Soliton Solutions and Conservation Laws of a (3+1)-Dimensional Nonlinear Evolution Equation

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This work studies a new (3+1)-dimensional nonlinear model, which was introduced by Abdul-Majid Wazwaz in 2014. This new physical model describes the shallow-water waves and short waves in nonlinear dispersive models. Analytical traveling wave solutions including the solitons and plane wave solutions are derived by using the G'/G-expansion technique. Moreover, the conserved quantities of this model are also given.

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1. Introduction

Studying the nonlinear evolution equation arising in natural science is very important for us to understand and explain the nonlinear phenomena [1–14]. By the analytical and numerical methods, many researchers constructed travelling wave solutions of some famous nonlinear mathematical physics equations, such as the nonlinear Schrödinger equation in nonlinear optics, the nonlinear Gross–Pitaevskii equation in the Bose–Einstein condensates, the Korteweg–de Vries (KdV) equation in fluid mechanics, and so on. It can be noted that a new (3+1)-dimensional nonlinear equation was proposed by Abdul-Majid Wazwaz in 2014, it is an extension version of the (3+1)-dimensional KdV equation. This work studies this model by a different method, which is the G'/G-expansion technique. Some new soliton solutions is reported, and finally the conserved quantities are discussed.

The new (3+1)-dimensional nonlinear model that is going to be studied in this paper is given by [1]:

$$3w_{xz}-\left(2w_t+w_{xxx}-2ww_x\right)_y+2\left(w_x\partial_x^{-1}w_y\right)_x$$
$$+2\left(w\partial_x^{-1}w_{yy}\right)_y+w_{yz}=0, \tag{1}$$

where the inverse operator ∂_x^{-1} is defined by:

$$\left(\partial_x^{-1}f\right)(x)=\int\limits_{-\infty}^{x}f(t)\,\mathrm{d}t, \tag{2}$$

under the decaying condition at infinity. It should be noted that

$$\partial_x\partial_x^{-1}=\partial_x^{-1}\partial_x=1. \tag{3}$$

We first remove the integral term in (1) by introducing the potential

$$w(x,y,z,t)=u_x(x,y,z,t), \tag{4}$$

to convert (1) to the equation

$$3u_{xxz}-\left(2u_{xt}+u_{xxxx}-2u_xu_{xx}\right)_y+2\left(u_{xx}u_y\right)_x$$
$$+2\left(u_xu_{yy}\right)_y+u_{xyz}=0. \tag{5}$$

In order to secure soliton solutions to Eq. (5), the following wave variable is employed:

$$u(x,y,z,t)=U(\eta),\quad \eta=\kappa_1x+\kappa_2y+\kappa_3z-vt, \tag{6}$$

where κ_i $(i=1,\,2,\,3)$ and v are constants, which are to be determined. Next, inserting (6) into (5), and then integrating the result twice with respect to η, and choosing constants of integration to zero yields

$$\left(2v\kappa_1\kappa_2+3\kappa_1^2\kappa_3+\kappa_1\kappa_2\kappa_3\right)U'$$
$$+\left(2\kappa_1^3\kappa_2+\kappa_1\kappa_2^3\right)\left(U'\right)^2-\kappa_1^4\kappa_2U'''=0. \tag{7}$$

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2. Extended G'/G-expansion scheme

In this section, extended G'/G-expansion technique [2–5] is employed to analyze the (3+1)-dimensional nonlinear model given by (1). We assume that Eq. (7) has the solution in the form

$$U(\eta)=\alpha_0+\sum_{i=1}^{M}\left[\alpha_i\left(\frac{G'}{G}\right)^{i}+\beta_i\left(\frac{G'}{G}\right)^{i-1}\sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^{2}\right)}+\gamma_i\left(\frac{G'}{G}\right)^{-i}+\delta_i\frac{\left(\frac{G'}{G}\right)^{-i+1}}{\sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^{2}\right)}}\right], \tag{8}$$

where α_0, α_i, β_i, γ_i, δ_i $(i=1,\ldots,M)$ are constants to be determined later, $\sigma=\pm1$, M is a positive integer, and $G=G(\eta)$ satisfies the following second order linear ODE:

$$G''+\mu G=0, \tag{9}$$

where μ is a constant to be determined later. Using the balance method leads to $M=1$. Therefore, the extended G'/G-expansion approach admits the use of

$$U(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)+\beta_1\sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^{2}\right)}+\gamma_1\left(\frac{G'}{G}\right)^{-1}+\delta_1\frac{1}{\sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^{2}\right)}}. \tag{10}$$

Substituting (9) and (10) into the reduced ODE (7), collecting the coefficients of $(\frac{G'}{G})^j$ and $(\frac{G'}{G})^j\sqrt{\sigma(1+\frac{1}{\mu}(\frac{G'}{G})^2)}$, and solving the resulting system, following sets of solutions are procured where μ is arbitrary constant:

The first set of parameters is given by

$$\beta_1=\gamma_1=\delta_1=0,\quad \alpha_0=\alpha_0,\quad \alpha_1=-\frac{6\kappa_1^3}{2\kappa_1^2+\kappa_2^2},\qquad v=-2\mu\kappa_1^3-\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}. \tag{11}$$

The second set of parameters is given by

$$\alpha_1=\beta_1=\delta_1=0,\quad \alpha_0=\alpha_0,\quad \gamma_1=\frac{6\mu\kappa_1^3}{2\kappa_1^2+\kappa_2^2},\qquad v=-2\mu\kappa_1^3-\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}. \tag{12}$$

The third set of parameters is given by

$$\beta_1=\delta_1=0,\quad \alpha_0=\alpha_0,\quad \alpha_1=-\frac{6\kappa_1^3}{2\kappa_1^2+\kappa_2^2},\qquad \gamma_1=\frac{6\mu\kappa_1^3}{2\kappa_1^2+\kappa_2^2},\quad v=-8\mu\kappa_1^3-\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}. \tag{13}$$

The fourth set of parameters is given by

$$\gamma_1=\delta_1=0,\quad \alpha_0=\alpha_0,\quad \alpha_1=-\frac{3\kappa_1^3}{2\kappa_1^2+\kappa_2^2},\qquad \beta_1=\pm\frac{3\kappa_1^3\sqrt{\mu}}{(2\kappa_1^2+\kappa_2^2)\sqrt{\sigma}},\qquad v=-\frac{1}{2}\left(\mu\kappa_1^3+\kappa_3+\frac{3\kappa_1\kappa_3}{\kappa_2}\right). \tag{14}$$

Using these results, the following exact solutions to Eq. (5) are derived:

When $\mu<0$, the hyperbolic traveling wave solutions are

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3\sqrt{-\mu}}{2\kappa_1^2+\kappa_2^2}\times\left[\frac{A_1\sinh(\sqrt{-\mu}\eta)+A_2\cosh(\sqrt{-\mu}\eta)}{A_1\cosh(\sqrt{-\mu}\eta)+A_2\sinh(\sqrt{-\mu}\eta)}\right], \tag{15}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+(2\mu\kappa_1^3+\frac{\kappa_3(3\kappa_1+\kappa_2)}{2\kappa_2})t$.

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3\sqrt{-\mu}}{2\kappa_1^2+\kappa_2^2}\times\left[\frac{A_1\sinh(\sqrt{-\mu}\eta)+A_2\cosh(\sqrt{-\mu}\eta)}{A_1\cosh(\sqrt{-\mu}\eta)+A_2\sinh(\sqrt{-\mu}\eta)}\right]^{-1}, \tag{16}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+(2\mu\kappa_1^3+\frac{\kappa_3(3\kappa_1+\kappa_2)}{2\kappa_2})t$.

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3\sqrt{-\mu}}{2\kappa_1^2+\kappa_2^2}\times\left\{\frac{A_1\sinh(\sqrt{-\mu}\eta)+A_2\cosh(\sqrt{-\mu}\eta)}{A_1\cosh(\sqrt{-\mu}\eta)+A_2\sinh(\sqrt{-\mu}\eta)}+\left[\frac{A_1\sinh(\sqrt{-\mu}\eta)+A_2\cosh(\sqrt{-\mu}\eta)}{A_1\cosh(\sqrt{-\mu}\eta)+A_2\sinh(\sqrt{-\mu}\eta)}\right]^{-1}\right\}, \tag{17}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+(8\mu\kappa_1^3+\frac{\kappa_3(3\kappa_1+\kappa_2)}{2\kappa_2})t$.

$$u(x,y,z,t)=\alpha_0-\frac{3\kappa_1^3\sqrt{-\mu}}{2\kappa_1^2+\kappa_2^2}\times\left\{\frac{A_1\sinh(\sqrt{-\mu}\eta)+A_2\cosh(\sqrt{-\mu}\eta)}{A_1\cosh(\sqrt{-\mu}\eta)+A_2\sinh(\sqrt{-\mu}\eta)}\mp \mathrm{i}\sqrt{1-\left[\frac{A_1\sinh(\sqrt{-\mu}\eta)+A_2\cosh(\sqrt{-\mu}\eta)}{A_1\cosh(\sqrt{-\mu}\eta)+A_2\sinh(\sqrt{-\mu}\eta)}\right]^{2}}\right\}, \tag{18}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+\frac{1}{2}(\mu\kappa_1^3+\kappa_3+\frac{3\kappa_1\kappa_3}{\kappa_2})t$.

If, however, $\mu>0$, the trigonometric traveling wave solutions are

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3\sqrt{\mu}}{2\kappa_1^2+\kappa_2^2}\times\left[\frac{A_1\cos(\sqrt{\mu}\eta)-A_2\sin(\sqrt{\mu}\eta)}{A_1\sin(\sqrt{\mu}\eta)+A_2\cos(\sqrt{\mu}\eta)}\right], \tag{19}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+(2\mu\kappa_1^3+\frac{\kappa_3(3\kappa_1+\kappa_2)}{2\kappa_2})t$.

$$u(x,y,z,t)=\alpha_0+\frac{6\kappa_1^3\sqrt{\mu}}{2\kappa_1^2+\kappa_2^2}\times\left[\frac{A_1\cos(\sqrt{\mu}\eta)-A_2\sin(\sqrt{\mu}\eta)}{A_1\sin(\sqrt{\mu}\eta)+A_2\cos(\sqrt{\mu}\eta)}\right]^{-1}, \tag{20}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+(2\mu\kappa_1^3+\frac{\kappa_3(3\kappa_1+\kappa_2)}{2\kappa_2})t$.

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3\sqrt{\mu}}{2\kappa_1^2+\kappa_2^2}\times\left\{\frac{A_1\cos(\sqrt{\mu}\eta)-A_2\sin(\sqrt{\mu}\eta)}{A_1\sin(\sqrt{\mu}\eta)+A_2\cos(\sqrt{\mu}\eta)}-\left[\frac{A_1\cos(\sqrt{\mu}\eta)-A_2\sin(\sqrt{\mu}\eta)}{A_1\sin(\sqrt{\mu}\eta)+A_2\cos(\sqrt{\mu}\eta)}\right]^{-1}\right\}, \tag{21}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+(8\mu\kappa_1^3+\frac{\kappa_3(3\kappa_1+\kappa_2)}{2\kappa_2})t$.

$$u(x,y,z,t)=\alpha_0-\frac{3\kappa_1^3\sqrt{\mu}}{2\kappa_1^2+\kappa_2^2}\times\left\{\frac{A_1\cos(\sqrt{\mu}\eta)-A_2\sin(\sqrt{\mu}\eta)}{A_1\sin(\sqrt{\mu}\eta)+A_2\cos(\sqrt{\mu}\eta)}\mp\sqrt{1+\left[\frac{A_1\cos(\sqrt{\mu}\eta)-A_2\sin(\sqrt{\mu}\eta)}{A_1\sin(\sqrt{\mu}\eta)+A_2\cos(\sqrt{\mu}\eta)}\right]^2}\right\}, \tag{22}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+\frac{1}{2}(\mu\kappa_1^3+\kappa_3+\frac{3\kappa_1\kappa_3}{\kappa_2})t$.

Finally, if $\mu=0$, the plane wave solutions are

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3}{2\kappa_1^2+\kappa_2^2}\left(\frac{A_1}{A_1\eta+A_2}\right), \tag{23}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+(\frac{\kappa_3(3\kappa_1+\kappa_2)}{2\kappa_2})t$.

$$u(x,y,z,t)=\alpha_0-\frac{3\kappa_1^3}{2\kappa_1^2+\kappa_2^2}\left(\frac{A_1}{A_1\eta+A_2}\right), \tag{24}$$

where A_1, A_2 are arbitrary constants and $\eta=\kappa_1x+\kappa_2y+\kappa_3z+\frac{1}{2}(\kappa_3+\frac{3\kappa_1\kappa_3}{\kappa_2})t$.

The special cases are as follows:
When $\mu<0$ and $A_1^2>A_2^2$, the following bright and singular soliton solutions emerged from (15) to (18), respectively:

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3\sqrt{-\mu}}{2\kappa_1^2+\kappa_2^2}\times\tanh\left(\sqrt{-\mu}\left(\kappa_1x+\kappa_2y+\kappa_3z+\left(2\mu\kappa_1^3+\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}\right)t\right)+\eta_0\right), \tag{25}$$

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3\sqrt{-\mu}}{2\kappa_1^2+\kappa_2^2}\times\coth\left(\sqrt{-\mu}\left(\kappa_1x+\kappa_2y+\kappa_3z+\left(2\mu\kappa_1^3+\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}\right)t\right)+\eta_0\right), \tag{26}$$

$$u(x,y,z,t)=\alpha_0-\frac{12\kappa_1^3\sqrt{-\mu}}{2\kappa_1^2+\kappa_2^2}\times\coth 2\left(\sqrt{-\mu}\left(\kappa_1x+\kappa_2y+\kappa_3z+\left(8\mu\kappa_1^3+\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}\right)t\right)+\eta_0\right), \tag{27}$$

$$u(x,y,z,t)=\alpha_0-\frac{3\kappa_1^3\sqrt{-\mu}}{2\kappa_1^2+\kappa_2^2}\left(\tanh\left(\sqrt{-\mu}\left(\kappa_1x+\kappa_2y+\kappa_3z+\frac{1}{2}\left(\mu\kappa_1^3+\kappa_3+\frac{3\kappa_1\kappa_3}{\kappa_2}\right)t\right)+\eta_0\right)\mp\mathrm{i}\,\mathrm{sech}\left(\sqrt{-\mu}\left(\kappa_1x+\kappa_2y+\kappa_3z+\frac{1}{2}\left(\mu\kappa_1^3+\kappa_3+\frac{3\kappa_1\kappa_3}{\kappa_2}\right)t\right)+\eta_0\right)\right), \tag{28}$$

where $\eta_0=\tanh^{-1}(A_2/A_1)$. Also, setting $A_1=0$, $A_2\neq0$ and $A_2=0$, $A_1\neq0$ in (15)–(18), addition soliton solutions to the model (5) can be secured. However, these are ignored for convenience.

If, however, $\mu>0$, the following periodic wave solutions emerged from (19)–(22), respectively:

$$u(x,y,z,t)=\alpha_0+\frac{6\kappa_1^3\sqrt{\mu}}{2\kappa_1^2+\kappa_2^2}\tan\left(\sqrt{\mu}\left(\kappa_1x+\kappa_2y+\kappa_3z+\left(2\mu\kappa_1^3+\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}\right)t\right)-\eta_0\right), \tag{29}$$

$$u(x,y,z,t)=\alpha_0-\frac{6\kappa_1^3\sqrt{\mu}}{2\kappa_1^2+\kappa_2^2}\left(\sqrt{\mu}\left(\kappa_1x+\kappa_2y+\kappa_3z+\left(2\mu\kappa_1^3+\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}\right)t\right)-\eta_0\right), \tag{30}$$

$$u(x,y,z,t)=\alpha_0-\frac{12\kappa_1^3\sqrt{\mu}}{2\kappa_1^2+\kappa_2^2}2\left(\sqrt{\mu}\left(\kappa_1x+\kappa_2y+\kappa_3z+\left(8\mu\kappa_1^3+\frac{\kappa_3\left(3\kappa_1+\kappa_2\right)}{2\kappa_2}\right)t\right)-\eta_0\right), \tag{31}$$

$$u(x,y,z,t)=\alpha_0+\frac{3\kappa_1^3\sqrt{\mu}}{2\kappa_1^2+\kappa_2^2}\left(\tan\left(\sqrt{\mu}\left(\kappa_1 x+\kappa_2 y+\kappa_3 z+\frac{1}{2}\left(\mu\kappa_1^3+\kappa_3+\frac{3\kappa_1\kappa_3}{\kappa_2}\right)t\right)-\eta_0\right)\right.$$
$$\left.\pm\sec\left(\sqrt{\mu}\left(\kappa_1 x+\kappa_2 y+\kappa_3 z+\frac{1}{2}\left(\mu\kappa_1^3+\kappa_3+\frac{3\kappa_1\kappa_3}{\kappa_2}\right)t\right)-\eta_0\right)\right), \tag{32}$$

where $\eta_0=\tan^{-1}(A_1/A_2)$. Moreover, setting $A_1=0$, $A_2\neq 0$ and $A_2=0$, $A_1\neq 0$ in (19)–(22), more periodic wave solutions to the model (5) can be acquired. But they are ignored for convenience.

Remark-1: By using (4), soliton and other solutions to the governing Eq. (1) can be obtained.

3. Conservation laws of (5)

A conservation law [6, 7] for Eq. (5) is a space-time divergence

$$D_tT^t+D_xT^x+D_yT^y+D_zT^z=0,$$

which holds for all formal solutions $u(t,x,y,z)$ of Eq. (5) where the conserved density T^t, and the spatial fluxes T^x, T^y, T^z are functions of t,x,y,z,u, and derivatives of u. Furthermore, if there exists a non-trivial differential function Λ, called a "multiplier" such that $E_u(\Lambda G)=0$, then ΛG is a total divergence, i.e. $\Lambda G=D_tT^t+D_xT^x+D_yT^y$, for some (conserved) vector $[T^t,T^x,T^y,T^z]$, and E_u is the Euler–Lagrange operator. Thus, knowledge of each multiplier Λ leads to a conserved vector computed by a homotopy operator. If u and its derivatives tend to zero as x, y, z approaches infinity, the conserved quantities are obtained by $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T^t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

For (5), we obtain a multiplier Λ, that is given by

$$\Lambda=f_1(t)u+xf_4(t,z)+yf_3(t,z)+f_2(t,z)$$
$$+\frac{1}{2}f_5(t)\left(y^2-x^2\right)+\frac{1}{4}x^2f_1',$$

where f_i, $i=1,2,3,4,5$ are arbitrary functions. Thus, corresponding to the above multiplier we have the following conservation laws of (5):

$$D_tT_1^t+D_xT_1^x+D_yT_1^y+D_zT_1^z=0,$$

where

$$T_1^t=\frac{1}{6}\left(2f_1(t)w\int w_y\,\mathrm{d}x-4f_1(t)w_y\int w\,\mathrm{d}x\right.$$
$$\left.+x^2\left(-f_1'\right)w_y+xf_1'\int w_y\,\mathrm{d}x\right), \tag{33}$$

$$T_1^x=\frac{1}{360}\left[30f_1'x^2\int w_{yz}\,\mathrm{d}x+45f_1'x^2\left(\int w_y\,\mathrm{d}x\right)\left(\int w_{yy}\,\mathrm{d}x\right)+45f_1'x^2\left(\int w\,\mathrm{d}x\right)\left(\int w_{yyy}\,\mathrm{d}x\right)\right.$$
$$-60f_1'x^2\int w_{ty}\,\mathrm{d}x+30f_1''x^2\int w_y\,\mathrm{d}x+180f_1'w_zx^2+120wf_1'w_yx^2+150f_1'w_xx^2\int w_y\,\mathrm{d}x$$
$$-30f_1'w_{xy}x^2\int w\,\mathrm{d}x-72f_1'w_{xxy}x^2-180f_1'x\int w_z\,\mathrm{d}x-240wf_1'x\int w_y\,\mathrm{d}x+120f_1'w_yx\int w\,\mathrm{d}x$$
$$+108f_1'w_{xy}x-400w^2f_1(t)\int w_y\,\mathrm{d}x-60f_1(t)\left(\int w_z\,\mathrm{d}x\right)\left(\int w_y\,\mathrm{d}x\right)$$
$$+120f_1(t)\left(\int w\,\mathrm{d}x\right)\left(\int w_{yz}\,\mathrm{d}x\right)+120f_1(t)\left(\int w\,\mathrm{d}x\right)^2\left(\int w_{yyy}\,\mathrm{d}x\right)$$
$$+120f_1(t)\left(\int w_y\,\mathrm{d}x\right)\left(\int w_t\,\mathrm{d}x\right)-240f_1(t)\left(\int w\,\mathrm{d}x\right)\left(\int w_{ty}\,\mathrm{d}x\right)-360wf_1(t)\int w_z\,\mathrm{d}x$$
$$+120f_1'\left(\int w\,\mathrm{d}x\right)\left(\int w_y\,\mathrm{d}x\right)+720f_1(t)w_z\int w\,\mathrm{d}x+640wf_1(t)w_y\int w\,\mathrm{d}x-72f_1'w_y$$
$$+560f_1(t)w_x\left(\int w\,\mathrm{d}x\right)\left(\int w_y\,\mathrm{d}x\right)-144f_1(t)w_yw_x-80f_1(t)w_{xy}\left(\int w\,\mathrm{d}x\right)^2+216wf_1(t)w_{xy}$$
$$\left.+72f_1(t)w_{xx}\int w_y\,\mathrm{d}x-288f_1(t)w_{xxy}\int w\,\mathrm{d}x\right], \tag{34}$$

$$T_1^y = \frac{1}{360}\bigg[90x^2 f_1' w_x w + 135x^2 f_1' w \int w_{yy}\,\mathrm{d}x - 45x^2 f_1' w_{yy}\int w\,\mathrm{d}x + 30x^2 f_1' w_{xx}\int w\,\mathrm{d}x$$
$$-90x f_1'\left(\int w_{yy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) - 60x f_1' w_x \int w\,\mathrm{d}x + 72 f_1(t) w_{xx} w - 240 f_1(t) w \left(\int w_y\,\mathrm{d}x\right)^2$$
$$-60 f_1(t) w \int w_z\,\mathrm{d}x + 480 f_1(t) w \left(\int w_{yy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) + 120 f_1(t) w \int w_t\,\mathrm{d}x + 240 f_1(t) w_x w \int w\,\mathrm{d}x$$
$$+120 f_1(t) w_z \int w\,\mathrm{d}x + 240 f_1(t) w_y \left(\int w_y\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) - 120 f_1(t) w_{yy}\left(\int w\,\mathrm{d}x\right)^2 + 80 f_1(t) w_{xx}\left(\int w\,\mathrm{d}x\right)^2$$
$$-72 f_1(t) w_{xxx}\int w\,\mathrm{d}x - 240 f_1(t) w_t \int w\,\mathrm{d}x + 30x^2 f_1'' w - 60x f_1' w^2 - 120x f_1''\int w\,\mathrm{d}x - 80 f_1(t) w^3 - 60x^2 f_1' w_t$$
$$+60x f_1'\int w_t\,\mathrm{d}x - 36 f_1(t) w_x^2 + 30x^2 f_1' w_z - 18x^2 f_1' w_{xxx} - 30x f_1'\int w_z\,\mathrm{d}x - 36 f_1' w_x + 36x f_1' w_{xx}\bigg], \tag{35}$$

$$T_1^z = \frac{1}{12}\bigg(-2 f_1(t) w \int w_y\,\mathrm{d}x + 4 f_1(t) w_y \int w\,\mathrm{d}x + 12 f_1(t) w_x \int w\,\mathrm{d}x - 6x f_1' w + 6 f_1'\int w\,\mathrm{d}x \tag{36}$$
$$-6 f_1(t) w^2 + x^2 f_1' w_y + 3x^2 f_1' w_x - x f_1'\int w_y\,\mathrm{d}x\bigg),$$

$$D_t T_2^t + D_x T_2^x + D_y T_2^y + D_z T_2^z = 0, \quad \text{where} \quad T_2^t = -\frac{2}{3} w_y f_2(t,z), \tag{37}$$

$$T_2^x = \frac{1}{30}\bigg[40 w_y f_2(t,z) w - 24 f_2(t,z) w_{xxy} - 30 f_{2z} w + 10 f_2(t,z)\int w_{yz}\,\mathrm{d}x + 15 f_2(t,z)\left(\int w_y\,\mathrm{d}x\right)\left(\int w_{yy}\,\mathrm{d}x\right)$$
$$+15 f_2(t,z)\left(\int w_{yyy}\,\mathrm{d}x\right)(\int w\,\mathrm{d}x) - 20 f_2(t,z)\int w_{ty}\,\mathrm{d}x + 50 w_x f_2(t,z)\int w_y\,\mathrm{d}x - 10 f_2(t,z) w_{xy}\int w\,\mathrm{d}x$$
$$+60 w_z f_2(t,z) + 10 f_{2t}\int w_y\,\mathrm{d}x - 5 f_{2z}\int w_y\,\mathrm{d}x\bigg], \tag{38}$$

$$T_2^y = \frac{1}{30}\bigg(30 w_x f_2(t,z) w - 5 f_{2z} w + 10 f_{2t} w + 45 f_2(t,z) w \int w_{yy}\,\mathrm{d}x - 15 w_{yy} f_2(t,z)\int w\,\mathrm{d}x$$
$$+10 w_{xx} f_2(t,z)\int w\,\mathrm{d}x - 6 w_{xxx} f_2(t,z) + 10 w_z f_2(t,z) - 20 w_t f_2(t,z)\bigg), \tag{39}$$

$$T_2^z = \frac{1}{3}\left(3 w_x f_2(t,z) + w_y f_2(t,z)\right), \tag{40}$$

$$D_t T_3^t + D_x T_3^x + D_y T_3^y + D_z T_3^z = 0, \quad \text{where} \quad T_3^t = \frac{1}{3}\left(f_3(t,z) w - 2y w_y f_3(t,z)\right), \tag{41}$$

$$T_3^x = \frac{1}{30}\bigg[-30y f_{3z} w + 40y w_y f_3(t,z) w - 24y f_3(t,z) w_{xxy} + 10y f_3(t,z)\int w_{yz}\,\mathrm{d}x$$
$$-15 f_3(t,z)\left(\int w_{yy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) + 15y f_3(t,z)\left(\int w_y\,\mathrm{d}x\right)\left(\int w_{yy}\,\mathrm{d}x\right) + 15y f_3(t,z)\left(\int w_{yyy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right)$$
$$-20y f_3(t,z)\int w_{ty}\,\mathrm{d}x + 10 f_{3z}\int w\,\mathrm{d}x - 20 f_{3t}\int w\,\mathrm{d}x - 10 w_x f_3(t,z)\int w\,\mathrm{d}x + 50y w_x f_3(t,z)\int w_y\,\mathrm{d}x$$
$$-10y f_3(t,z) w_{xy}\int w\,\mathrm{d}x + 6 w_{xx} f_3(t,z) - 5 f_3(t,z)\int w_z\,\mathrm{d}x + 10 f_3(t,z)\int w_t\,\mathrm{d}x + 60y w_z f_3(t,z) - 10 f_3(t,z) w^2$$
$$+10y f_{3t}\int w_y\,\mathrm{d}x - 5y f_{3z}\int w_y\,\mathrm{d}x\bigg], \tag{42}$$

$$T_3^y = \frac{1}{30}\bigg(30yw_xf_3(t,z)w - 6yw_{xxx}f_3(t,z) - 5yf_{3z}w + 10yf_{3t}w - 30f_3(t,z)w\int w_y\,\mathrm{d}x + 45yf_3(t,z)w\int w_{yy}\,\mathrm{d}x$$

$$+30w_yf_3(t,z)\int w\,\mathrm{d}x - 15yw_{yy}f_3(t,z)\int w\,\mathrm{d}x + 10yw_{xx}f_3(t,z)\int w\,\mathrm{d}x + 10yw_zf_3(t,z) - 20yw_tf_3(t,z)\bigg), \quad (43)$$

$$T_3^z = \frac{1}{6}\big(6yw_xf_3(t,z) + 2yw_yf_3(t,z) + f_3(t,z)(-w)\big), \quad (44)$$

$$D_tT_4^t + D_xT_4^x + D_yT_4^y + D_zT_4^z = 0,$$

$$\text{where } T_4^t = \frac{1}{3}\left(f_4(t,z)\int w_y\,\mathrm{d}x - 2xw_yf_4(t,z)\right), \quad (45)$$

$$T_4^x = \frac{1}{30}\bigg[40xw_yf_4(t,z)w + 18f_4(t,z)w_{xy} - 24xf_4(t,z)w_{xxy} - 30xf_{4z}w + 10xf_4(t,z)\int w_{yz}\,\mathrm{d}x$$

$$+15xf_4(t,z)\left(\int w_y\,\mathrm{d}x\right)\left(\int w_{yy}\,\mathrm{d}x\right) + 15xf_4(t,z)\left(\int w_{yyy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) - 20xf_4(t,z)\int w_{ty}\,\mathrm{d}x$$

$$-40f_4(t,z)w\int w_y\,\mathrm{d}x + 20w_yf_4(t,z)\int w\,\mathrm{d}x + 50xw_xf_4(t,z)\int w_y\,\mathrm{d}x - 10xf_4(t,z)w_{xy}\int w\,\mathrm{d}x$$

$$+60f_{4z}\int w\,\mathrm{d}x + 60xw_zf_4(t,z) - 30f_4(t,z)\int w_z\,\mathrm{d}x + 10xf_{4t}\int w_y\,\mathrm{d}x - 5xf_{4z}\int w_y\,\mathrm{d}x\bigg], \quad (46)$$

$$T_4^y = \frac{1}{30}\bigg[-5xf_{4z}w + 10xf_{4t}w + 30xw_xf_4(t,z)w + 45xf_4(t,z)w\int w_{yy}\,\mathrm{d}x - 15f_4(t,z)\left(\int w_{yy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right)$$

$$+10f_{4z}\int w\,\mathrm{d}x - 20f_{4t}\int w\,\mathrm{d}x - 15xw_{yy}f_4(t,z)\int w\,\mathrm{d}x - 10w_xf_4(t,z)\int w\,\mathrm{d}x + 10xw_{xx}f_4(t,z)\int w\,\mathrm{d}x$$

$$+10xw_zf_4(t,z) + 6w_{xx}f_4(t,z) - 6xw_{xxx}f_4(t,z) - 20xw_tf_4(t,z) - 5f_4(t,z)\int w_z\,\mathrm{d}x + 10f_4(t,z)\int w_t\,\mathrm{d}x$$

$$-10f_4(t,z)w^2\bigg], \quad (47)$$

$$T_4^z = \frac{1}{6}\bigg(2xw_yf_4(t,z) - f_4(t,z)\int w_y\,\mathrm{d}x + 6xw_xf_4(t,z) - 6f_4(t,z)w\bigg), \quad (48)$$

$$D_tT_5^t + D_xT_5^x + D_yT_5^y + D_zT_5^z = 0,$$

$$\text{where } T_5^t = \frac{1}{3}\bigg(yf_5(t)w + x^2f_5(t)w_y - xf_5(t)\int w_y\,\mathrm{d}x - y^2f_5(t)w_y\bigg), \quad (49)$$

$$T_5^x = \frac{1}{60}\bigg[-40x^2f_5(t)w_yw - 15x^2f_5(t)\left(\int w_{yyy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) + 10x^2f_5(t)w_{xy}\int w\,\mathrm{d}x + 40y^2f_5(t)w_yw$$

$$+15y^2f_5(t)\left(\int w_{yyy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) - 10y^2f_5(t)w_{xy}\int w\,\mathrm{d}x + 80xf_5(t)w\int w_y\,\mathrm{d}x - 40xf_5(t)w_y\int w\,\mathrm{d}x$$

$$-60f_5(t)\left(\int w_y\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) - 30yf_5(t)\left(\int w_{yy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right) - 20yf_5(t)w_x\int w\,\mathrm{d}x - 40yf_5'\int w\,\mathrm{d}x$$

$$-20yf_5(t)w^2 - 10x^2f_5(t)\int w_{yz}\,\mathrm{d}x + 24x^2f_5(t)w_{xxy} - 15x^2f_5(t)\left(\int w_y\,\mathrm{d}x\right)\left(\int w_{yy}\,\mathrm{d}x\right) + 20x^2f_5(t)\int w_{ty}\,\mathrm{d}x$$

$$
\begin{aligned}
&-50x^2 f_5(t)w_x \int w_y \,\mathrm{d}x - 60x^2 f_5(t)w_z + 10y^2 f_5(t)\int w_{yz}\,\mathrm{d}x - 24y^2 f_5(t)w_{xxy}\\
&+15y^2 f_5(t)\left(\int w_y\,\mathrm{d}x\right)\left(\int w_{yy}\,\mathrm{d}x\right) - 20y^2 f_5(t)\int w_{ty}\,\mathrm{d}x + 50y^2 f_5(t)w_x\int w_y\,\mathrm{d}x - 10yf_5(t)\int w_z\,\mathrm{d}x\\
&-36xf_5(t)w_{xy} + 12yf_5(t)w_{xx} + 20yf_5(t)\int w_t\,\mathrm{d}x + 60xf_5(t)\int w_z\,\mathrm{d}x + 60y^2 f_5(t)w_z + 24f_5(t)w_y\\
&-10x^2 f_5'\int w_y\,\mathrm{d}x + 10y^2 f_5'\int w_y\,\mathrm{d}x\Bigg],
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
T_5^y = \frac{1}{60}\Bigg[&-30x^2 f_5(t)w_x w - 45x^2 f_5(t)w\int w_{yy}\,\mathrm{d}x + 15x^2 f_5(t)w_{yy}\int w\,\mathrm{d}x - 10x^2 f_5(t)w_{xx}\int w\,\mathrm{d}x + 30y^2 f_5(t)w_x w\\
&+45y^2 f_5(t)w\int w_{yy}\,\mathrm{d}x - 15y^2 f_5(t)w_{yy}\int w\,\mathrm{d}x + 10y^2 f_5(t)w_{xx}\int w\,\mathrm{d}x + 30xf_5(t)\left(\int w_{yy}\,\mathrm{d}x\right)\left(\int w\,\mathrm{d}x\right)\\
&+20xf_5(t)w_x\int w\,\mathrm{d}x - 60yf_5(t)w\int w_y\,\mathrm{d}x + 60yf_5(t)w_y\int w\,\mathrm{d}x - 10x^2 f_5' w + 10y^2 f_5' w + 40xf_5'\int w\,\mathrm{d}x\\
&+20xf_5(t)w^2 + 40f_5(t)w\int w\,\mathrm{d}x - 10x^2 f_5(t)w_z + 6x^2 f_5(t)w_{xxx} + 20x^2 f_5(t)w_t - 6y^2 f_5(t)w_{xxx}\\
&+10xf_5(t)\int w_z\,\mathrm{d}x - 12xf_5(t)w_{xx} + 12f_5(t)w_x - 20xf_5(t)\int w_t\,\mathrm{d}x + 10y^2 f_5(t)w_z - 20y^2 f_5(t)w_t\Bigg],
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
T_5^z = \frac{1}{6}\Bigg(&6xf_5(t)w - yf_5(t)w - 6f_5(t)\int w\,\mathrm{d}x - x^2 f_5(t)w_y - 3x^2 f_5(t)w_x + 3y^2 f_5(t)w_x\\
&+xf_5(t)\int w_y\,\mathrm{d}x + y^2 f_5(t)w_y\Bigg).
\end{aligned}
\tag{52}
$$

Remark-2: Due to the presence of the arbitrary functions, in the multiplier, one can obtain an infinitely many conservation laws of (5).

4. Conclusions

The new (3+1)-dimensional nonlinear model has been investigated analytically. Based on the G'/G-expansion method, the hyperbolic travelling wave solutions, trigonometric travelling wave solutions, plane wave solutions, bright and singular soliton solutions, and periodic wave solutions are presented. Finally, the conservation laws are discussed.

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