Optical Soliton in Nonlocal Nonlinear Medium with Cubic-Quintic Nonlinearities and Spatio-Temporal Dispersion

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In this article, we have constructed some new soliton solutions of the nonlocal nonlinear medium equation via three reliable approaches. The executed approaches are modified Kudryashov method, sinh-Gordon equation expansion method and extended sinh-Gordon equation expansion method. In this case, there are three types of competing nonlinearities that are taken into account in our model. They are nonlocal nonlinearity, cubic nonlinearity, and quintic nonlinearity. By means of the aforementioned methods, dark, bright, combined dark-bright, singular, combined singular, periodic, and other soliton solutions are obtained from the nonlocal nonlinear medium equation, and their respective existence conditions.

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1. Introduction

Observation of soliton solutions have a pivotal role in the study of integer and fractional order partial differential equations. To form a variety of soliton solutions of the nonlinear integer or fractional order partial differential equations has become a vital task because the solutions of their equations can clarify the thorough description of nonlinear phenomena of various real-life problems in the field of nonlinear optics, plasma physics, communications and electronic engineering, fluid mechanics, ocean engineering, signal processing and so on. In this context, it is quite important to establish and adopt a new analytical method for exploring a general and newer soliton solutions for any nonlinear partial differential equations related to fiber optics with the help of symbolic computation based software maple. As we know, solitons play a significant role in many physical systems and it appears in various forms like as kink, pulse, envelope, dark, bright, breather,cusp, combined soliton, and many others. Generally, soliton is a localized wave form that travels along the system with constant velocity and undeformed shape.

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Because of the growing progress of computer and computation technologies and artificial intelligence based symbolic computation like as Maple, Mathematica and MATLAB, several analytical approaches have attempt and aptly applied to look for more general and newer exact solutions of nonlinear integer and fractional order partial differential equations (NPDEs) such as the Lie symmetry analysis [1], extended trial equation method [2, 3], functional variable method [4], Kudryashov's method [4], Jacobian elliptic equations expansion method [5, 6], exp $(-\phi(\xi))$-expansion method [7], semi-inverse variational principle [8], ansatz method [9], (G'/G)-expansion method [10], modified Kudryashov's method [11, 12], sine-Gordon expansion method [13, 14], extended sinh-Gordon expansion method [14–18].

In this article, we will investigate some new complex hyperbolic and complex trigonometric function solutions, especially dark, bright, combined dark–bright, singular, combined singular, periodic soliton and other soliton solutions from nonlocal nonlinear medium equation via modified Kudryashov's method, sine-Gordon expansion method, and extended sinh-Gordon equation expansion method.

The residue of the paper is arranged the following way. In Sect. 2, we will discuss the studied mathematical model. Methodology will be elaborated in Sect. 3. As an application of the aforementioned methods, we will solve the nonlinear nonlocal medium equation with three media in Sect. 4. Finally, we conclude our applied method and generated results will be discussed in Sect. 5.

2. Mathematical model

The dimensionless form of the nonlocal nonlinearity and cubic-quintic nonlinearities with spatio-temporal dispersion is given by [1]:

$$iq_t + \rho_1 q_{xx} + \rho_2 q_{xt} + \left(b_1 |q|^2 + b_2 \ |q|^4\right) q + b_3 \left(|q|^2\right)_{xx} q = 0, \tag{1}$$

where $u(x,t)$ is the slowly varying amplitude, while x and t are dimensionless transverse and propagation coordinates.

In Eq. (1), the first term gives the linear evolution, while the second term represents diffraction and finally the last three terms that are weakly nonlocal nonlinearity, cubic nonlinearity, and quintic nonlinearity are due to competing nonlinearities. It should be noted that, in previous studies, the Lie group analysis were employed to extract solitons to Eq. (1).

3. Methodology

In this regard, we consider a general form of nonlinear partial differential equation as

$$F\left(q, q_x, q_t, q_{xx}, q_{xt}, q_{tt}, \ldots\right) = 0, \tag{2}$$

where $q = q(x,t)$ is an unknown complex function and F is a polynomial function with respect to some functions or specified variables, which contains nonlinear terms and highest order derivatives of the $q(x,t)$. Introducing the transformation $q(x,t) = P(\eta)\mathrm{e}^{\mathrm{i}\,\phi(x,t)}$, where $\xi = x - vt$, and $\phi(x,t) = -\kappa x + \omega t + \theta$. Then, Eq. (2) reduces to the following ODE:

$$G(P, P', P'', \ldots) = 0, \tag{3}$$

where G is a polynomial of P and its derivatives and the superscripts suggest the ordinary derivatives with respect to ξ.

3.1. Algorithm of modified Kudryashov method

We present a succinct about the modified Kudryashov method [11, 12] producing new exact solutions for a given nonlinear partial differential equation.

• *Step-1:* It is supposed that the solution $P(\xi)$ of the nonlinear Eq. (3) can be presented as

$$P(\xi) = a_0 + \sum_{l=1}^{N} a_l Q^l(\xi), \tag{4}$$

where the arbitrary constants $a_l (l = 1, 2, \ldots, N)$ are determined latter but $a_N \neq 0$ and is a positive integer, N which is determined by using balancing principle on Eq. (4) and satisfies the following ansatz equation:

$$Q'(\xi) = \left(Q^2(\xi) - Q(\xi)\right)\ln(a), \tag{5}$$

where $a \neq 0, 1$ and the general solution of Eq. (5) is $Q(\xi) = \frac{1}{1+da^{\xi}}$.

• *Step 2:* By inserting Eq. (4) along with Eq. (5) into Eq. (3) and equating the coefficients of powers of $Q^i(\xi)$ to zero, we receive a system of algebraic equations.

• *Step 3:* Solving these system, we secure the value of free parameters $a_0,\ a_1, k$ and v. After that, putting the obtained values in Eq. (4), finally generates new exact solutions for Eq. (2).

3.2. Algorithm of sine-Gordon equation expansion method

The sine-Gordon equation expansion method [13, 14] is one of the most efficient technique for investigating the bright, dark, dark-bright, singular soliton, and combined singular soliton solutions of the nonlinear differential equations. Basic ideas of the sine-Gordon equation expansion method can be abbreviated as follows:
Consider the following sine-Gordon equation [13]:

$$u_{xx} - u_{tt} = m^2 \sin(u), \tag{6}$$

where $u = u(x,t)$ and m is a constant. Applying the transformation $u(x,t) = U(\xi)$ where $\xi = k(x - vt)$, reduces Eq. (6) to the following nonlinear ordinary differential equation:

$$U'' = \frac{m^2}{k^2(1-c^2)} \sin(U). \tag{7}$$

Multiplying U' on the both sides of Eq. (7) and integrating it once gives

$$\left[\left(\frac{U}{2}\right)'\right]^2 = \frac{m^2}{k^2(1-c^2)} \sin^2\left(\frac{U}{2}\right) + C, \tag{8}$$

where C is an integration constant.
By setting $C = 0$, $\frac{U}{2} = w(\xi)$, and $\frac{m^2}{\mu^2(1-c^2)} = a^2$ in Eq. (8), we obtain

$$w' = a\sin(w). \tag{9}$$

If we take $a = 1$ in Eq. (9), we find

$$w' = \sin(w). \tag{10}$$

This is a simplified form of the sine-Gordon equation. Therefore, Eq. (10) has the following solutions:

$$\sin(w) = \operatorname{sech}(\xi), \quad \cos(w) = \tanh(\xi) \tag{11}$$

and

$$\sin(w) = \mathrm{i}\operatorname{csch}(\xi), \quad \cos(w) = \coth(\xi). \tag{12}$$

Now, we assume the formal solution of Eq. (3):

$$P(w) = \sum_{j=1}^{N} \cos^{j-1}(w)\left[B_j \sin(w) + A_j \cos(w)\right] + A_0. \tag{13}$$

It is assumed that the solution $P(\xi)$ of the nonlinear Eq. (13) along with Eq. (11) and Eq. (12) can be presented as follows:

$$P(\xi) = \sum_{j=1}^{N} \tanh^{j-1}(\xi)\left[B_j \operatorname{sech}(\xi) + A_j \tanh(\xi)\right] + A_0, \tag{14}$$

and

$$P(\xi) = \sum_{j=1}^{N} \coth^{j-1}(\xi)\left[\mathrm{i}B_j \operatorname{csch}(\xi) + A_j \coth(\xi)\right] + A_0. \tag{15}$$

By determining the value of N using the homogeneous balance principle, substituting the value of N into Eq. (13) and putting the result into the reduced ordinary differential equation using Eq. (10) yield a nonlinear algebraic system. Equating the coefficients of $\sin^j(w)$ and $\cos^j(w)$ equal to zero and solving the acquired system give the values of A_j, B_j, k, and v. Finally, after substituting the values of A_j, B_j, k, and v into Eq. (14) and Eq. (15), we can retrieve the dark, bright, combined dark-bright, singular and combined singular soliton solutions of Eq. (1).

3.3. Algorithm of extended sinh-Gordon equation expansion method

The extended sinh-Gordon equation expansion method [15–18] is a new robust technique for constructing the bright, dark, dark-bright, singular, combined singular, and other soliton solutions of the nonlinear differential equations. The fundamentals of the extended sinh-Gordon equation expansion method can be abbreviated as follows: Consider the following sinh-Gordon equation [15]:

$$u_{xt} = m \sinh(u), \tag{16}$$

where $u = u(x,t)$ and m is a constant. Introducing the transformation $u(x,t) = U(\xi)$ where $\xi = k(x - vt)$, reduces Eq. (16) to the following nonlinear ordinary differential equation:

$$U'' = -\frac{m}{k^2 v} \sinh(U). \tag{17}$$

Multiplying U' on the both sides of Eq. (17) and integrating it once gives

$$\left[\left(\frac{U}{2}\right)'\right]^2 = -\frac{m}{k^2 v} \sinh^2\left(\frac{U}{2}\right) + p, \tag{18}$$

where p is an integration constant.

By setting $\frac{U}{2} = w(\xi)$, and $-\frac{m}{k^2 v} = q$ in Eq. (18), we obtain

$$w' = \sqrt{p + q \sinh^2(w)}. \tag{19}$$

For different values of parameters p and q, Eq. (19) possesses the following set of solutions [15]:

• *Case-I*: When we take $p = 0$ and $q = 1$, Eq. (19) becomes

$$w' = \sinh(w). \tag{20}$$

This is a simplified form of the sinh-Gordon equation. Simplifying Eq. (20), the following solutions are obtained [15]:

$$\sinh(w) = \pm \mathrm{i} \operatorname{sech}(\xi), \quad \cosh(w) = -\tanh(\xi) \tag{21}$$

and

$$\sinh(w) = \pm \operatorname{csch}(\xi), \quad \cosh(w) = -\coth(\xi), \tag{22}$$

where $\mathrm{i} = \sqrt{-1}$ represents an imaginary number.

• *Case-II*: When we take $p = 1$ and $q = 1$, Eq. (19) becomes

$$w' = \cosh(w). \tag{23}$$

This is also a simplified form of the sinh-Gordon equation. Simplifying Eq. (23), the following solutions are obtained [15]:

$$\sinh(w) = \tan(\xi), \quad \cosh(w) = \pm \sec(\xi) \tag{24}$$

and

$$\sinh(w) = -\cot(\xi), \quad \cosh(w) = \pm \csc(\xi). \tag{25}$$

Now, we assume the formal solution of Eq. (3):

$$P(w) = \sum_{j=1}^{N} \cosh^{j-1}(w)\left[B_j \sinh(w) + A_j \cosh(w)\right] + A_0. \tag{26}$$

It is assumed that the solution $P(\xi)$ of the nonlinear Eq. (26) along with Eq. (20), Eq. (21) and Eq. (22) can be presented as follows:

$$P(\xi) = \sum_{j=1}^{N} (-\tanh(\xi))^{j-1}\left[\pm \mathrm{i} B_j \operatorname{sech}(\xi) - A_j \tanh(\xi)\right] + A_0, \tag{27}$$

and

$$P(\xi) = \sum_{j=1}^{N} (-\coth(\xi))^{j-1}\left[\pm B_j \operatorname{csch}(\xi) - A_j \coth(\xi)\right] + A_0. \tag{28}$$

Similarly, it is supposed that the solution $P(\xi)$ of the nonlinear Eq. (26) along with Eq. (23), Eq. (24) and Eq. (25) can be presented as follows:

$$P(\xi) = \sum_{j=1}^{N} (\pm \sec(\xi))^{j-1}\left[B_j \tan(\xi) \pm A_j \sec(\xi)\right] + A_0, \tag{29}$$

and

$$V(\xi) = \sum_{j=1}^{N} (\pm \csc(\xi))^{j-1}\left[-B_j \cot(\xi) \pm A_j \csc(\xi)\right] + A_0. \tag{30}$$

By determining the value of N using the homogeneous balance principle, inserting the value of N into Eq. (26) along with Eq. (20) give a nonlinear algebraic system. Equating the coefficients of $\sinh^j(w)$ and $\cosh^j(w)$ equal to zero and solving the acquired system give the values of A_j, B_j, k, and v. Finally, we substitute the values of A_j, B_j, k, and v into Eq. (27), Eq. (28), we retrieve dark, bright, combined dark-bright, singular and combined singular solutions of Eq. (1) (as for example, case-I). Similarly, we can proceed the same way for case-II and we receive the explicit trigonometric function solutions of Eq. (1).

4. Mathematical analysis

In order to solve the model, the following hypothesis is selected:

$$q(x,t) = P(\xi)\,\mathrm{e}^{\mathrm{i}\,\phi(x,t)}, \tag{31}$$

where $P(\xi)$ represents the shape of the pulse and $\phi(x,t)$ represents the phase component which is defined as the following transformation:

$$\xi = x - vt, \quad \phi(x,t) = -\kappa x + \omega t + \theta. \tag{32}$$

Substituting Eq. (31) and Eq. (32) into Eq. (1) and decomposing into real and imaginary parts, give

$$(\rho_1 - v\rho_2)P'' - (\omega + \rho_1\kappa^2 - \kappa\omega\rho_2)P + b_1P^3 + b_2P^5 + 2b_3\left\{P(P')^2 + P^2P''\right\} = 0, \tag{33}$$

and

$$v - \rho_2(\kappa v + \omega) + 2\rho_1\kappa = 0. \tag{34}$$

From Eq. (34), setting the coefficients of the linearly independent functions to zero gives the speed of the soliton as

$$v = \frac{\rho_2\omega - 2\rho_1\kappa}{1-\rho_2\kappa}. \tag{35}$$

The constraint condition $\rho_2\kappa \neq 1$. By applying Eq. (35) in Eq. (33), we get

$$\begin{aligned}&(\rho_1(1-\rho_2\kappa) - (\rho_2\omega - 2\rho_1\kappa)\rho_2)P'' \\ &-(1-\rho_2\kappa)(\omega + \rho_1\kappa^2 - \kappa\omega\rho_2)P \\ &+b_1(1-\rho_2\kappa)P^3 + b_2(1-\rho_2\kappa)P^5 \\ &+2b_3(1-\rho_2\kappa)\left\{P(P')^2 + P^2P''\right\} = 0.\end{aligned} \tag{36}$$

Balancing P^5 with P^2P'' in Eq. (36), then we get $N = 1$.

4.1. Application of modified Kudryashov method

Assuming solution of Eq. (36) is

$$P(\xi) = a_0 + a_1Q(\xi). \tag{37}$$

By inserting Eq. (37) along with its first and second derivatives into Eq. (36) and comparing the terms in the resulting equation, a nonlinear system is gained which by solving it, we determined the following sets:

- *Set 1:* $A = \ln(a)$, $a_0 = \mp\frac{1}{2}\sqrt{-\frac{6b_3}{b_2}}A$, $a_1 = \pm\sqrt{-\frac{6b_3}{b_2}}A$, $\omega = -\frac{3b_3}{4b_2}\left(8\kappa^2A^2b_3{\rho_2}^2 - 3A^4b_3{\rho_2}^2 - 4\kappa^2b_1{\rho_2}^2 - 16A^2\kappa b_3\rho_2 + 2A^2b_1{\rho_2}^2 + 8\kappa b_1\rho_2 + 8A^2b_3 - 4b_1\right)$, and $\rho_1 = -\frac{3b_3}{4b_2}\left(8\kappa^2A^2b_3{\rho_2}^2 - 3A^4b_3{\rho_2}^2 - 4\kappa^2b_1{\rho_2}^2 - 16A^2\kappa b_3\rho_2 + 2A^2b_1{\rho_2}^2 + 8\kappa b_1\rho_2 + 8A^2b_3 - 4b_1\right)$.

Therefore, Set 1 corresponds to the following solutions of Eq. (1):

$$q_{1,2}(x,t) = \mp\sqrt{-\frac{6b_3}{b_2}}A\left(\frac{1}{2} - \frac{1}{1+da^{x-vt}}\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)}, \quad b_2b_3 < 0. \tag{38}$$

For simplification of Eq. (38), we may apply $a^x = \mathrm{e}^{x\ln(a)}$, and as we know $\mathrm{e}^y = \cosh y + \sinh y$, $\mathrm{e}^{-y} = \cosh y - \sinh y$, and $d = 1$.

Rewriting Eq. (38), we obtain the hyperbolic function solutions

$$q_{1,2}(x,t) = \mp\sqrt{-\frac{6b_3}{b_2}}A \times\left(\frac{1}{2} - \frac{1}{1+\cosh((x-vt)A)+\sinh((x-vt)A)}\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)}, \quad b_2b_3 < 0.$$

where $A = \ln(a)$ and $v = \frac{3b_3}{4b_2}\left(3A^4b_3\rho_2 - 8A^2\kappa^2b_3\rho_2 + 16A^2\kappa b_3 - 2A^2b_1\rho_2 + 4\kappa^2b_1\rho_2 - 8\kappa b_1\right)$.

4.2. Application of sine-Gordon equation expansion method

As before, we know $N = 1$ by balancing the linear terms of highest order in Eq. (36) with the highest order nonlinear terms. As a result, Eq. (14) and Eq. (15) take the sine-Gordon equation expansion approach in the finite expansion form

$$P(\xi) = B_1\mathrm{sech}(\xi) + A_1\tanh(\xi) + A_0, \tag{39}$$

and

$$P(\xi) = \mathrm{i}B_1\mathrm{csch}(\xi) + A_1\coth(\xi) + A_0, \tag{40}$$

and so from Eq. (13)

$$P(w) = B_1\sin(w) + A_1\cos(w) + A_0, \tag{41}$$

where either A_1 or B_1 may be zero, but both A_1 and B_1 cannot be zero simultaneously.

By substituting Eq. (41) along with Eq. (10) into Eq. (36) and using some mathematical operations, we arrive at a nonlinear algebraic system. Solving the resulting system with the help of symbolic computation package, results in:

- *Set 1:* $A_0 = 0$, $A_1 = \pm\sqrt{-\frac{6b_3}{b_2}}$, $B_0 = 0$, $\omega = \frac{3b_3}{b_2}\left(\kappa^3b_1\rho_2 - 8\kappa^3b_3\rho_2 - \kappa^2b_1 + 8\kappa^2b_3 - 2\kappa b_1\rho_2 + 12\kappa b_3\rho_2 - 2b_1 + 12b_3\right)$, and $\rho_1 = \frac{3b_3}{b_2}\left(\kappa^2b_1{\rho_2}^2 - 8\kappa^2b_3{\rho_2}^2 - 2\kappa b_1\rho_2 + 16\kappa b_3\rho_2 - 2b_1{\rho_2}^2 + 12b_3{\rho_2}^2 + b_1 - 8b_3\right)$.

Therefore, Set 1 corresponds to the following solutions of Eq. (1):

$$q_{3,4}(x,t) = \mp\sqrt{-\frac{6b_3}{b_2}}\tanh(x-vt)\,\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)}, \quad b_2b_3 < 0. \tag{42}$$

and

$$q_{5,6}(x,t) = \mp\sqrt{-\frac{6b_3}{b_2}}\coth(x-vt)\,\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)}, \quad b_2b_3 < 0. \tag{43}$$

where $v = \frac{3b_3}{b_2}\left(\kappa^2b_1\rho_2 - 8\kappa^2b_3\rho_2 - 2\kappa b_1 + 16\kappa b_3 - 2b_1\rho_2 + 12b_3\rho_2\right)$.

- *Set 2:* $A_0 = 0$, $A_1 = 0$, $B_1 = \pm\sqrt{-\frac{6b_3}{b_2}}$, $\omega = \frac{3b_3}{b_2}\left(\kappa^3b_1\rho_2 + 4\kappa^3b_3\rho_2 - \kappa^2b_1 - 4\kappa^2b_3 + \kappa b_1\rho_2 + 4\kappa b_3\rho_2 + b_1 + 4b_3\right)$, and $\rho_1 = \frac{3b_3}{b_2}\left(\kappa^2b_1{\rho_2}^2 + 4\kappa^2b_3{\rho_2}^2 - 2\kappa b_1\rho_2 - 8\kappa b_3\rho_2 + b_1{\rho_2}^2 + 4b_3{\rho_2}^2 + b_1 + 4b_3\right)$.

Therefore, Set 2 corresponds to the following solutions of Eq. (1):

$$q_{7,8}(x,t) = \pm\sqrt{-\frac{6b_3}{b_2}}\mathrm{sech}(x-vt)\,\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)}, \quad b_2b_3 < 0 \tag{44}$$

and

$$q_{9,10}(x,t) = \pm\mathrm{i}\sqrt{-\frac{6b_3}{b_2}}\mathrm{csch}(x-vt)\,\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)}, \quad b_2b_3 < 0, \tag{45}$$

where $v = \frac{3b_3}{b_2}\left(\kappa^2b_1\rho_2 + 4\kappa^2b_3\rho_2 - 2\kappa b_1 - 8\kappa b_3 + b_1\rho_2 + 4b_3\rho_2\right)$.

• *Set 3-1:* $A_0 = 0$, $A_1 = \sqrt{-\frac{3b_3}{2b_2}}$, $B_1 = \pm\frac{1}{2}\sqrt{\frac{6b_3}{b_2}}$, $\omega = \frac{3b_3}{b_2}\left(4\kappa^3 b_1\rho_2 - 8\kappa^3 b_3\rho_2 - 4\kappa^2 b_1 + 8\kappa^2 b_3 - 2\kappa b_1\rho_2 + 3\kappa b_3\rho_2 - 2b_1 + 3b_3\right)$, and $\rho_1 = \frac{3b_3}{4b_2}\left(4\kappa^2 b_1{\rho_2}^2 - 8\kappa^2 b_3{\rho_2}^2 - 8\kappa b_1\rho_2 + 16\kappa b_3\rho_2 - 2b_1{\rho_2}^2 + 3b_3{\rho_2}^2 + 4b_1 - 8b_3\right)$.

Therefore, Set 3-1 corresponds to the following solutions of Eq. (1):

$$q_{11,12}(x,t) = \left(\sqrt{-\frac{3b_3}{2b_2}}\tanh(x - vt) \pm\frac{1}{2}\sqrt{\frac{6b_3}{b_2}}\operatorname{sech}(x - vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x + \omega t + \theta)}, \quad b_2 b_3 < 0 \tag{46}$$

and

$$q_{13,14}(x,t) = \left(\sqrt{-\frac{3b_3}{2b_2}}\coth(x - vt) \pm\mathrm{i}\frac{1}{2}\sqrt{\frac{6b_3}{b_2}}\operatorname{csch}(x - vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x + \omega t + \theta)}, \quad b_2 b_3 < 0, \tag{47}$$

where $v = \frac{3b_3}{4b_2}\left(4\kappa^2 b_1\rho_2 - 8\kappa^2 b_3\rho_2 - 8\kappa b_1 + 16\kappa b_3 - 2b_1\rho_2 + 3b_3\rho_2\right)$.

• *Set 3-2:* $A_0 = 0$, $A_1 = -\sqrt{-\frac{3b_3}{2b_2}}$, $B_1 = \pm\frac{1}{2}\sqrt{\frac{6b_3}{b_2}}$, $\omega = \frac{3b_3}{b_2}\left(4\kappa^3 b_1\rho_2 - 8\kappa^3 b_3\rho_2 - 4\kappa^2 b_1 + 8\kappa^2 b_3 - 2\kappa b_1\rho_2 + 3\kappa b_3\rho_2 - 2b_1 + 3b_3\right)$, and $\rho_1 = \frac{3b_3}{4b_2}\left(4\kappa^2 b_1{\rho_2}^2 - 8\kappa^2 b_3{\rho_2}^2 - 8\kappa b_1\rho_2 + 16\kappa b_3\rho_2 - 2b_1{\rho_2}^2 + 3b_3{\rho_2}^2 + 4b_1 - 8b_3\right)$.

Therefore, Set 3-2 corresponds to the following solutions of Eq. (1):

$$q_{15,16}(x,t) = \left(-\sqrt{-\frac{3b_3}{2b_2}}\tanh(x - vt) \pm\frac{1}{2}\sqrt{\frac{6b_3}{b_2}}\operatorname{sech}(x - vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x + \omega t + \theta)}, \quad b_2 b_3 < 0 \tag{48}$$

and

$$q_{17,18}(x,t) = \left(-\sqrt{-\frac{3b_3}{2b_2}}\coth(x - vt) \pm\mathrm{i}\frac{1}{2}\sqrt{\frac{6b_3}{b_2}}\operatorname{csch}(x - vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x + \omega t + \theta)}, \quad b_2 b_3 < 0, \tag{49}$$

where $v = \frac{3b_3}{4b_2}\left(4\kappa^2 b_1\rho_2 - 8\kappa^2 b_3\rho_2 - 8\kappa b_1 + 16\kappa b_3 - 2b_1\rho_2 + 3b_3\rho_2\right)$.

4.3. Application of extended sinh-Gordon equation expansion method

In this section, a new and effective version of the extended ShGEEM is applied to generate new solitary wave and other solutions of Eq. (1) in non-linear optics. As before, we know $N = 1$ by balancing the linear terms of highest order in Eq. (36) with the highest order nonlinear terms.

4.3.1. For case-I: $w' = \sinh(w)$

With the help of Eqs. (26)–(28), the extended ShGEEM has the solution in the form of Eq. (36):

$$P(\xi) = \pm\mathrm{i}B_1\operatorname{sech}(\xi) - A_1\tanh(\xi) + A_0, \tag{50}$$

and

$$P(\xi) = \pm B_1\operatorname{csch}(\xi) - A_1\coth(\xi) + A_0, \tag{51}$$

and so

$$P(w) = B_1\sinh(w) + A_1\cosh(w) + A_0, \tag{52}$$

where either A_1 or B_1 may be zero, but both A_1 and B_1 cannot be zero simultaneously.

By substituting Eq. (52) into Eq. (36) and using some mathematical operations, we arrive at a nonlinear algebraic system. Solving the resulting system with the help of symbolic computation package, results in:

• *Set 1:* $A_0 = 0$, $A_1 = \pm\sqrt{-\frac{6b_3}{b_2}}$, $B_0 = 0$, $\omega = \frac{3b_3}{b_2}\left(\kappa^3 b_1\rho_2 - 8\kappa^3 b_3\rho_2 - \kappa^2 b_1 + 8\kappa^2 b_3 - 2\kappa b_1\rho_2 + 12\kappa b_3\rho_2 - 2b_1 + 12b_3\right)$, and $\rho_1 = \frac{3b_3}{b_2}\left(\kappa^2 b_1{\rho_2}^2 - 8\kappa^2 b_3{\rho_2}^2 - 2\kappa b_1\rho_2 + 16\kappa b_3\rho_2 - 2b_1{\rho_2}^2 + 12b_3{\rho_2}^2 + b_1 - 8b_3\right)$.

Therefore, Set 1 corresponds to the following solutions of Eq. (1):

$$q_{19,20}(x,t) = \mp\sqrt{-\frac{6b_3}{b_2}}\tanh(x - vt)\,\mathrm{e}^{\mathrm{i}(\kappa x + \omega t + \theta)}, \quad b_2 b_3 < 0, \tag{53}$$

and

$$q_{21,22}(x,t) = \mp\sqrt{-\frac{6b_3}{b_2}}\coth(x - vt)\,\mathrm{e}^{\mathrm{i}(\kappa x + \omega t + \theta)}, \quad b_2 b_3 < 0, \tag{54}$$

where $v = \frac{3b_3}{b_2}\left(\kappa^2 b_1\rho_2 - 8\kappa^2 b_3\rho_2 - 2\kappa b_1 + 16\kappa b_3 - 2b_1\rho_2 + 12b_3\rho_2\right)$.

• *Set 2:* $A_0 = 0$, $A_1 = 0$, $B_1 = \pm\sqrt{-\frac{6b_3}{b_2}}$, $\omega = \frac{3b_3}{b_2}\left(\kappa^3 b_1\rho_2 + 4\kappa^3 b_3\rho_2 - \kappa^2 b_1 - 4\kappa^2 b_3 + \kappa b_1\rho_2 + 4\kappa b_3\rho_2 + b_1 + 4b_3\right)$, and $\rho_1 = \frac{3b_3}{b_2}\left(\kappa^2 b_1{\rho_2}^2 + 4\kappa^2 b_3{\rho_2}^2 - 2\kappa b_1\rho_2 - 8\kappa b_3\rho_2 + b_1{\rho_2}^2 + 4b_3{\rho_2}^2 + b_1 + 4b_3\right)$.

Therefore, Set 2 corresponds to the following solutions of Eq. (1):

$$q_{23,24}(x,t) = \pm\mathrm{i}\sqrt{-\frac{6b_3}{b_2}}\operatorname{sech}(x - vt)\,\mathrm{e}^{\mathrm{i}(\kappa x + \omega t + \theta)}, \quad b_2 b_3 < 0, \tag{55}$$

and

$$q_{25,26}(x,t) = \pm\sqrt{-\frac{6b_3}{b_2}}\operatorname{csch}(x - vt)\,\mathrm{e}^{\mathrm{i}(\kappa x + \omega t + \theta)}, \quad b_2 b_3 < 0, \tag{56}$$

where $v = \frac{3b_3}{b_2}\left(\kappa^2 b_1\rho_2 + 4\kappa^2 b_3\rho_2 - 2\kappa b_1 - 8\kappa b_3 + b_1\rho_2 + 4b_3\rho_2\right)$.

• *Set 3-1:* $A_0 = 0$, $A_1 = \sqrt{-\frac{3b_3}{2b_2}}$, $B_1 = \pm\sqrt{-\frac{3b_3}{2b_2}}$, $\omega = \frac{3b_3}{4b_2}\left(4\kappa^3 b_1\rho_2 - 8\kappa^3 b_3\rho_2 - 4\kappa^2 b_1 + 8\kappa^2 b_3 - 2\kappa b_1\rho_2 + 3\kappa b_3\rho_2 - 2b_1 + 3b_3\right)$, and $\rho_1 = \frac{3b_3}{4b_2}\left(4\kappa^2 b_1{\rho_2}^2 - 8\kappa^2 b_3{\rho_2}^2 - 8\kappa b_1\rho_2 + 16\kappa b_3\rho_2 - 2b_1{\rho_2}^2 + 3b_3{\rho_2}^2 + 4b_1 - 8b_3\right)$.

Therefore, Set 3-1 corresponds to the following solutions of Eq. (1):

$$q_{27,28}(x,t)=\left(-\sqrt{-\frac{3b_3}{2b_2}}\tanh(x-vt)\right.$$
$$\left.\mp\mathrm{i}\sqrt{-\frac{3b_3}{2b_2}}\mathrm{sech}(x-vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{57}$$

and

$$q_{29,30}(x,t)=\left(-\sqrt{-\frac{3b_3}{2b_2}}\coth(x-vt)\right.$$
$$\left.\mp\sqrt{-\frac{3b_3}{2b_2}}\mathrm{csch}(x-vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{58}$$

where $v = \frac{3b_3}{4b_2}\left(4\kappa^2b_1\rho_2-8\kappa^2b_3\rho_2-8\kappa b_1+16\kappa b_3-2b_1\rho_2+3b_3\rho_2\right)$.

- *Set 3-2:* $A_0=0$, $A_1=-\sqrt{-\frac{3b_3}{2b_2}}$, $B_1=\pm\sqrt{-\frac{3b_3}{2b_2}}$, $\omega=\frac{3b_3}{4b_2}\left(4\kappa^3b_1\rho_2-8\kappa^3b_3\rho_2-4\kappa^2b_1+8\kappa^2b_3-2\kappa b_1\rho_2+3\kappa b_3\rho_2-2b_1+3b_3\right)$, and $\rho_1=\frac{3b_3}{4b_2}\left(4\kappa^2b_1\rho_2{}^2-8\kappa^2b_3\rho_2{}^2-8\kappa b_1\rho_2+16\kappa b_3\rho_2-2b_1\rho_2{}^2+3b_3\rho_2{}^2+4b_1-8b_3\right)$.

Therefore, Set 3-2 corresponds to the following solutions of Eq. (1):

$$q_{31,32}(x,t)=\left(\sqrt{-\frac{3b_3}{2b_2}}\tanh(x-vt)\right.$$
$$\left.\mp\mathrm{i}\sqrt{-\frac{3b_3}{2b_2}}\mathrm{sech}(x-vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{59}$$

and

$$q_{33,34}(x,t)=\left(\sqrt{-\frac{3b_3}{2b_2}}\coth(x-vt)\right.$$
$$\left.\mp\sqrt{-\frac{3b_3}{2b_2}}\mathrm{csch}(x-vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{60}$$

where $v=\frac{3b_3}{4b_2}\left(4\kappa^2b_1\rho_2-8\kappa^2b_3\rho_2-8\kappa b_1+16\kappa b_3-2b_1\rho_2+3b_3\rho_2\right)$.

4.3.2. For case-II: $w'=\cosh(w)$

With the help of Eq. (26), Eq. (29) and Eq. (30), the extended ShGEEM has the solution in the form of Eq. (36):

$$P(\xi)=B_1\tan(\xi)\pm A_1\sec(\xi)+A_0, \tag{61}$$

and

$$P(\xi)=-B_1\cot(\xi)\pm A_1\csc(\xi)+A_0, \tag{62}$$

and so

$$P(w)=B_1\sinh(w)+A_1\cosh(w)+A_0, \tag{63}$$

where either A_1 or B_1 may be zero, but both A_1 and B_1 cannot be zero simultaneously.

By substituting Eq. (63) into Eq. (36) and using some mathematical operations, we arrive at a nonlinear algebraic system. Solving the resulting system with the help of symbolic computation package, results in:

- *Set 1:* $A_0=0$, $A_1=\pm\sqrt{-\frac{6b_3}{b_2}}$, $B_0=0$, $\omega=\frac{3b_3}{b_2}\left(\kappa^3b_1\rho_2-4\kappa^3b_3\rho_2-\kappa^2b_1+4\kappa^2b_3-\kappa b_1\rho_2+4\kappa b_3\rho_2-b_1+4b_3\right)$, and $\rho_1=\frac{3b_3}{b_2}\left(\kappa^2b_1\rho_2{}^2-4\kappa^2b_3\rho_2{}^2-2\kappa b_1\rho_2+8\kappa b_3\rho_2-b_1\rho_2{}^2+4b_3\rho_2{}^2+b_1-4b_3\right)$.

Therefore, Set 1 corresponds to the following solutions of Eq. (1):

$$q_{35,36}(x,t)=\mp\sqrt{-\frac{6b_3}{b_2}}\sec(x-vt)\,\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{64}$$

and

$$q_{37,38}(x,t)=\mp\sqrt{-\frac{6b_3}{b_2}}\csc(x-vt)\,\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{65}$$

where $v=\frac{3b_3}{b_2}\left(\kappa^2b_1\rho_2-4\kappa^2b_3\rho_2-2\kappa b_1+8\kappa b_3-b_1\rho_2+4b_3\rho_2\right)$.

- *Set 2:* $A_0=0$, $A_1=0$, $B_0=\pm\sqrt{-\frac{6b_3}{b_2}}$, $\omega=\frac{3b_3}{b_2}\left(\kappa^3b_1\rho_2+8\kappa^3b_3\rho_2-\kappa^2b_1-8\kappa^2b_3+2\kappa b_1\rho_2+12\kappa b_3\rho_2+2b_1+12b_3\right)$, and $\rho_1=\frac{3b_3}{b_2}\left(\kappa^2b_1\rho_2{}^2+8\kappa^2b_3\rho_2{}^2-2\kappa b_1\rho_2-16\kappa b_3\rho_2+2b_1\rho_2{}^2+12b_3\rho_2{}^2+b_1+8b_3\right)$.

Therefore, Set 2 corresponds to the following solutions of Eq. (1):

$$q_{39,40}(x,t)=\pm\sqrt{-\frac{6b_3}{b_2}}\tan(x-vt)\,\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{66}$$

and

$$q_{41,42}(x,t)=\mp\sqrt{-\frac{6b_3}{b_2}}\cot(x-vt)\,\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{67}$$

where $v=\frac{3b_3}{b_2}\left(\kappa^2b_1\rho_2+8\kappa^2b_3\rho_2-2\kappa b_1-16\kappa b_3+2b_1\rho_2+12b_3\rho_2\right)$.

- *Set 3-1:* $A_0=0$, $A_1=\sqrt{-\frac{3b_3}{2b_2}}$, $B_1=\pm\sqrt{-\frac{3b_3}{2b_2}}$, $\omega=\frac{3b_3}{4b_2}\left(4\kappa^3b_1\rho_2+8\kappa^3b_3\rho_2-4\kappa^2b_1-8\kappa^2b_3+2\kappa b_1\rho_2+3\kappa b_3\rho_2+2b_1+3b_3\right)$, and $\rho_1=\frac{3b_3}{4b_2}\left(4\kappa^2b_1\rho_2{}^2+8\kappa^2b_3\rho_2{}^2-8\kappa b_1\rho_2-16\kappa b_3\rho_2+2b_1\rho_2{}^2+3b_3\rho_2{}^2+4b_1+8b_3\right)$.

Therefore, Set 3-1 corresponds to the following solutions of Eq. (1):

$$q_{43,44}(x,t)=\left(-\sqrt{-\frac{3b_3}{2b_2}}\sec(x-vt)\right.$$
$$\left.\pm\sqrt{-\frac{3b_3}{2b_2}}\tan(x-vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{68}$$

and

$$q_{45,46}(x,t) = \left(-\sqrt{-\frac{3b_3}{2b_2}}\csc(x-vt) \mp\sqrt{-\frac{3b_3}{2b_2}}\cot(x-vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{69}$$

where $v = \frac{3b_3}{4b_2}\left(4\kappa^2b_1\rho_2+8\kappa^2b_3\rho_2-8\kappa b_1-16\kappa b_3 +2b_1\rho_2+3b_3\rho_2\right)$.

• *Set 3-2:* $A_0=0$, $A_1=-\sqrt{-\frac{3b_3}{2b_2}}$, $B_1=\pm\sqrt{-\frac{3b_3}{2b_2}}$, $\omega = \frac{3b_3}{4b_2}\left(4\kappa^3b_1\rho_2+8\kappa^3b_3\rho_2-4\kappa^2b_1-8\kappa^2b_3+2\kappa b_1\rho_2 +3\kappa b_3\rho_2+2b_1+3b_3\right)$, and $\rho_1=\frac{3b_3}{4b_2}\left(4\kappa^2b_1{\rho_2}^2+8\kappa^2b_3{\rho_2}^2 -8\kappa b_1\rho_2-16\kappa b_3\rho_2+2b_1{\rho_2}^2+3b_3{\rho_2}^2+4b_1+8b_3\right)$.

Therefore, Set 3-2 corresponds to the following solutions of Eq. (1):

$$q_{47,48}(x,t) = \left(\sqrt{-\frac{3b_3}{2b_2}}\sec(x-vt) \pm\sqrt{-\frac{3b_3}{2b_2}}\tan(x-vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{70}$$

and

$$q_{49,50}(x,t) = \left(\sqrt{-\frac{3b_3}{2b_2}}\csc(x-vt) \mp\sqrt{-\frac{3b_3}{2b_2}}\cot(x-vt)\right)\mathrm{e}^{\mathrm{i}(\kappa x+\omega t+\theta)},$$
$$b_2b_3<0, \tag{71}$$

where $v = \frac{3b_3}{4b_2}\left(4\kappa^2b_1\rho_2+8\kappa^2b_3\rho_2-8\kappa b_1-16\kappa b_3 +2b_1\rho_2+3b_3\rho_2\right)$.

Remarks: To the best of our knowledge some of the derived solutions have never been reported so far by other authors in the literature [1] and our executed approaches are different. In this case, combined soliton solutions are new and we verified the all solutions by putting back into original equation via the symbolic software maple and found them correct.

5. Conclusion

The nonlinear dynamical model Eq. (1) that describes the optical solitons propagating in a nonlinear medium with nonlocal nonlinearity, cubic nonlinearity and quintic nonlinearity is investigated analytically. Though the modified Kudrashov method, sine-Gordon equation expansion method and extended sinh-Gordon equation expansion method, we successfully computed bright, dark, combined bright-dark, singular, combined singular, periodic and other solitons from the studied model. Form our generated solitons show that the executed methods are new and robust for solving any other nonlinear differential equations.

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