, which is omitted here. These equations can be solved with the aid of Maple or Mathematica to get the following two cases:

Case 1.

$$\beta_1 = \beta_1, \quad \alpha_1 = \alpha_1, \quad a_0 = \frac{3\beta_1}{\alpha_1}\left(q + \sqrt{q^2 - \frac{3}{2}cr}\right),$$

$$a_1 = b_1 = b_2 = 0, \quad a_2 = \frac{9\beta_1 c}{\alpha_1},$$

$$\rho_2 = \rho_3 = \frac{1}{24\beta_1\sqrt{q^2 - \frac{3}{2}cr}}, \quad \rho_1 = \frac{1}{12\beta_1\sqrt{q^2 - \frac{3}{2}cr}},$$

$$c_1 = \left(1 + 4\beta_1\sqrt{q^2 - \frac{3}{2}cr}\right),$$

provided $\left(q^2 - \frac{3}{2}cr\right) > 0$, $\beta_1 \neq 0$ and $\alpha_1 \neq 0$ In this case, Eq. (1.1) has the solution

$$u(x,t) = \frac{3\ \beta_1}{\alpha_1}\left[q + \sqrt{q^2 - \frac{3}{2}cr} + 3cf^2(\xi)\right], \qquad (4.5)$$

where $\xi = \frac{x^\beta}{\beta} - \left(1 + 4\beta_1\sqrt{q^2 - \frac{3}{2}cr}\right)\frac{t^\alpha}{\alpha}$.

Case 2.

$$\beta_1 = -\frac{3}{50q\rho_1}, \quad a_0 = a_1 = b_1 = 0, \quad a_2 = -\frac{3c}{10q\rho_1\alpha_1},$$

$$b_2 = \frac{3\sqrt{-c}}{q\rho_1\alpha_1}, \quad c_1 = \frac{12q^2 + 200q^2\rho_1 + 9cr}{200q^2\rho_1},$$

$$\rho_2 = \frac{-5}{3}\rho_1, \quad \rho_1 = \rho_1, \quad \rho_3 = 5\rho_1,$$

provided $c < 0$, $\rho_1 \neq 0$, $\alpha_1 \neq 0$ and $q \neq 0$. In this case, Eq. (1.1) has the solution

$$u(x,t) = \frac{-3c}{10q\rho_1\alpha_1}f^2(\xi) + \frac{3\sqrt{-c}}{q\rho_1\alpha_1}f(\xi)\,g(\xi), \qquad (4.6)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{12q^2+200q^2\rho_1+9cr}{200q^2\rho_1}\right)\frac{t^\alpha}{\alpha}$.

4.2 The solutions of Eq. (1.1) with the aid of case 1

For case 1, Eq. (1.1) has the following results:

Result 1. If we substitute $q = \left(1 + m^2\right)$, $r = -2m^2$, $c = -1$ in (4.5) and use (1) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\left[1{+}m^2 + \sqrt{1{-}m^2 + m^4} - 3\text{ns}^2(\xi,m)\right]. \qquad (4.7)$$

In particular, if $m \to 0$, then we have the periodic solution

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\left[2{-}3\csc^2\left(\frac{x^\beta}{\beta} - [1{+}4\beta_1]\frac{t^\alpha}{\alpha}\right)\right], \quad (4.8)$$

while if $m \to 1$, then we have the singular soliton solution

$$u(x,t) = \frac{-9\beta_1}{\alpha_1}\text{csch}^2\left[\frac{x^\beta}{\beta} - (1 + 4\beta_1)\frac{t^\alpha}{\alpha}\right]. \qquad (4.9)$$

Result 2. If we substitute $q = \left(1 - 2m^2\right)$, $r = 2m^2$, $c = \left(m^2 - 1\right)$ in (4.5) and use (2) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\left[1 - 2m^2 + \sqrt{1 - m^2 + m^4} + 3\left(m^2 - 1\right)nc^2(\xi,m)\right]. \qquad (4.10)$$

In particular, if $m \to 0$, then we have the periodic solution

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\left[2{-}3\sec^2\left(\frac{x^\beta}{\beta}{-}\,(1{+}4\beta_1)\frac{t^\alpha}{\alpha}\right)\right]. \quad (4.11)$$

Result 3. If we substitute $q = \left(-2 + m^2\right)$, $r = 2$, $c = \left(1 - m^2\right)$ in (4.5) and use (3) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\left[m^2 - 2 + \sqrt{1 - m^2 + m^4} + 3\left(1 - m^2\right)\text{nd}^2(\xi,m)\right]. \qquad (4.12)$$

Result 4. If we substitute $q = \left(1 + m^2\right)$, $r = -2$, $c = -m^2$ in (4.5) and use (4) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\left[1 + m^2 + \sqrt{1 - m^2 + m^4} - 3m^2\text{sn}^2(\xi,m)\right]. \qquad (4.13)$$

In particular, if $m \to 1$, then we have the solitary wave solution

$$u(x,t) = \frac{9\beta_1}{\alpha_1}\text{sech}^2\left[\frac{x^\beta}{\beta} - (1 + 4\beta_1)\frac{t^\alpha}{\alpha}\right] \qquad (4.14)$$

Result 5. If we substitute $q = \left(1 - 2m^2\right)$, $r = \left(-2 + 2m^2\right)$, $c = m^2$ in (4.5) and use (5) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\left[1 - 2m^2 + \sqrt{1 - m^2 + m^4} + 3m^2\text{cn}^2(\xi,m)\right]. \qquad (4.15)$$

In particular, if $m \to 1$, then we have the same bright soliton solution (4.14).

Result 6. If we substitute $q = \left(-2 + m^2\right)$, $r = \left(2 - 2m^2\right)$, $c = 1$ in (4.5) and use (6) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\left[m^2 - 2 + \sqrt{1 - m^2 + m^4} + 3\text{dn}^2(\xi,m)\right]. \qquad (4.16)$$

In particular, if m→1, then we have the same bright soliton solution (4.14).

Result 7. If we substitute $q = \left(-2 + m^2\right)$, $r = \left(-2 + 2m^2\right)$, $c = -1$ in (4.5) and use (7) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\Bigg[m^2 - 2 + \sqrt{1-m^2+m^4}$$

$$-3\mathrm{cs}^2(\xi,m)\Bigg]. \tag{4.17}$$

In particular, if $m \to 1$, then we have the singular soliton solution

$$u(x,t) = \frac{-9\beta_1}{\alpha_1}\mathrm{csch}^2\left(\frac{x^\beta}{\beta} - [1+4\beta_1]\frac{t^\alpha}{\alpha}\right), \tag{4.17a}$$

while if $m \to 0$, then we have the periodic solution

$$u(x,t) =$$

$$\frac{-3\beta_1}{\alpha_1}\left[1 + 3\cot^2\left(\frac{x^\beta}{\beta} - (1+4\beta_1)\frac{t^\alpha}{\alpha}\right)\right]. \tag{4.18}$$

Result 8. If we substitute $q = (1-2m^2)$, $r = (2m^2 - 2m^4)$, $c = -1$ in (4.5) and use (8) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\Bigg[1 - 2m^2 + \sqrt{1-m^2+m^4}$$

$$-3\mathrm{ds}^2(\xi,m)\Bigg]. \tag{4.19}$$

In particular, if $m \to 0$, then we have the same periodic solution (4.8), while if $m \to 1$, then we have the same singular soliton solution (4.9).

Result 9. If we substitute $q = (-2+m^2)$, $r = -2$, $c = (-1+m^2)$ in (4.5) and use (9) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\Bigg[m^2 - 2 + \sqrt{1-m^2+m^4}$$

$$+3(-1+m^2)\mathrm{sc}^2(\xi,m)\Bigg]. \tag{4.20}$$

In particular, if $m \to 0$, then we have the periodic solution

$$u(x,t) =$$

$$\frac{-3\beta_1}{\alpha_1}\left[1 + 3\tan^2\left(\frac{x^\beta}{\beta} - (1+4\beta_1)\frac{t^\alpha}{\alpha}\right)\right]. \tag{4.21}$$

Result 10. If we substitute $q = (1+m^2)$, $r = -2m^2$, $c = -1$ in (4.5) and use (10) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\Bigg[1 + m^2 + \sqrt{1-m^2+m^4}$$

$$-3\mathrm{dc}^2(\xi,m)\Bigg]. \tag{4.22}$$

In particular, if $m \to 0$, then we have the same periodic solution (4.11).

Result 11. If we substitute $q = (1-2m^2)$, $r = -2$, $c = (m^2 - m^4)$ in (4.5) and use (11) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{\alpha_1}\Bigg[1 - 2m^2 + \sqrt{1-m^2+m^4}$$

$$+3(m^2 - m^4)\mathrm{sd}^2(\xi,m)\Bigg]. \tag{4.23}$$

In the above results (1)–(11), we have $\xi = \frac{x^\beta}{\beta} - \left[1 + 4\beta_1\sqrt{1-m^2+m^4}\right]\frac{t^\alpha}{\alpha}$.

Result 12. If we substitute $q = \frac{1}{2}(-1+2m^2)$, $r = -\frac{1}{2}$, $c = -\frac{1}{4}$ in (4.5) and use (12) of step 5 in Sect. 3, then Eq. (1.1) has the Jacobi elliptic solutions

$$u(x,t) = \frac{3\beta_1}{4\alpha_1}\Bigg[4m^2 - 2 + \sqrt{1-16m^2+16m^4}$$

$$-3\left(\frac{\mathrm{cn}(\xi,m) \pm 1}{\mathrm{sn}(\xi,m)}\right)^2\Bigg] \tag{4.24}$$

where $\xi = \frac{x^\beta}{\beta} - \left[1 + \beta_1\sqrt{1-16m^2+16m^4}\right]\frac{t^\alpha}{\alpha}$. In particular, if $m \to 0$, then we have the periodic solution

$$u(x,t) = \frac{-3\beta_1}{4\alpha_1}\Bigg[1 + 3\Bigg(\cot\left(\frac{x^\beta}{\beta} - (1+\beta_1)\frac{t^\alpha}{\alpha}\right)$$

$$\pm\csc\left(\frac{x^\beta}{\beta} - (1+\beta_1)\frac{t^\alpha}{\alpha}\right)\Bigg)^2\Bigg], \tag{4.25}$$

while if $m \to 1$, then we have the following singular soliton solution

$$u(x,t) = \frac{9\beta_1}{4\alpha_1}\Bigg[1 - \Bigg(\mathrm{csch}\left(\frac{x^\beta}{\beta} - (1+\beta_1)\frac{t^\alpha}{\alpha}\right)$$

$$\pm\coth\left(\frac{x^\beta}{\beta} - (1+\beta_1)\frac{t^\alpha}{\alpha}\right)\Bigg)^2\Bigg]. \tag{4.26}$$

Similarly, we can find many other solutions of Eq. (1.1) using (4.6) which are omitted here for simplicity.

4.3. Higher-order wave equations of KdV type (1.2)

To solve Eq. (1.2), we first use the same conformable space-time wave transformation (4.1), to reduce Eq. (1.2) into the following nonlinear ODE:

$$(1-c_1)\phi' + \beta_1\left(1 + \alpha_1\rho_2\phi + \alpha_1^2\rho_5\phi^2\right)\phi'''$$

$$+\alpha_1\phi\phi'\left(1 + \alpha_1\rho_1^2\phi + \rho_4\phi^2\right)$$

$$+\alpha_1\beta_1(\rho_3 + \alpha_1\rho_6\phi)\phi'\phi'' + \alpha_1^2\beta_1\rho_7\phi'^3 = 0. \tag{4.27}$$

Balancing $\phi^2\phi'''$ with $\phi^3\phi'$ in Eq. (4.27), we get $N = 2$. According to the form (3.4), Eq. (4.27) has the same formal solution (4.4). Substituting (4.4) along with (3.5)–(3.7) into Eq. (4.27), collecting all terms of the same order of $f^i(\xi)g^j(\xi)$, $(i = 0, \pm1, \pm2, \ldots, j = 0, 1)$, and setting them to zero, then we get a system of algebraic equations which is omitted here. These equations can be solved with the aid of Maple or Mathematica to get the following two cases:

Case 3.

$$\beta_1=\beta_1,\quad \alpha_1=\alpha_1,\quad a_0=\frac{4q\beta_1}{\alpha_1},$$
$$a_1=b_1=b_2=\rho_6=\rho_7=0,\quad a_2=\frac{12\beta_1 c}{\alpha_1},$$
$$\rho_2=\rho_2,\quad \rho_3=\left(2\rho_1^2-2\rho_2\right),\quad \rho_4=\alpha_1^2\rho_5,\quad \rho_5=\rho_5,$$
$$c_1=\left[1+8\beta^2\left(\rho_1^2-\rho_2\right)\left(2q^2-3cr\right)\right],$$

provided $\alpha_1\neq 0$. In this case, Eq. (1.2) has the solution

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[q+3cf^2(\xi)\right], \tag{4.28}$$

where $\xi=\frac{x^\beta}{\beta}-\left[1+8\beta^2\left(\rho_1^2-\rho_2\right)\left(2q^2-3cr\right)\right]\frac{t^\alpha}{\alpha}$.

Case 4.

$$r_1=\frac{-1}{162}\frac{q^2\left(63\rho_1^2\rho_6+64\rho_3^3\right)}{c\rho_1^2\rho_6},\quad \alpha_1=\alpha_1,$$
$$\beta_1=\frac{-9\rho_1^2}{8q\rho_3^2},\quad a_0=\frac{3}{8\alpha_1\rho_3},\quad a_1=\frac{9}{4\rho_3\alpha_1}\sqrt{\frac{-2c}{q}},$$
$$b_1=b_2=\rho_7=0,\quad a_2=\frac{9c}{4q\rho_3\alpha_1},\quad \rho_1=\rho_1,$$
$$\rho_2=\frac{3\rho_6-2\rho_3^2}{3\rho_3},\quad \rho_4=\frac{\alpha_1^2\rho_1^2\rho_6}{\rho_3},\quad \rho_5=\frac{-2\rho_6}{3},$$
$$c_1=\left(1-\frac{\rho_3}{\rho_6}\right),\quad \rho_6=\rho_6,$$

provided $\rho_3\neq 0$, $\rho_1\neq 0$, $\rho_6\neq 0$, $\alpha_1\neq 0$, $qc<0$. In this case, Eq. (1.2) has the solution

$$u(x,t)=\left[\frac{3}{8\alpha_1\rho_3}+\frac{9}{4\rho_3\alpha_1}\sqrt{\frac{-2c}{q}}f(\xi)+\frac{9c}{4q\rho_3\alpha_1}f^2(\xi)\right], \tag{4.29}$$

where $\xi=\frac{x^\beta}{\beta}-\left(1-\frac{\rho_3}{\rho_6}\right)\frac{t^\alpha}{\alpha}$.

4.4. The solutions of Eq. (1.2) with the aid of case 3

For case 3, Eq. (1.2) has the following results:

Result 1. If we substitute $q=\left(1+m^2\right), r=-2m^2, c=-1$ in (4.28) and use (1) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(1+m^2\right)-3\mathrm{ns}^2(\xi,m)\right]. \tag{4.30}$$

In particular, if $m\to 0$, then we have the periodic solution

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[1-3\csc^2\left(\frac{x^\beta}{\beta}-\left[1+16\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right], \tag{4.31}$$

while if $m\to 1$, then we have the singular soliton solution

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[2-3\coth^2\left(\frac{x^\beta}{\beta}-\left[1+16\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right]. \tag{4.32}$$

Result 2. If we substitute $q=\left(1-2m^2\right)$, $r=2m^2$, $c=\left(m^2-1\right)$ in (4.28) and use (2) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(1-2m^2\right)+3\left(m^2-1\right)nc^2(\xi,m)\right]. \tag{4.33}$$

In particular, if $m\to 0$, then we have the periodic solution

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[1-3\sec^2\left(\frac{x^\beta}{\beta}-\left[1+16\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right]. \tag{4.34}$$

Result 3. If we substitute $q=\left(-2+m^2\right)$, $r=2$, $c=\left(1-m^2\right)$ in (4.28) and use (3) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(-2+m^2\right)+3\left(1-m^2\right)\mathrm{nd}^2(\xi,m)\right]. \tag{4.35}$$

Result 4. If we substitute $q=\left(1+m^2\right)$, $r=-2$, $c=-m^2$ in (4.28) and use (4) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(1+m^2\right)-3m^2\mathrm{sn}^2(\xi,m)\right]. \tag{4.36}$$

In particular, if $m\to 1$, then we have the shock wave solution

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\times\left[2-3\tanh^2\left(\frac{x^\beta}{\beta}-\left[1+16\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right]. \tag{4.37}$$

Result 5. If we substitute $q=\left(1-2m^2\right)$, $r=\left(-2+2m^2\right)$, $c=m^2$ in (4.28) and use (5) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(1-2m^2\right)+3m^2\mathrm{cn}^2(\xi,m)\right]. \tag{4.38}$$

In particular, if $m\to 1$, then we have the bright soliton solution

$$u(x,t)=\frac{-4\beta_1}{\alpha_1}\times\left[1-3\mathrm{sech}^2\left(\frac{x^\beta}{\beta}-\left[1+16\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right]. \tag{4.39}$$

Result 6. If we substitute $q=\left(-2+m^2\right)$, $r=\left(2-2m^2\right)$, $c=1$ in (4.28) and use (6) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(-2+m^2\right)+3\mathrm{dn}^2(\xi,m)\right]. \tag{4.40}$$

In particular, if $m\to 1$, then we have the same bright soliton solution (4.39).

Result 7. If we substitute $q=\left(-2+m^2\right)$, $r=\left(-2+2m^2\right)$, $c=-1$ in (4.28) and use (7) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(-2+m^2\right)-3\mathrm{cs}^2(\xi,m)\right]. \quad (4.41)$$

In particular, if $m\to 1$, then we have the singular solitary wave solution

$$u(x,t)=\frac{-4\beta_1}{\alpha_1} \quad (4.42)$$
$$\times\left[1+3\mathrm{csch}^2\left(\frac{x^\beta}{\beta}-\left[1+16\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right],$$

while if $m\to 0$, then we have the periodic solution

$$u(x,t)=\frac{-4\beta_1}{\alpha_1} \quad (4.43)$$
$$\times\left[2+3\cot^2\left(\frac{x^\beta}{\beta}-\left[1+16\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right].$$

Result 8. If we substitute $q=\left(1-2m^2\right)$, $r=\left(2m^2-2m^4\right)$, $c=-1$ in (4.28) and use (8) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(1-2m^2\right)-3\mathrm{ds}^2(\xi,m)\right]. \quad (4.44)$$

In particular, if $m\to 0$, then we have the same periodic solution (4.31), while if $m\to 1$, then we have the same singular soliton solution (4.42).

Result 9. If we substitute $q=\left(-2+m^2\right)$, $r=-2$, $c=\left(-1+m^2\right)$ in (4.28) and use (9) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(-2+m^2\right)+3\left(-1+m^2\right)\mathrm{sc}^2(\xi,m)\right]. \quad (4.45)$$

In particular, if $m\to 0$, then we have the periodic solution

$$u(x,t)=\frac{-4\beta_1}{\alpha_1} \quad (4.46)$$
$$\times\left[2+3\tan^2\left(\frac{x^\beta}{\beta}-\left[1+16\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right].$$

Result 10. If we substitute $q=\left(1+m^2\right)$, $r=-2m^2$, $c=-1$ in (4.28) and use (10) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(m^2+1\right)-3\mathrm{dc}^2(\xi,m)\right]. \quad (4.47)$$

In particular, if $m\to 0$, then we have the same periodic solution (4.34).

Result 11. If we substitute $q=\left(1-2m^2\right)$, $r=-2$, $c=\left(m^2-m^4\right)$ in (4.28) and use (11) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=\frac{4\beta_1}{\alpha_1}\left[\left(1-2m^2\right)+3\left(m^2-m^4\right)\mathrm{sd}^2(\xi,m)\right]. \quad (4.48)$$

In the above results (1)–(11), we have $\xi=\frac{x^\beta}{\beta}-\left[1+16\beta_1^2\left(\rho_1^2-\rho_2\right)\left(1-m^2+m^4\right)\right]\frac{t^\alpha}{\alpha}$.

Result 12. If we substitute $q=\frac{1}{2}\left(-1+2m^2\right)$, $r=\frac{-1}{2}$, $c=\frac{-1}{4}$ in (4.28) and use (12) of step 5 in Sect. 3, then Eq. (1.2) has the Jacobi elliptic solutions

$$u(x,t)=$$
$$\frac{\beta_1}{\alpha_1}\left[2\left(-1+2m^2\right)-3\left(\frac{\mathrm{cn}(\xi,m)\pm 1}{\mathrm{sn}(\xi,m)}\right)^2\right], \quad (4.49)$$

where $\xi=\frac{x^\beta}{\beta}-\left[1+\beta^2\left(\rho_1^2-\rho_2\right)\left(1-16m^2+16m^4\right)\right]\frac{t^\alpha}{\alpha}$. In particular, if $m\to 0$, then we have the periodic solution

$$u(x,t)=$$
$$\frac{-\beta_1}{\alpha_1}\left(2+3\left[\cot\left(\frac{x^\beta}{\beta}-\left[1+\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right.\right.$$
$$\left.\left.\pm\csc\left(\frac{x^\beta}{\beta}-\left[1+\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right]^2\right), \quad (4.50)$$

while if $m\to 1$, then we have the following singular soliton solution:

$$u(x,t)=$$
$$\left(\frac{\beta_1}{\alpha_1}\left[2-3\left[\mathrm{csch}\left(\frac{x^\beta}{\beta}-\left[1+\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right.\right.\right.$$
$$\left.\left.\left.\pm\coth\left(\frac{x^\beta}{\beta}-\left[1+\beta^2\left(\rho_1^2-\rho_2\right)\right]\frac{t^\alpha}{\alpha}\right)\right]^2\right]\right). \quad (4.51)$$

Similarly, we can find many other solutions of Eq. (1.2) using (4.29) which are omitted here for simplicity.

5. The graphical representations of some solutions

In this section, we present some graphs of the solitons and other solutions of Eqs. (1.1) and (1.2). Let us now examine Figs. 1–4 illustrating some of our solutions obtained in this article. To this aim, we select some special values of the parameters obtained, for example, in the solutions (4.14), (4.26), (4.37) and (4.46) of Eqs. (1.1) and (1.2). For more convenience the graphical representations of these solutions are shown in the following figures.

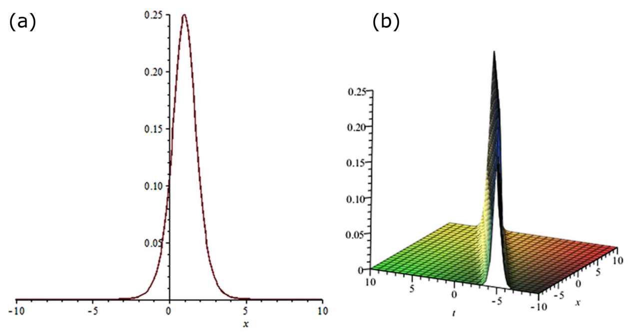

Fig. 1. (a) 2D graph of solitary wave solution (4.14) with $\beta_1=1/4$, $\alpha_1=9$, $\alpha=1$, $\beta=1$, $t=1/2$ and $-10\le x\le 10$; (b) 3D graph of solitary wave solution (4.14) with $\beta_1=1/4$, $\alpha_1=9$, $\alpha=1$, $\beta=1$ and $-10\le x,t\le 10$.

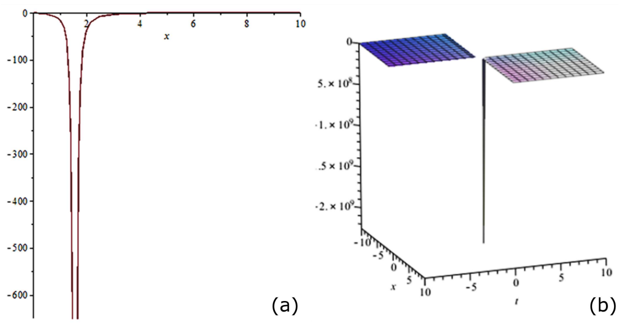

Fig. 2. (a) 2D graph of conformable singular solitary wave solution (4.26) with$\beta_1 = 1/4$, $\alpha_1 = 1/9$, $\alpha = 1/2$, $\beta = 1/2$, $t = 1$ and $-10 \leq x \leq 10$; (b) 3D graph of conformable singular solitary wave solution (4.26) with $\beta_1 = 1/4$, $\alpha_1 = 1/9$, $\alpha = 1/2$, $\beta = 1/2$ and $-10 \leq x, t \leq 10$.

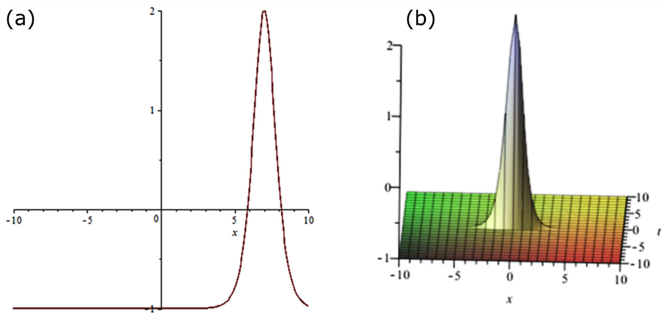

Fig. 3. (a) 2D graph of shock wave solution (4.37) with $\beta_1 = 1/8$, $\alpha_1 = 1/2$, $\rho_1 = 2$, $\rho_2 = 1$, $\alpha = 1$, $\beta = 1$, $t = 1/7$ and $-10 \leq x \leq 10$; (b) 3D graph of shock wave solution (4.37) with $\beta_1 = 1/8$, $\alpha_1 = 1/2$, $\rho_1 = 2$, $\rho_2 = 1$, $\alpha = 1$, $\beta = 1$ and $-10 \leq x, t \leq 10$.

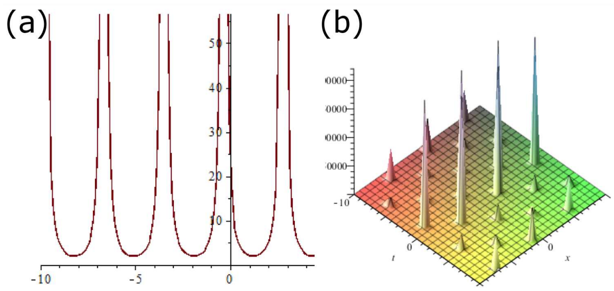

Fig. 4. (a) 2D graph of conformable periodic solution (4.46) with $\beta_1 = -1/4$, $\alpha_1 = 1$, $\rho_1 = 1$, $\rho_2 = 2$, $\alpha = 1/3$, $\beta = 1/3$, $t = 1/2$ and $-10 \leq x \leq 10$; (b) 3D graph of conformable periodic solution (4.46) with $\beta_1 = -1/4$, $\alpha_1 = 1$, $\rho_1 = 1$, $\rho_2 = 2$, $\alpha = 1/3$, $\beta = 1/3$ and $-10 \leq x, t \leq 10$.

From the above figures, one can see that Figs. 1–4 possess the solitary wave solution, the conformable singular solitary wave solution, the shock wave solution and the conformable periodic wave solution, respectively of the Eqs. (1.1) and (1.2). Also, these Figures express the behaviour of these solutions which give some perspective readers how the behaviour solutions are produced.

6. Conclusions

The originality of this manuscript is to derive for the first time, many families of Jacobi elliptic function solutions and other solutions for two higher-order nonlinear wave equations of KdV type (1.1) and (1.2) using the unified auxiliary equation method combined with the conformable space-time fractional derivatives. Different types of these solutions via solitary wave solutions, shock wave solutions, singular solitary wave solutions and trigonometric wave solutions are found analytically and represented successfully via graphics. By comparing our new results in this article with the well known results obtained in [43–47], we conclude that our results are new and not found elsewhere. From these discussions, we conclude that the proposed method of Sect. 3, is direct, concise and an effective powerful mathematical tool for obtaining the exact solutions of other nonlinear evolution equations. Finally, our results in this article have been checked using the Maple by putting them back into the original Eqs. (1.1) and (1.2).

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