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Hall Conductivity of Strongly Interacting Bosons in Optical Lattice

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We study the Hall conductivity of ultra-cold bosonic atoms in optical lattice described by the Bose–Hubbard model. We use the quantum rotor approximation, which allows to describe lattices with non-zero Chern numbers. As examples of such systems we consider square lattice in a synthetic magnetic field as well as the Haldane model. We derive formula for the Hall conductivity, which strongly resembles the famous TKNN formula. We investigate the behavior of conductivity depending on temperature and model parameters. It appears that bosonic systems substantially differ from the fermionic ones, e.g. the presence of non-zero Hall conductivity is not directly related to non-zero value of the Chern number.

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1. Introduction

Ultracold atomic systems are well known for their easy tunability and exceptional cleanliness. In the recent years, possibilities of creating artificial gauge fields in these systems have seen great development. Techniques like the Floquet engineering and photon-assisted tunneling [1, 2] allowed to realize Harper [3] and Haldane models [4]. So far, such topological properties like the Chern number or transverse drift velocity were studied experimentally in ultracold atom systems. Recently, a new type of experiment appeared, which is closer to condensed matter physics, namely measurements of current–current correlation functions [5, 6]. This quantity can be directly related to the conductivity of the system. These developments motivate us to study the Hall conductivity of ultracold bosons in optical lattices.

In this paper we analyze the influence of model parameters on the Hall conductivity in the Harper- and Haldane–Bose–Hubbard models. We use quantum rotor approach, which allows us to determine the impact of the Berry curvature and its symmetries on the calculated quantities. We also investigate the effects of superfluid–normal state phase transition on the Hall conductivity.

The paper is organized as follows: first we briefly introduce Bose–Hubbard model and quantum rotor approach. Next, we give formula Hall conductivity within this approximation. In Sects. 4 and 5, we present results for square lattice in magnetic field and Haldane model. Finally, in the last section, we summarize the results.

2. Model and method

Physics of bosons in optical lattice is well captured by the Bose–Hubbard model [7, 8]:

$$\hat{H} = \hat{H}_{\mathrm{kin}} + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \mu\sum_i \hat{n}_i, \tag{1}$$

where

$$\hat{H}_{\mathrm{kin}} = -\sum_{i,j}\left(t_{ij}\,\mathrm{e}^{\mathrm{i}\,\varphi_{ij}}\hat{a}_i^\dagger\hat{a}_j + \mathrm{H.c.}\right), \tag{2}$$

where $\hat{a}_i$ ($\hat{a}_i^\dagger$) is bosonic annihilation (creation) operator and $\hat{n}_i$ is particle number operator on the lattice site i. The first term describes hopping between lattice sites with coupling t_{ij} and additional phase change φ_{ij}[¶] from the gauge fields. The second term describes on-site repulsive interaction. The last term contains chemical potential and controls the number of particles in the system.

2.1. Method

In order to solve the Bose–Hubbard model, we use quantum rotor approach. It is based on the path integral formalism in imaginary time τ [9]. Using coherent state representation [10], we get the partition function in the form

$$Z = \int [\mathcal{D}\overline{a}\mathcal{D}a]\,\mathrm{e}^{-S[\overline{a},a]}, \tag{3}$$

where

$$S[\overline{a},a] = \int_0^\beta \mathrm{d}\tau H(\overline{a},a) + \int_0^\beta \mathrm{d}\tau \sum_i \overline{a}_i\frac{\mathrm{d}}{\mathrm{d}\tau}a_i. \tag{4}$$

The Hamiltonian $H(\overline{a},a)$ is of the same form as Eq. (1) with creation and annihilation operators replaced by complex fields $\overline{a}$, a. Next, we perform gauge transformation to separate amplitudes and phases of the fields [11–13]:

$$a_i = b_i\,\mathrm{e}^{\mathrm{i}\,\phi_i}. \tag{5}$$

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[¶]Actually, in the original Bose–Hubbard model phase factor is not included, i.e. $\varphi_{ij} = 0$.

In the strong coupling regime, the phase degrees of freedom are responsible for the phase transition in the system (see Fig. 2 in Ref. [14]). Here, they are also directly modified by the presence of gauge potential. Hence, we integrate out the amplitudes, which yields phase only action

$$S[\phi] = \int_0^\beta \mathrm{d}\tau \left[-\sum_{i,j} J_{ij} \cos(\phi_j - \phi_i - \varphi_{ij}) + \sum_i \left(\frac{\dot{\phi}_i^2}{2U} + \mathrm{i}\,\frac{\overline{\mu}}{U}\dot{\phi}_i \right) \right], \tag{6}$$

where $J_{ij} = 2t_{ij}b_0^2$ is a coupling modified by the average on-site amplitudes b_0 and $\overline{\mu} = \mu + U/2$ is the shifted chemical potential.

Next step is an application of the spherical approximation [15, 16]. This allows us to calculate phase diagrams and correlations function, which are necessary to determine the Hall conductivity. The details of these derivation are presented in Refs. [11, 17].

3. Hall conductivity

The Kubo linear response theory connects the conductivity to the current–current correlation function

$$\sigma_{ij}(\omega_\nu) = \frac{1}{N\beta\omega_\nu} \int_0^\beta \mathrm{d}\tau\,\mathrm{d}\tau' \left. \frac{\delta^2 \ln Z}{\delta A_j(\tau')\,\delta A_i(\tau)} \right|_{\boldsymbol{A}=0} \times \mathrm{e}^{-\mathrm{i}\,\omega_\nu(\tau-\tau')}, \tag{7}$$

where $\boldsymbol{A}(\tau)$ is a vector potential of an external probe field and $\omega_\nu = 2\pi\nu/\beta$ is Matsubara frequency. The vector potential is introduced to the model through Peierls factor [18]. Following the derivation presented in [19] with minor modifications to account for the transverse transport, we obtain the formula for the Hall conductivity

$$\sigma_H = \left(\frac{2\pi}{\Phi_0}\right)^2 \frac{1}{N} \sum_{\boldsymbol{k},b} \mathcal{F}^b_{xy} \left(n^p_{\mathrm{BE}} - n^h_{\mathrm{BE}}\right), \tag{8}$$

where

$$\mathcal{F}^b_{xy} = \mathrm{Im} \sum_{b' \neq b} \frac{\left\langle b \left| \frac{\partial \mathcal{H}(\boldsymbol{k})}{\partial k_y} \right| b' \right\rangle \left\langle b' \left| \frac{\partial \mathcal{H}(\boldsymbol{k})}{\partial k_x} \right| b \right\rangle}{\left[\varepsilon_b(\boldsymbol{k}) - \varepsilon_{b'}(\boldsymbol{k})\right]^2} \tag{9}$$

is the Berry curvature of the single-particle band b and Φ_0 is the magnetic flux quantum. The Hamiltonian kernel $\mathcal{H}$ and its eigenstates $|b\rangle$ come from single particle tight-binding model of considered lattice. The Bose–Einstein distributions $n^{p(h)}_{\mathrm{BE}}$ define the occupation of the quasiparticle (hole) states and take the following form:

$$n^{p(h)}_{\mathrm{BE}} = \frac{1}{\mathrm{e}^{\beta U\left[\Xi \mp v\left(\frac{\mu}{U}\right)\right]} - 1}, \tag{10}$$

where

$$\Xi = \sqrt{\frac{J}{U}(\varepsilon_b(\boldsymbol{k}) - \varepsilon_1(0)) + \frac{\delta\lambda}{U} + v\left(\frac{\mu}{U}\right)}, \tag{11}$$

$$v\left(\frac{\mu}{U}\right) = \frac{\mu}{U} - \left\lfloor \frac{\mu}{U} \right\rfloor - \frac{1}{2}, \tag{12}$$

with $\delta\lambda$ being a self-consistently calculated parameter, which is equal to 0 in the ordered state and positive in the disordered state, and $\lfloor x \rfloor$ being a floor function.

Due to the lack of thermal excitations at $T = 0$ the Hall conductivity in bosonic system vanishes

$$\sigma_H(T = 0) = 0. \tag{13}$$

4. Harper–Bose–Hubbard model

Particles in a lattice moving in a uniform magnetic field are described by the Harper model [18] (see Fig. 1). It leads to a fractal energy structure of Hofstadter butterfly [20] characterized by quantity

$$\alpha = \frac{Ba^2}{\Phi_0} = \frac{\Phi}{\Phi_0}, \tag{14}$$

where B is magnetic field, a is lattice constant, Φ is flux through elementary cell. If the interaction between particles are included, we arrive at the Harper–Bose–Hubbard model, where the phases in $\hat{H}_{\mathrm{kin}}$ are implemented as shown in Fig. 1. Then, the increase of the magnetic field does not necessarily lead to the increase in the Hall

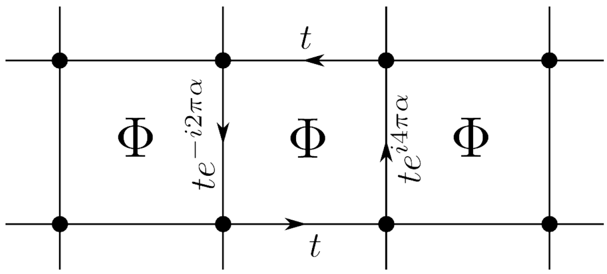

Fig. 1. Square lattice with hoppings modified by vector potential of uniform magnetic field in Landau gauge. Each elementary cell is pierced by flux Φ.

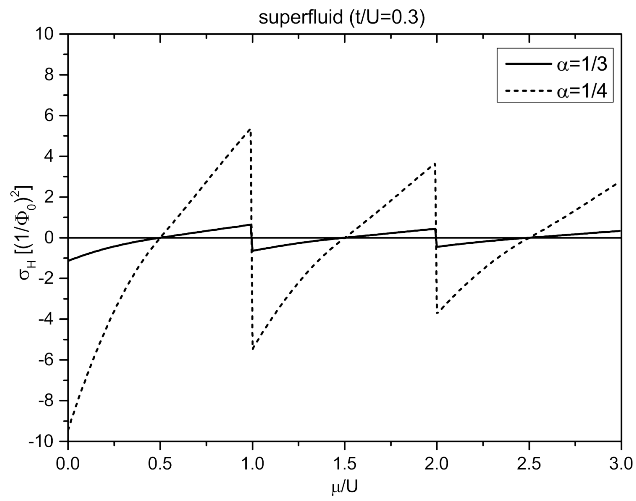

Fig. 2. The Hall conductivity as a function of chemical potential for two values of magnetic field $\alpha = \frac{1}{3}, \frac{1}{4}$.

conductivity in bosonic systems. This is due to the higher occupation of the flatter lowest band (according to the Bose–Einstein distribution), which results in higher contribution to the conductivity for smaller values of magnetic field α. Example of such behavior is presented in Fig. 2. This effect, as well as non-monotonic dependence of critical hopping [21], are direct consequence of the inner energy structure of the Hofstadter butterfly.

5. Haldane–Bose–Hubbard model

The Hall conductivity can be also observed in systems with zero net magnetic flux. A model with such properties was proposed by Haldane [22]. It describes a honeycomb lattice with phase change accompanying next-nearest-neighbors hopping, which breaks time-reversal symmetry, and a mass M distinguishing between two sublattices, which breaks the inversion symmetry (see Fig. 3). Depending on the values of the parameters M and φ, the Haldane model can lead to either topologically trivial or non-trivial band structure. The Hamiltonian of the Haldane model is of the following form:

$$\hat{H}_{\text{kin}} = -t \sum_{\langle i,j \rangle} \left(\hat{a}_j^\dagger \hat{a}_i + \text{H.c.} \right)$$
$$-t' \sum_{\langle\langle i,j \rangle\rangle} \left(\mathrm{e}^{\pm \mathrm{i}\varphi} \hat{a}_j^\dagger \hat{a}_i + \text{H.c.} \right) \pm M \sum_i \hat{a}_i^\dagger \hat{a}_i. \qquad (15)$$

Addition of the interaction term yields the Haldane–Bose–Hubbard model. For small values of the hopping integral t, we observe a decrease in the number of excitations due to the increase of the energy gap in the disordered state. On the other hand, for large values of t, the available energy of the excitation increases, which means that the occupation $n_{\text{BE}}^{p(h)}$ is lower. Competition of these two effects leads to extremum of the Hall conductivity below the critical value of the hopping integral t_c (see Fig. 4).

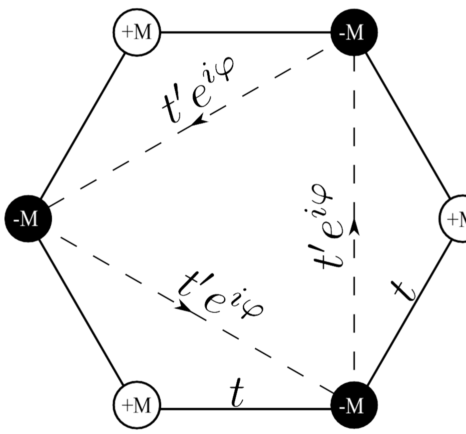

Fig. 3. Haldane honeycomb lattice with direction of positive phase φ indicated for one of the sublattices and “mass” M which breaks inversion symmetry.

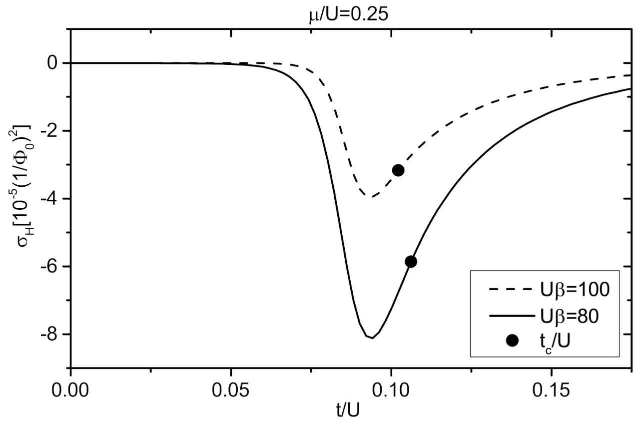

Fig. 4. Hall conductivity of Haldane–Bose–Hubbard model as a function of hopping for two different temperatures. The extremum of the dependence lies below critical hopping t_c.

The sign of the Hall conductivity depends on the type of the majority carriers and the sign of the Berry curvature. In the particle-hole-symmetric points, i.e. $\mu/U = m + 1/2$ with m being integer, the Hall conductivity vanishes.

In order to obtain a non-zero Hall conductivity, the Berry curvature cannot be antisymmetric. This ensures that contributions from $\boldsymbol{k}$ and $-\boldsymbol{k}$ do not cancel each other. The non-antisymmetric Berry curvature can be achieved by breaking the time-reversal symmetry. This is true for both, trivial and non-trivial band topology (see Figs. 5, 6).

In the normal state, there can be observed a flat region in the Hall conductivity around the half-integer values of the chemical potential, which can be related to the incompressibility of the Mott insulator ground state.

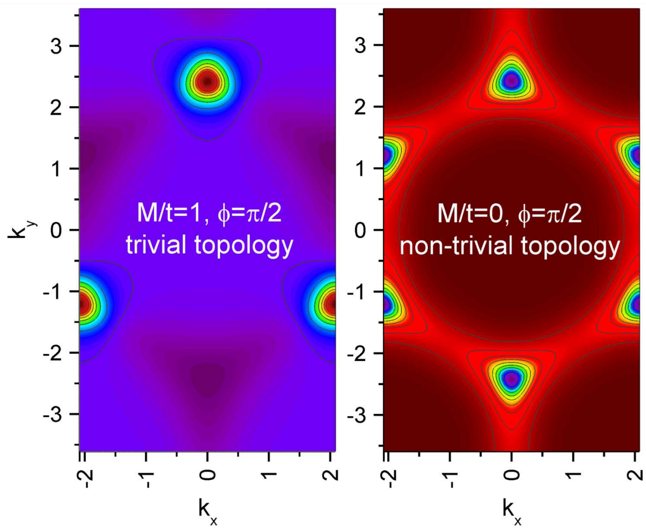

Fig. 5. Berry curvature of Haldane honeycomb lattice for trivial and non-trivial topology of the band structure. In both cases the time-reversal symmetry is broken, which leads to non-zero Hall conductivity.

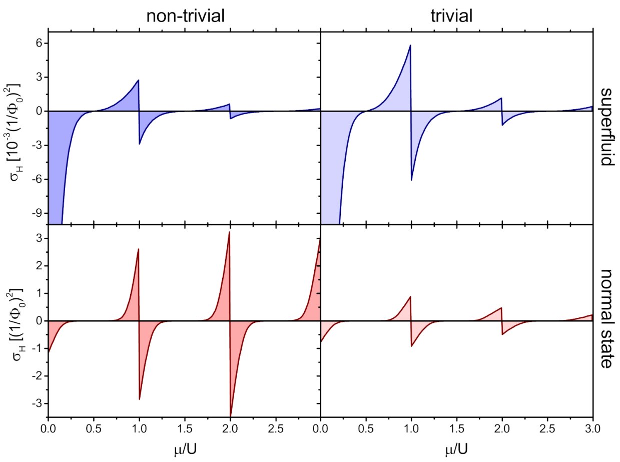

Fig. 6. Hall conductivity in superfluid and normal state for trivial and non-trivial band topology.

6. Summary

In this paper, we have studied the Hall conductivity of strongly interacting bosons in optical lattice. We have used the Bose–Hubbard model in quantum rotor approach in order to capture the influence of the Berry curvature on the properties of the system. We have modified the Bose–Hubbard model by adding appropriate Peierls phase factors corresponding to square lattice in a uniform magnetic field and the Haldane honeycomb lattice. We have obtained formula for the Hall conductivity of bosons analogous to the famous TKNN formula [23].

In the zero temperature limit our model does not exhibit transverse transport. This is due to the dependence of the Hall conductivity on quasiparticles imbalance, while in the ground state these excitations are not present. We have shown that the extremum in the dependence of conductivity on the hopping integral can be explained by competition between increasing energy gap and widening of quasiparticle bands. For the Harper–Bose–Hubbard model, we have found that the magnitude of the Hall conductivity does not directly depend on the value of magnetic field, but rather on the inner energy structure of the Hofstadter butterfly. For the Haldane–Bose–Hubbard model, we have shown that the sign of Hall conductivity depends on the type of majority carriers and the shape of the Berry curvature. Especially, it vanishes at the particle–hole symmetry points. Non-zero Hall conductivity is ensured by non-antisymmetric Berry curvature, which can be achieved by breaking the time-reversal symmetry. This scenario is possible in both topologically trivial and non-trivial band structure.

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