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Interaction of Electromagnetic Waves with Multi Periodic Modulated Dielectric Filling of a Regular Waveguide

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The propagation of transverse electric (TE) and transverse magnetic (TM) signal waves in the waveguide with arbitrary cross-section is considered. It is assumed that the dielectric filling of the waveguide is modulated on z coordinate (the axis Oz is the axis of the waveguide) under multi periodic law with small modulation indexes $m_{q\varepsilon} \ll 1$ ($q = 1, 2, 3, 4, 5$). The wave equations for H_z and E_z describing TE and TM fields in the waveguide are received. With the help of change of variables these differential equations are reduced to the Mathieu–Hill equations with periodic coefficients. Analytic solutions of these equations are found up to small modulation indexes in the first degree in the region of "weak" interaction between the signal wave and the modulation wave, when the first-order Wolf–Bragg condition for the waves reflected from seals at their interference is not satisfied. The results show that TE and TM fields in the waveguide represent the sum of spatial harmonics (zero, plus, and minus one) with various amplitudes.

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1. Introduction

The interaction of electromagnetic signal wave with periodically modulated filling of the waveguide of arbitrary cross-section was considered in our early articles (see, for example, [1–7]). This article is devoted to the study of the interaction of the signal wave with multi periodically modulated non-magnetic filling of the waveguide, when the signal wave propagates along the Oz axis (Oz axis coincides with the axis of the waveguide). Note that the similar investigation has both theoretical and practical interest [8].

2. Statement of the problem and its solution

Let the electromagnetic signal wave with a frequency ω propagate in a regular waveguide of arbitrary cross-section, the axis of which coincides with Oz axis of some rectangular coordinate system. It is assumed that the dielectric permittivity ε of non-magnetic filling of the waveguide is modulated in space by harmonic law

$$\varepsilon\left(z\right) = \varepsilon^0\left(1 + \sum_{q=1}^{5} m_{q\varepsilon \cos k_q z}\right), \tag{1}$$

where ε^0 is the dielectric permittivity of the filling in the absence of modulation, k_1, k_2, k_3, k_4, k_5 are the wave numbers of the modulation waves, and $m_{1\varepsilon} \approx m_{2\varepsilon} \approx m_{3\varepsilon} \approx m_{4\varepsilon} \approx m_{5\varepsilon} << 1$ are the small modulation indexes. As is known (see, for example, [1]), the TE and TM fields in the waveguide are described by longitudinal components of magnetic and electric vectors (H_z and $\tilde{E}_z = \varepsilon(z)E_z$) and they satisfy the following wave equations:

$$\Delta_\perp H_z + \frac{\partial^2 H_z}{\partial z^2} - \varepsilon_0\mu_0\varepsilon\left(z\right)\frac{\partial^2 H_z}{\partial t^2} = 0, \tag{2}$$

$$\Delta_\perp \tilde{E}_z + \varepsilon\left(z\right)\frac{\partial}{\partial z}\left[\frac{1}{\varepsilon\left(z\right)}\frac{\partial \tilde{E}_z}{\partial z}\right] - \varepsilon_0\mu_0\varepsilon\left(z\right)\frac{\partial^2 \tilde{E}_z}{\partial t^2} = 0, \tag{3}$$

where ε_0 and μ_0 are the dielectric and the magnetic constants, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We look for the solutions of Eqs. (2) and (3) in the form

$$H_z\left(x,y,z,t\right) = \mathrm{e}^{\mathrm{i}\omega t}\sum_{n=0}^{\infty} H_n\left(z\right)\widehat{\psi}_n\left(x,y\right), \tag{4}$$

$$\tilde{E}_z\left(x,y,z,t\right) = \mathrm{e}^{\mathrm{i}\omega t}\sum_{n=0}^{\infty} \tilde{E}_n\left(z\right)\psi_n\left(x,y\right), \tag{5}$$

where the functions $\widehat{\psi}_n(x,y)$ and $\psi_n(x,y)$ satisfy the following equations and boundary conditions:

$$\Delta_\perp\widehat{\psi}_n\left(x,y\right) + \widehat{\lambda}_n\widehat{\psi}_n\left(x,y\right) = 0, \quad \left.\frac{\partial\widehat{\psi}_n\left(x,y\right)}{\partial \boldsymbol{n}}\right|_\Sigma = 0, \tag{6}$$

$$\Delta_\perp\psi_n\left(x,y\right) + \lambda_n\psi_n\left(x,y\right) = 0, \quad \psi_n\left(x,y\right)\big|_\Sigma = 0. \tag{7}$$

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Note that in (6) and (7) $\widehat{\lambda}_n$ and λ_n are the eigenvalues of the second and first boundary value problems for the cross-section of the waveguide with a contour Σ and a normal vector $\boldsymbol{n}$ to Σ. From Maxwell's equations it is possible to receive analytical expressions for the transverse components of the TE and TM fields in the waveguide. They are expressed by following formulae:

$$\boldsymbol{H}_\tau^{(\mathrm{TE})} = \sum_{n=0}^{\infty} \widehat{\lambda}_n^{-2} \frac{\partial H_n(z,t)}{\partial z} \nabla \widehat{\psi}_n(x,y), \tag{8}$$

$$\boldsymbol{E}_\tau^{(\mathrm{TE})} = \mu_0 \sum_{n=0}^{\infty} \widehat{\lambda}_n^{-2} \frac{\partial H_n(z,t)}{\partial t} \left[\boldsymbol{z}_0 \nabla \widehat{\psi}_n(x,y)\right], \tag{9}$$

$$\boldsymbol{H}_\tau^{(\mathrm{TM})} = -\varepsilon_0 \sum_{n=0}^{\infty} \lambda_n^{-2} \frac{\partial \tilde{E}_n(z,t)}{\partial t} \left[\boldsymbol{z}_0 \nabla \psi_n(x,y)\right], \tag{10}$$

$$\boldsymbol{E}_\tau^{(\mathrm{TM})} = \frac{1}{\varepsilon(z)} \sum_{n=0}^{\infty} \lambda_n^{-2} \frac{\partial \tilde{E}_n(z,t)}{\partial z} \nabla \psi_n(x,y), \tag{11}$$

where $\nabla = \boldsymbol{i}(\partial/\partial x) + \boldsymbol{j}(\partial/\partial y)$, $\boldsymbol{z}_0$ is the unit vector of the Oz axis, and τ indicates the transverse component. If Eqs. (4) and (5) are now substituted into (2) and (3) and some transformations are performed, taking into account (6) and (7), then for $H_n(z)$ and $\tilde{E}_n(z)$, we obtain the following differential equations with periodic coefficients:

$$\frac{\mathrm{d}H_n(z)}{\mathrm{d}z^2} + \widehat{\phi}_n(z) H_n(z) = 0, \tag{12}$$

$$\varepsilon(z) \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\varepsilon(z)} \frac{\mathrm{d}\tilde{E}_n(z)}{\mathrm{d}z}\right] + \phi_n(z) \tilde{E}_n(z) = 0, \tag{13}$$

where

$$\widehat{\phi}_n(z) = \varepsilon_0 \mu_0 \varepsilon^0 \left(1 + \sum_{q=1}^{5} m_{q\varepsilon} \cos k_q z\right) \omega^2 - \widehat{\lambda}^2, \tag{14}$$

$$\phi_n(z) = \varepsilon_0 \mu_0 \varepsilon^0 \left(1 + \sum_{q=1}^{5} m_{q\varepsilon} \cos k_q z\right) \omega^2 - \lambda^2. \tag{15}$$

In Eq. (12) we make the change of variable according to formula $\widehat{s} = (\sum_{l=1}^{5} k_l) z/2$, and in (13) — according to formula $s = (\sum_{l=1}^{5} k_l)(2\varepsilon^0)^{-1} \int_0^z \varepsilon(z)\,\mathrm{d}z$. Even if we do not take into account the terms that depend on the modulation indexes in the arguments of the cosines, then from (12) and (13), we obtain the following equations:

$$\frac{\mathrm{d}^2 H_n(\widehat{s})}{\mathrm{d}\widehat{s}^2} + \widehat{F}_n(\widehat{s}) H_n(\widehat{s}) = 0, \tag{16}$$

$$\frac{\mathrm{d}^2 \tilde{E}_n(\widehat{s})}{\mathrm{d}\widehat{s}^2} + F_n(\widehat{s}) \tilde{E}_n(\widehat{s}) = 0, \tag{17}$$

where

$$\widehat{F}_n(\widehat{s}) = 4 \left(\sum_{l=1}^{5} k_l\right)^{-2} \times \left[\varepsilon_0 \mu_0 \varepsilon^0 \omega^2 f(\widehat{s}) - \widehat{\lambda}^2\right], \tag{18}$$

$$F_n(\widehat{s}) = 4 \left(\sum_{l=1}^{5} k_l\right)^{-2} \times \left[\varepsilon_0 \mu_0 \varepsilon^0 \omega^2 - \lambda^2 - \left(\varepsilon_0 \mu_0 \varepsilon^0 \omega^2 - 2\lambda_n^2\right) f(\widehat{s})\right], \tag{19}$$

$$f(\widehat{s}) = 1 + \sum_{q=1}^{5} \cos\left[2 \left(\sum_{l=1}^{5} k_l\right)^{-1} m_{q\varepsilon}\right]. \tag{20}$$

Note that (16) and (17) are ordinary differential equations with periodic coefficients of the Mathieu–Hill type. If we take into account only the first three harmonics, then we can rewrite them in the form

$$\frac{\mathrm{d}^2 H_n(\widehat{s})}{\mathrm{d}\widehat{s}^2} + \left(\sum_{k=-1}^{1} \widehat{\theta}_k^n e^{2ik\widehat{s}}\right) H_n(\widehat{s}) = 0, \tag{21}$$

$$\frac{\mathrm{d}^2 \tilde{E}_n(\widehat{s})}{\mathrm{d}\widehat{s}^2} + \left(\sum_{k=-1}^{1} \theta_k^n e^{2ik\widehat{s}}\right) \tilde{E}_n(\widehat{s}) = 0, \tag{22}$$

where $\widehat{\theta}_k^n$ and θ_k^n are the Fourier coefficients of the Fourier representations of the functions $\widehat{F}_n(\widehat{s})$ and $F_n(\widehat{s})$ and have the form

$$\widehat{\theta}_0^n = 4 \left(\sum_{l=1}^{5} k_l\right)^{-2} \left(\varepsilon_0 \mu_0 \varepsilon^0 \omega^2 - \widehat{\lambda}_n^2\right), \tag{23}$$

$$\theta_0^n = 4 \left(\sum_{l=1}^{5} k_l\right)^{-2} \left(\varepsilon_0 \mu_0 \varepsilon^0 \omega^2 - \lambda_n^2\right), \tag{24}$$

$$\widehat{\theta}_{\pm 1}^n = \varepsilon_0 \mu_0 \varepsilon^0 \pi^{-1} \left(\sum_{l=1}^{5} k_l\right)^{-2} D_\varepsilon, \tag{25}$$

$$\theta_{\pm 1}^n = \left(2\lambda_n^2 - \varepsilon_0 \mu_0 \varepsilon^0 \omega^2\right) \pi^{-1} \left(\sum_{l=1}^{5} k_l\right)^{-2} D_\varepsilon, \tag{26}$$

$$D_\varepsilon = B_\varepsilon m_{1\varepsilon} + B_\beta m_{2\varepsilon} + B_\gamma m_{3\varepsilon} + B_\eta m_{4\varepsilon} + B_\nu m_{5\varepsilon}, \tag{27}$$

$$B_\alpha = \frac{\sin 2(\alpha - 1)\pi}{\alpha - 1} + \frac{\sin 2(\alpha + 1)\pi}{\alpha + 1},$$

$$B_\beta = \frac{\sin 2(\beta - 1)\pi}{\beta - 1} + \frac{\sin 2(\beta + 1)\pi}{\beta + 1},$$

$$B_\gamma = \frac{\sin 2(\gamma - 1)\pi}{\gamma - 1} + \frac{\sin 2(\gamma + 1)\pi}{\gamma + 1}, \tag{28}$$

$$B_\eta = \frac{\sin 2(\eta - 1)\pi}{\eta - 1} + \frac{\sin 2(\eta + 1)\pi}{\eta + 1},$$

$$B_\nu = \frac{\sin 2(\nu - 1)\pi}{\nu - 1} + \frac{\sin 2(\nu + 1)\pi}{\nu + 1}, \tag{29}$$

$$\alpha = k_1\left(\sum_{l=1}^{5} k_l\right)^{-1}, \quad \beta = k_2\left(\sum_{l=1}^{5} k_l\right)^{-1},$$
$$\gamma = k_3\left(\sum_{l=1}^{5} k_l\right)^{-1}, \quad \eta = k_4\left(\sum_{l=1}^{5} k_l\right)^{-1},$$
$$\nu = k_5\left(\sum_{l=1}^{5} k_l\right)^{-1}. \tag{30}$$

The solutions of the Mathieu–Hill Eqs. (21) and (22) will be sought in the form

$$H_n\left(\widehat{s}\right) = \mathrm{e}^{\mathrm{i}\,\widehat{\mu}_n \widehat{s}} \sum_{k=-1}^{1} \widehat{C}_k^n \mathrm{e}^{2\mathrm{i}\,k\widehat{s}},$$
$$\tilde{E}_n\left(\widehat{s}\right) = \mathrm{e}^{\mathrm{i}\,\mu_n \widehat{s}} \sum_{k=-1}^{1} C_k^n \mathrm{e}^{2\mathrm{i}\,k\widehat{s}}. \tag{31}$$

By substituting (31) in (21) and (22) to determine the characteristic numbers $\widehat{\mu}_n$ and μ_n, we obtain the dispersion equations and to determine the coefficients $\widehat{C}_k^n$ and C_k^n we obtain the system of algebraic equations. By solving the obtained equations in the region of weak interaction between the signal wave and modulation wave and by limiting ourselves to terms proportional to the modulation indexes in the first power, we obtain

$$\widehat{\mu}_n \cong \sqrt{\widehat{\theta}_0^n} = 2\left(\sum_{l=1}^{5} k_l\right)^{-1} \sqrt{\varepsilon_0\mu_0\varepsilon^0\omega^2 - \widehat{\lambda}_n^2},$$
$$\mu_n \cong \sqrt{\theta_0^n} = 2\left(\sum_{l=1}^{5} k_l\right)^{-1} \sqrt{\varepsilon_0\mu_0\varepsilon^0\omega^2 - \lambda_n^2}, \tag{32}$$

$$\widehat{C}_{\pm 1}^n \cong \frac{\widehat{\theta}_1^n \widehat{C}_0^n}{4\left(1 \pm \sqrt{\widehat{\theta}_0^n}\right)}, \quad C_{\pm 1}^n \cong \frac{\theta_1^n C_0^n}{4\left(1 \pm \sqrt{\theta_0^n}\right)}, \tag{33}$$

where $\widehat{C}_0^n$ and C_0^n are determined from the normalization conditions. Now in (31) passing to the variable z and substituting them in (4) and (5) we get

$$H_z\left(x, y, z, t\right) = \sum_{n=0}^{\infty} \widehat{\psi}_n\left(x, y\right) \mathrm{e}^{\mathrm{i}\left(\omega t + \widehat{p}_0^n z\right)}$$
$$\times \sum_{k=-1}^{1} \widehat{C}_k^n \mathrm{e}^{\mathrm{i}\,k(k_1+k_2+k_3+k_4+k_5)z}, \tag{34}$$

$$\tilde{E}_z\left(x, y, z, t\right) = \sum_{n=0}^{\infty} \psi_n\left(x, y\right) \mathrm{e}^{\mathrm{i}\,(\omega t + p_0^n z)}$$
$$\times \sum_{k=-1}^{1} C_k^n \mathrm{e}^{\mathrm{i}\,k(k_1+k_2+k_3+k_4+k_5)z}, \tag{35}$$

where

$$\widehat{p}_0^n = \sqrt{\varepsilon_0\mu_0\varepsilon^0\omega^2 - \widehat{\lambda}_n^2}, \quad p_0^n = \sqrt{\varepsilon_0\mu_0\varepsilon^0\omega^2 - \lambda_n^2}. \tag{36}$$

As formulae (34) and (35) show, the transverse electric and transverse magnetic fields in the waveguide with multi periodically modulated filling represent a set of harmonics (zero, plus and minus one) with different amplitudes. Moreover, if the amplitude on the fundamental (zero) harmonic does not depend on the modulation indexes, then the amplitudes on the lateral harmonics (plus and minus one) depend on the modulation indexes in the first degree.

3. Conclusion

We note that the results obtained in this paper make it possible in the future to investigate certain features of the propagation of a signal wave in the waveguide with multi periodically modulated filling in the region of "strong" (resonant) interaction of a signal wave with a modulation wave. We also note that, using the method developed in this paper, it is possible to solve the problems of radiation of a charged particle moving with constant velocity along and perpendicular to the axis of the waveguide with a multi periodically modulated filling.

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