

Fractional Euler–Bernoulli Beam Theory Based on the Fractional Strain–Displacement Relation and its Application in Free Vibration, Bending and Buckling Analyses of Micro/Nanobeams

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Applications of fractional calculus in the constitutive relation lead to the fractional derivatives models. They are stately generalization of the integer derivatives models — this general form makes fractional derivatives models more flexible and suitable to describe properties and behavior of different materials/structures. In the present work, the general strain deformation gradient has been presented by using the modified conformable fractional derivatives definition. Within this approach the fractional Euler–Bernoulli beam theory has been formulated and applied to the analysis of free vibration, bending and buckling of micro/nanobeams which exhibit strong scale effect.

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1. Introduction

Fractional calculus has become an exciting new mathematical method of solution of diverse problems in mathematics, science, and engineering [1–3]. Recent advances of fractional calculus are dominated by modern examples of applications in differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology and electrochemistry [4].

Applications of fractional calculus have been brought to applied mechanics' problems and leads to fractional derivatives models (FDMs). Many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials [5]. These models are the generalized form of classical models, therefore, they are able to describe the behavior of materials better than integer derivatives models (IDMs). For instance, Challamel et al. [6] by using fractional derivatives (FDs) generalized the Eringen nonlocal theory and showed that the optimized fractional derivative model has a perfect matching with the dispersive wave properties of the Born–Kármán model of lattice dynamics and is better than the classical Eringen theory. Also Demir et al. [7] studied vibration of viscoelastic beam that obeys a fractional differentiation constitutive law and stated that FDs are useful for describing the occurrence of vibrations in engineering practice. FDMs have been introduced in different problems, Sumelka [8] applied fractional calculus to a classical problem of the

structural mechanics and introduced nonlocal Kirchhoff–Love plate theory. Atanackovic and Stankovic [9] modified the kinematics strain–displacement relationship with an alternative nonlocal formulation and by using the Caputo fractional derivatives generalized wave equation in nonlocal elasticity. Lazopoulos [10] assumed that strain energy density depends not only on the local strain but also on a fractional derivative of the strain. Carpinteri et al. [11] by means of attenuation function of strain and the Caputo fractional derivatives introduced a fractional calculus approach to the nonlocal elasticity.

In the present work, the general strain deformation gradient has been presented by using modified conformable fractional derivatives definition (CFDD). The fractional Euler–Bernoulli beam theory (FEBBT) has been presented based on this general form and has two free parameters: fractional parameter (which control the displacement's derivative in strain–displacement relation) and length scale parameter to consider size effects in the micron and the sub-micron scales. Finally, vibration of clamped–clamped (C–C) microbeams under axial force and free vibration, bending and buckling of nanobeam have been studied by the theory. It should be emphasized that compared to the previous papers [12–14] the novelty lays in the definition of the constitutive relation which is based on the fractional strain, as mentioned, whereas in [12, 13] the constitutive relation is based on the general form of the Eringen nonlocal elasticity theory, and in [14] the formulation is based on the fractional strain energy.

This paper has been divided into four sections: 1. Using fractional Taylor series expansion and the CFDD, the general form of the deformation gradient has been introduced. 2. The FEBBT based on the general form of

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the strain has been presented which has two free parameters: fractional parameter to control the displacement derivative in the constitutive relation and the length scale parameter to consider the size effects in micron and sub-micron scales. 3. To demonstrate the functionality of FEBBT the non-dimensional frequency of the microbeams under axial force has been compared with the experimental data and EBT. 4. Bending, buckling and vibration of nanobeams has been studied based on the FEBBT.

2. Basic definitions and tools

In this section, the basic definitions that are necessary for the theory are introduced namely the fractional derivatives definition and the fractional Taylor series expansion.

2.1. Modified conformable fractional derivatives definition

Modified CFDD is a definition, which has been presented by Khalil et al. [15] and Tallafha et al. [16] is:

Let $f, g : R \rightarrow R$. Then modified CFDD is

$$D^\alpha(f)(x) = \frac{\partial^\alpha f}{\partial x^\alpha} = |x|^{([\alpha]-\alpha)} \frac{d^{[\alpha]}f(x)}{|dx|^{[\alpha]}}. \tag{1}$$

According to the modified CFDD, its application to the multi-variable functions (Appendix A) reads:

$$D_x^\alpha(g)(x, y) = \frac{\partial^\alpha g}{\partial x^\alpha} = |x|^{([\alpha]-\alpha)} \frac{\partial^{[\alpha]}g(x, y)}{|\partial x|^{[\alpha]}}$$

$$D_y^\alpha(g)(x, y) = \frac{\partial^\alpha g}{\partial y^\alpha} = |y|^{([\alpha]-\alpha)} \frac{\partial^{[\alpha]}g(x, y)}{|\partial y|^{[\alpha]}} \tag{2}$$

where $\alpha \in (0, \infty]$, f is $(n + 1)$ -differentiable at $x > 0$ and $[\alpha]$ is the smallest integer greater than or equal to α .

2.2. Fractional Taylor series expansion

Fractional Taylor series expansion (FTSE) is a generalization of the one presented in [17].

Assume that f is an infinitely α -differentiable function, for some $0 < \alpha \leq 1$ at a neighborhood of a point x_0 . Then f has the fractional power series expansion in the form

$$f(x) = \sum_0^\infty \frac{(D_{x_0}^\alpha f)^{(k)} |x - x_0|^{k\alpha}}{\alpha^k k!},$$

$$x_0 - R^{1/\alpha} < x < x_0 + R^{1/\alpha}, \quad R > 0, \tag{3}$$

where $(D_{x_0}^\alpha f)^{(k)}$ means the application of the fractional k times derivative.

3. General form of strain in terms of fractional calculus

The general form of strains is obtained by generalization of the displacement derivatives in the strain-displacement relation, namely the first derivatives are substituted with derivatives of order α . Parameter α

can be integer and non-integer number between 0 and 1 ($0 < \alpha \leq 1$) [9] and leads to the appearance of the fractional calculus in the equation of the motion.

The general form of strain presented already by Sumelka [8, 18] and Atanackovic and Stankovic [9] where based on the Riesz–Caputo definition in which the integral form makes the numerical solution of the governing equation difficult. In the present work, the general form has been presented based on modified CFDD and unlike previous works, its usability has been studied from two part of view: 1. geometrical view, by using fractional Taylor series expansion method, 2. continuum mechanics view. In addition, the use of the modified CFDD makes the numerical solution of the governing equation simpler.

3.1. Geometrical derivation of the general strains

FDs allow us to make a generalization of integer derivatives, and therefore in this part by using FTSE we present a general form of deformation gradient and improve the strain tensor by assuming a general form as below.

Consider a two-dimensional deformation of an infinitesimal rectangular material element with dimensions dx^α by dy^α where $0 < \alpha \leq 1$ but with this proviso that value of alpha satisfies the small deformation theory. It means that the value of α must not be too small — after the deformation rectangular material element should take the form of a rhombus. Based on small deformation theory and fractional Taylor series expansion we have: $u(x + dx, y) \approx u(x, y) + \frac{1}{\alpha} \frac{\partial^\alpha u}{\partial x^\alpha} |dx|^\alpha$ with similar expansions for all other terms — for $\alpha = 1$ the classical form is obtained of course.

Next, based on Eq. (1):

$$u(x + dx, y) \approx u(x, y) + \frac{1}{\alpha} \left(|x|^{1-\alpha} \frac{\partial u}{\partial x} |dx|^\alpha \right), \tag{4}$$

so from the geometry of the deformed element ($\frac{1}{\alpha} \frac{\partial^\alpha v}{\partial x^\alpha} |dx|^\alpha \ll 1$) we have

$$A'B' = \sqrt{\left(|dx|^\alpha + \frac{1}{\alpha} \frac{\partial^\alpha u}{\partial x^\alpha} |dx|^\alpha \right)^2 + \left(\frac{1}{\alpha} \frac{\partial^\alpha v}{\partial x^\alpha} |dx|^\alpha \right)^2} \approx \left(1 + \frac{1}{\alpha} \frac{\partial^\alpha u}{\partial x^\alpha} \right) |dx|^\alpha. \tag{5}$$

Note that when $\alpha \rightarrow 0$ then $\frac{1}{\alpha} \frac{\partial^\alpha v}{\partial x^\alpha} |dx|^\alpha$ or $\frac{1}{\alpha} \frac{\partial^\alpha x}{\partial x^\alpha} |dx|^\alpha \rightarrow \infty$ ($\frac{1}{\alpha} \frac{\partial^\alpha v}{\partial x^\alpha} |dx|^\alpha \ll 1$) since $0 < \alpha \leq 1$. The normal strain in the x -direction of the rectangular element is defined by:

$$\varepsilon_x = \frac{A'B' - AB}{AB}, \tag{6}$$

and knowing that $AB = dx^\alpha$, we have

$$\varepsilon_x = (l^{\alpha-1}) \frac{1}{\alpha} \frac{\partial^\alpha u}{\partial x^\alpha}. \tag{7}$$

Similarly, the normal strain in the y -direction and z -direction becomes

$$\varepsilon_y = (l^{\alpha-1}) \frac{1}{\alpha} \frac{\partial^\alpha v}{\partial y^\alpha}, \quad \varepsilon_z = \frac{1}{\alpha} l^{\alpha-1} \frac{\partial^\alpha w}{\partial z^\alpha}, \tag{8}$$

where $l^{\alpha-1}$ is the length scale parameter similarly like in the classical non-local gradient methods and fractional

nonlocal Kirchhoff theory [8]. The introduction of the length scale, allows finally to obtain dimensionless quantity and lets one to consider the size-dependence effects for micron and sub-micron scales.

The engineering shear strain, or the change in the angle between two originally orthogonal material lines (in this case line AB and AC) is defined as:

$$\gamma_{xy} = \frac{\pi}{2} - \angle C' A' B' \tag{9}$$

where $C' A' B'$ is line's angle in the deformed element. For small deformations, $\alpha = \tan \alpha$ and $\beta = \tan \beta$, shear strain by neglecting higher-order terms in the displacement gradients (for small displacement gradient we have $\frac{1}{\alpha} \frac{\partial^\alpha u}{\partial x^\alpha} \ll 1$ and $\frac{1}{\alpha} \frac{\partial^\alpha u}{\partial y^\alpha} \ll 1$) can be then expressed as

$$\gamma_{xy} = \frac{\frac{1}{\alpha} \frac{\partial^\alpha v}{\partial x^\alpha} dx^\alpha}{dx^\alpha + \frac{1}{\alpha} \frac{\partial^\alpha u}{\partial x^\alpha} dx^\alpha} + \frac{\frac{1}{\alpha} \frac{\partial^\alpha u}{\partial y^\alpha} dy^\alpha}{dy^\alpha + \frac{1}{\alpha} \frac{\partial^\alpha u}{\partial y^\alpha} dy^\alpha} = \frac{1}{\alpha} l^{\alpha-1} \left(\frac{\partial^\alpha v}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial y^\alpha} \right). \tag{10}$$

By considering similar behaviors in the y - z and x - z planes, these results can be easily extended to the general three-dimensional case.

Now the fractional strain-displacement relations are

$$\varepsilon_x = \frac{1}{\alpha} l^{\alpha-1} \frac{\partial^\alpha u}{\partial x^\alpha}, \quad \varepsilon_y = \frac{1}{\alpha} l^{\alpha-1} \frac{\partial^\alpha v}{\partial y^\alpha}, \quad \varepsilon_z = \frac{1}{\alpha} l^{\alpha-1} \frac{\partial^\alpha w}{\partial z^\alpha},$$

$$\begin{aligned} \gamma_{xy} &= \frac{1}{\alpha} l^{\alpha-1} \left(\frac{\partial^\alpha v}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial y^\alpha} \right), \\ \gamma_{zx} &= \frac{1}{\alpha} l^{\alpha-1} \left(\frac{\partial^\alpha w}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial z^\alpha} \right), \\ \gamma_{yz} &= \frac{1}{\alpha} l^{\alpha-1} \left(\frac{\partial^\alpha v}{\partial z^\alpha} + \frac{\partial^\alpha w}{\partial y^\alpha} \right). \end{aligned} \tag{11}$$

For completeness the two-dimensional geometric fractional strain deformation has been shown in Fig. 1.

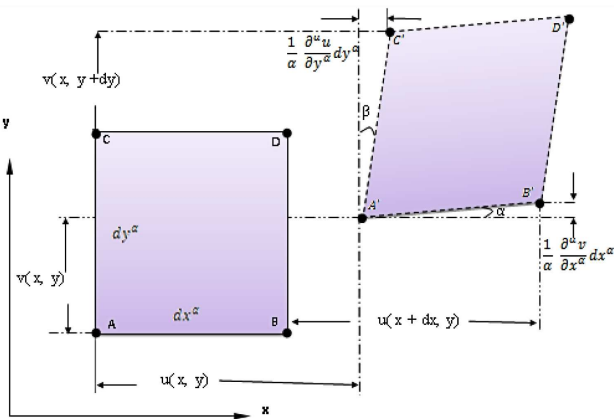


Fig. 1. Two-dimensional geometric fractional strain deformation.

3.2. Continuum view

One considers a continuum body B , as the set of elements X whom the reference and the current configura-

tions are R and C , respectively. The regular motion of the material body B can be written as

$$x = \aleph(X, t) \quad \text{or} \quad x_i = \aleph_i(X, t), \tag{12}$$

where the mapping function \aleph gives the position x for each particles X for all times t . Its inverse due to continuity assumption exists as

$$X = N(x, t) \quad \text{or} \quad X_A = N_A(x, t), \tag{12}$$

and states that the particle X is located at position x at time t . Forasmuch, as in the deformation just two stationary configurations have been considered and disregard any consideration by which the final deformed configuration is reached from the initial undeformed configuration, so the time variable is eliminated and $x = \aleph(X)$ and $X = N(x)$.

The mapping function is continuous, so

$$dx_i = \frac{\partial \aleph_i(X)}{\partial X_A} dX_A = x_{i,A} dX_A, \tag{14}$$

where the deformation gradient tensor is [19]:

$$x_{i,A} \equiv F_{iA} = \frac{\partial \aleph_i(X)}{\partial X_A} \tag{15}$$

Rewriting Eq. (14) leads to the following form:

$$dx = F \cdot dX. \tag{16}$$

F is invertible, so that the inverse F^{-1} exists, therefore

$$dX = F^{-1} \cdot dx, \quad F^{-1} = X_{A,i}. \tag{17}$$

We can generalize the deformation gradient and its inverse as follows:

$$\begin{aligned} {}^G F_{iA} &= \frac{\partial^\alpha \aleph_i(X)}{\partial X_A^\alpha} = \frac{l^{\alpha-1}}{\alpha} |X|^{1-\alpha} \frac{\partial \aleph_i(X)}{\partial X_A} = \\ &= \frac{l^{\alpha-1}}{\alpha} |X|^{1-\alpha} F_{iA}, \\ {}^G F_{Ai}^{-1} &= \frac{\partial^\alpha N_A(X)}{\partial x_i^\alpha} = \frac{l^{\alpha-1}}{\alpha} |x|^{1-\alpha} \frac{\partial N_A(X)}{\partial x_i} = \\ &= \frac{l^{\alpha-1}}{\alpha} |x_i|^{1-\alpha} F_{iA}, \end{aligned} \tag{18}$$

where l is the length scale of the isotropic material and leads to the dimensional less quantity similarly like in the non-local gradient methods.

3.2.1. Rigid body motion

It should be emphasized that it is crucial to observe how the fractional deformation gradients transforms under isomorphism (superimposed rigid-body motions). There is a condition for the rigid body motion, which is satisfied by classical strain.

The Lagrangian finite strain tensor is

$$2E = C - I, \tag{19}$$

where $C = F^T \cdot F$ and I is an identity matrix. For the classical rigid body motion

$$C = I \rightarrow F^T \cdot F = I. \tag{20}$$

Now, for the general form of the strains the following condition must be satisfied:

$${}^G F^T \cdot {}^G F = I. \tag{21}$$

Two categories have been considered for value of α : (1) $\alpha = 1$ (classical strain form), (2) $\alpha \neq 1$ (the fractional

strain form). In the case of $\alpha = 1$ Eq. (20) is valid, but for $\alpha \neq 1$:

$${}^G F^T \cdot {}^G F = I \rightarrow \left(\frac{l^{\alpha-1}}{\alpha} |X_A|^{\alpha-1}\right) F^T \cdot \left(\frac{l^{\alpha-1}}{\alpha} |X_B|^{\alpha-1}\right) F = I. \quad (22)$$

With respect to Eq. (20), the following condition must satisfy

$$\left(\frac{l^{\alpha-1}}{\alpha} |X_A|^{1-\alpha}\right) = \left(\frac{l^{\alpha-1}}{\alpha} |X_B|^{1-\alpha}\right) = 1 \rightarrow X = \alpha^{1-\alpha} l. \quad (23)$$

3.3. Fractional Euler–Bernoulli beam theory

In the previous sections, the general form of strain-displacement relations has been obtained based on the modified CFDD. Now, these relations will be used in the one of the structural mechanics problem — beam’s equations of the motion — and the FEBBT will be presented.

3.3.1. Equation of motion

A schematic view of a beam under distributed and axial forces has been shown in Fig. 2. Assume that its displacement field is

$$u_0 = -zw_x, \quad v_0 = 0, \quad w_0 = w(x, t), \quad (24)$$

where u_0 , v_0 and w_0 denote the components of displacement parallel to x -, y - and z -direction. The classical equation of motion of the Euler–Bernoulli beam is [20]:

$$M_{xx} + f(x, t) + pw_{xx} = \rho Aw_{tt}, \quad 0 < x < L, \quad (25)$$

where $M(x, t) = \int_{-h/2}^{h/2} \sigma z dz$, A is cross-section of the beam, ρ is density, p and f are axial and transverse forces. According to the fractional strain-displacement

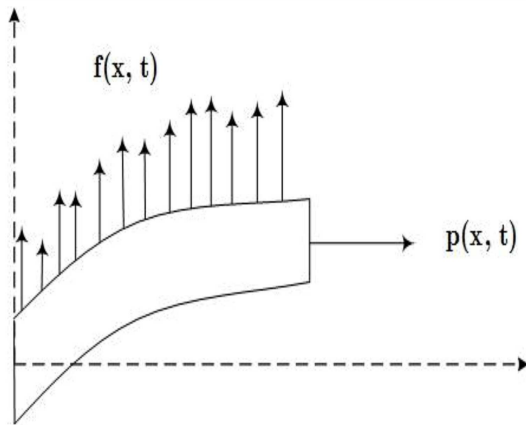


Fig. 2. A schematic view of a beam under distributed and axial forces.

relations and Eq. (24) the $M(x, t)$ obtains the form as below

$$M(x, t) = -EI \left(\frac{1}{\alpha} l^{\alpha-1} x^{1-\alpha} w_{xx}\right), \quad (26)$$

where E is Young’s modulus, h is the beam thickness and I is the effective moment of inertia. Now, by substituting the second derivative of Eq. (26) into Eq. (25) the fractional dynamics equation is obtained

$$EI \left(\frac{1}{\alpha} l^{\alpha-1} x^{1-\alpha} w_{xxxx}\right) + EI \left(\frac{2}{\alpha} l^{\alpha-1} (1-\alpha) x^{-\alpha} w_{xxx}\right) + EI \left(\frac{1}{\alpha} l^{\alpha-1} (1-\alpha) (-\alpha) x^{-1-\alpha} w_{xx}\right) + mw_{tt} = pw_{xx} + f(x, t), \quad 0 < x < L. \quad (27)$$

For convenience, the following non-dimensional parameters are utilized:

$$\hat{w} = \frac{w_0}{L}, \quad \hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{t}{T},$$

$$T = \sqrt{\frac{\rho AL^4}{EI}}, \quad S_1 = \left(\frac{1}{\alpha} l^{\alpha-1} L^{-\alpha+1}\right),$$

$$S_2 = \left(\frac{L^2}{EI}\right), \quad S_3 = \frac{L^3}{EI}.$$

Then the non-dimensional form of Eq. (27) is

$$S_1 [\hat{x}^{1-\alpha} \hat{w}_{xxxx} + 2(1-\alpha) \hat{x}^{-\alpha} \hat{w}_{xxx} + (1-\alpha)(-\alpha) \hat{x}^{-1-\alpha} \hat{w}_{xx}] + \hat{w}_{tt} = S_2 p \hat{w}_{xx} + S_3 f(\hat{x}, \hat{t}), \quad 0 < \hat{x} < 1 \quad (28)$$

It should be noted that when $\alpha = 1$ the classical equation of motion of the Euler–Bernoulli beam is obtained.

3.3.2. Vibration, bending and buckling of nanobeams based on the FEBBT

In this section, we show the effects of the fractional parameter and the length scale parameter on the free vibration, bending and buckling of nanobeams.

3.3.2.1. Vibration

In the vibration analyses, we neglect the effect of transverse loading on the beam, in the other words for analysis of the frequency of the beam under axial load, Eq. (28) reduces to the following form:

$$S_1 [\hat{x}^{1-\alpha} \hat{w}_{xxxx} + 2(1-\alpha) \hat{x}^{-\alpha} \hat{w}_{xxx} + (1-\alpha)(-\alpha) \hat{x}^{-1-\alpha} \hat{w}_{xx}] + w_{tt} - S_2 p \hat{w}_{xx} = 0. \quad (29)$$

Using modified CFDD the governing equations are easily solved by the classical numerical schemes, therefore here to calculate the non-dimensional frequency of the nanobeam we use the Galerkin weighted residual method. Based on this method we assume

$$w(x, t) \approx \sum_{i=1}^n \varphi_i(x) q_i(t), \quad (30)$$

where $\varphi_i(x)$ and $q_i(t)$ are the mode shape and a time dependent function to be determined, respectively. Substituting Eq. (30) into Eq. (29) yields

$$S_1 [\hat{x}^{1-\alpha} \sum_{i=1}^n \varphi_i^{(4)}(\hat{x}) q_i(\hat{t}) + 2(1-\alpha) \hat{x}^{-\alpha} \times \sum_{i=1}^n \varphi_i'''(\hat{x}) q_i(\hat{t}) + (1-\alpha)(-\alpha) \hat{x}^{-1-\alpha} \sum_{i=1}^n \varphi_i''(\hat{x}) q_i(\hat{t})] + \sum_{i=1}^n \varphi_i(\hat{x}) \ddot{q}_i(\hat{t}) - S_2 p \sum_{i=1}^n \varphi_i''(\hat{x}) q_i(\hat{t}) = \varepsilon. \quad (31)$$

Next, by multiplying the result by the mode shape and integrating outcome from 0 to 1 we have

$$M\ddot{q}(t) + Kq(t) = 0. \tag{32}$$

where

$$M = \int_0^1 \varphi_i \varphi_j d\hat{x},$$

$$K = \int_0^1 [S_1(\hat{x}^{1-\alpha} \varphi_j'''' + 2(1-\alpha)\hat{x}^{-\alpha} \varphi_j'' + (1-\alpha)(-\alpha)\hat{x}^{-1-\alpha} \varphi_j'') \varphi_i + S_2 \varphi_j'' \varphi_i] d\hat{x}. \tag{33}$$

The non-dimensional eigenproblem of the beam is then

$$[M\omega^2 + S_1 K] \bar{q} = 0. \tag{34}$$

3.3.2.2. *Bending*

The static deflection is obtained from Eq. (28) by setting axial force and the inertial term to zero as below

$$S_1[\hat{x}^{1-\alpha} \hat{w}_{xxxx} + 2(1-\alpha)\hat{x}^{-\alpha} \hat{w}_{xxx} + (1-\alpha)(-\alpha)\hat{x}^{-1-\alpha} \hat{w}_{xx}] = S_3 f(\hat{x}, \hat{t}). \tag{35}$$

To solve Eq. (35) the Galerkin method is used again. The transverse displacement of the beam is as below

$$w(x) \approx \sum_{i=1}^n a_i \varphi_i(x). \tag{36}$$

Substituting Eq. (36) into Eq. (35) one has

$$S_1[\hat{x}^{1-\alpha} \sum_{i=1}^n \varphi_i^{(4)}(\hat{x}) q_i(t) + 2(1-\alpha)\hat{x}^{-\alpha} \times \sum_{i=1}^n \varphi_i'''(\hat{x}) q_i(t) + (1-\alpha)(-\alpha)\hat{x}^{-1-\alpha} \times \sum_{i=1}^n \varphi_i''(\hat{x}) q_i(t)] = S_3 f(\hat{x}, \hat{t}). \tag{37}$$

Multiplying Eq. (37) by φ_i , as a weight function in the Galerkin method, and integrating the outcomes from $x = 0$ to 1, the set of algebraic equations is obtained (Eq. (38)). By solution of these algebraic equations, the deflection of nanobeam can be determined in a form

$$F_i = \sum_{j=1}^n K_{ij} a_j, i = 1, 2, \dots, n, \tag{38}$$

where

$$K_{ij} = S_1 \int_0^1 (\hat{x}^{1-\alpha} \varphi_j'''' + 2(1-\alpha)\hat{x}^{-\alpha} \varphi_j''' + (1-\alpha)(-\alpha)\hat{x}^{-1-\alpha} \varphi_j'') \varphi_i d\hat{x},$$

$$F_i = \int_0^1 S_3 f(x, t) \varphi_i d\hat{x}.$$

3.3.2.3. *Buckling*

The buckling load is obtained from Eq. (28) by setting the transverse force and the inertia term to zero, so the governing equation is

$$S_1[\hat{x}^{1-\alpha} \hat{w}_{xxxx} + 2(1-\alpha)\hat{x}^{-\alpha} \hat{w}_{xxx} + (1-\alpha)(-\alpha)\hat{x}^{-1-\alpha} \hat{w}_{xx}] = S_2 p \hat{w}_{xx}. \tag{39}$$

Next, by assuming the axial force $p = -p_{cr}$ and applying the Galerkin method, the buckling load reads

$$p_{cr} = \frac{K^f}{K},$$

where

$$K^f = S_1 \int_0^1 (\hat{x}^{1-\alpha} \varphi_j'''' + 2(1-\alpha)\hat{x}^{-\alpha} \varphi_j''' + (1-\alpha)(-\alpha)\hat{x}^{-1-\alpha} \varphi_j'') \varphi_i d\hat{x},$$

$$K = S_2 \int_0^1 \varphi_j'' \varphi_i d\hat{x}.$$

3.3.3. *Results*

This part contains two sections. In the first section, to demonstrate the functionality of FEBBT, the non-dimensional frequency of micro C-C beams under axial force has been studied and in the second section, the effects of the fractional and the non-dimensional length scale parameters have been shown on the free vibration, bending and buckling of nanobeams. It is important that the results of Sect. 3.3.3.1 are contrasted with the experimental data [20].

3.3.3.1. *Is the FEBBT is more efficient than the Euler-Bernoulli theory (EBT)?*

In this section, to validate our fractional theory and to show that FEBBT is more appropriate than EBT, the vibration of C-C micro beams under axial force has been studied. A comparison has been done between experimental results [21] and the FEBBT and EBT. In Table I the physical properties of the microbeam have been shown and the value of axial force is 0.0009 N.

Geometrical and physical properties of the micro C-C beams TABLE I

Property	Value
length [μm]	210,310,410,510
width [μm]	100
height [μm]	1.5
Young modulus [GPa]	$E/(1-\nu) = 166$
density [kg/m^3]	$\rho = 2329$

The fractional parameter and the length scale parameter give us the ability to describe the behavior of materials better than the classical EBT. Figure 3 shows the effects of different length scale parameter ($\lambda = l/L$) on the non-dimensional frequency for $L = 210 \mu\text{m}$ when $0 < \alpha \leq 1$. For $\lambda = 0.2$, $\lambda = 0.4$ and $\lambda = 0.6$ decreasing α from 1 to 0.1 causes the increase in the non-dimensional frequency. Table II shows the calculated non-dimensional frequency of different microbeams based on FEBBTs and EBT.

TABLE II

The compression of the non-dimensional frequency of C-C micro beams based on the FEBBT and EBT with experimental data [20] ($p = 0.0009$ N, $\lambda = 1$, error = absolute number of (experimental value – calculated value))

	α (fractional parameter)												Experimental [20]
	0.89	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1(EBT)	
$L = 210$	322.74	322.73	322.73	322.76	322.83	322.95	323.11	323.31	323.55	323.84	324.17	324.55	322.05
error1	0.69	0.68	0.68	0.71	0.78	0.90	1.06	1.26	1.50	1.79	2.12	2.50	-
$L = 310$	162.65	162.65	162.64	162.66	162.69	162.74	162.80	162.89	162.99	163.11	163.25	163.41	163.22
error2	0.57	0.57	0.58	0.56	0.53	0.48	0.42	0.33	0.23	0.11	0.03	0.19	-
$L = 410$	103.41	103.40	103.40	103.40	103.42	103.45	103.48	103.52	103.58	103.64	103.71	103.79	102.17
error3	1.24	1.23	1.23	1.23	1.25	1.28	1.31	1.35	1.41	1.47	1.54	1.62	-
$L = 510$	74.55	74.55	74.55	74.55	74.56	74.58	74.59	74.62	74.65	74.69	74.73	74.77	73.79
error4	0.76	0.76	0.76	0.76	0.77	0.79	0.80	0.83	0.86	0.90	0.94	0.98	-
\sum_1^4 error	3.26	3.24	3.24	3.27	3.34	3.44	3.59	3.77	4.00	4.26	4.62	5.29	-

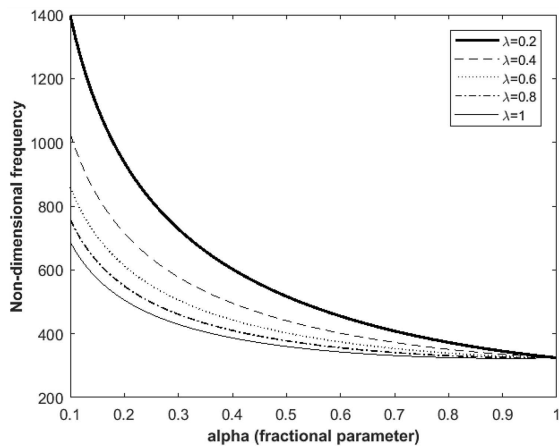


Fig. 3. Effects of fractional parameter α and length scale parameter λ on the non-dimensional frequency ω_1 of fractional Euler–Bernoulli micro C-C nanobeam ($p = 0.0009$ N, $L = 210$ μ m).

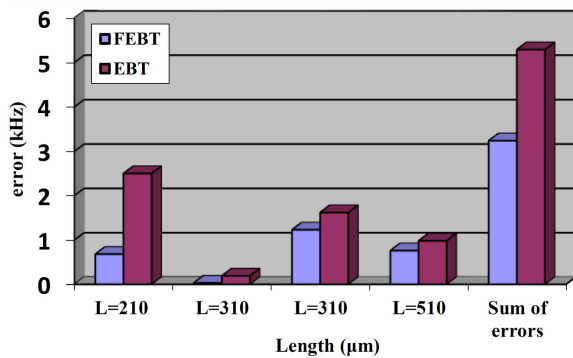


Fig. 4. The compression of calculated the errors of the FEBBT and EBT. Blue columns show error of FEBBT and red columns show the error of EBT.

The errors are the difference between experimentally measured frequencies and the calculated ones. It is visible from Table II that the errors of FEBBT are lower than the errors of EBT. As it can be seen, FEBBT reduces the sum of the error from 5.29 kHz to 3.24 kHz. In addition, FEBBT reduces the error for each lengths, for

instance in $L = 210$ μ m FEBBT reduces the error from 2.5 kHz to 0.68 kHz, for $L = 310$ μ m from 0.19 kHz to 0.03 kHz, for $L = 410$ μ m from 1.62 kHz to 1.23 kHz and for $L = 510$ μ m from 0.98 kHz to 0.76 kHz. In Fig. 4, the error of the calculated non-dimensional frequencies based on FEBBT and EBT has been compared. In Fig. 3 and Table II, the values of errors show that FEBBT could be more efficient than EBT.

3.3.3.2. Numerical results for free vibration, bending, and buckling of nanobeams

In this section, the effects of the fractional parameter α and the length scale parameter λ on the non-dimensional frequency, non-dimensional center deflection and non-dimensional buckling load of a simply supported nano beam have been shown.

Free vibration of nano beam means the axial force is zero ($p = 0$). In Table 3 and in Fig. 5, the effects of the fractional parameter α and length scale parameter λ have been shown on the first, second and third non-dimensional frequency of simply supported beam. As it can be seen from Table III and Fig. 4, decreasing of the fractional parameter from 1 to 0.1 leads to the increase of the first, the second and the third non-dimensional natural frequencies. This increase for ω_3 is larger than for ω_2 and ω_1 and for ω_2 is more than the ω_1 . In addition, it can be seen that at the constant α the increase of λ causes the decrease in the natural frequencies.

The non-dimensional center deflection of the simply supported nanobeam has been shown in Table IV for different values of λ and α , when L/h has different values. It is visible that for different L/h ratios when α decreases from 1 to 0.1 the maximum non-dimensional deflection decreases and this means that decrease of α increases the stiffness of simply supported beams. In Fig. 6, the variation of the center deflection versus α for 5 values of λ has been shown when $L/h = 50$. At the constant value of α , increase of λ causes the increase in the center deflection, so one can be said that it causes the decrease in the beam stiffness, and it can be seen also that the variation of the non-dimensional center deflection increases as the value of λ decreases.

TABLE III

Non-dimensional natural frequency of simply supported nanobeam ($L = 10$ nm, $b = h = 0.1L$, $\rho = 1$, $\lambda = l/L$)

Length scale	Frequency	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$ (classical theory)
$\lambda = 0.2$	ω_1	46.99	31.66	24.64	20.36	17.39	15.16	13.42	12.02	10.85	9.87
	ω_2	187.23	125.73	97.64	80.54	68.72	59.95	53.13	47.67	43.20	39.48
	ω_3	420.86	282.39	219.13	180.65	154.09	134.40	119.15	106.96	97.03	88.83
$\lambda = 0.4$	ω_1	34.40	23.99	19.34	16.54	14.62	13.20	12.10	11.21	10.48	9.87
	ω_2	137.06	95.29	76.60	65.42	57.79	52.19	47.88	44.48	41.72	39.48
	ω_3	308.09	214.01	171.92	146.73	129.57	117.00	107.38	99.80	93.73	88.83
$\lambda = 0.6$	ω_1	28.66	20.40	16.78	14.64	13.21	12.17	11.38	10.77	10.27	9.87
	ω_2	114.20	81.02	66.47	57.92	52.22	48.12	45.06	42.71	40.89	39.48
	ω_3	256.71	181.97	149.18	129.93	117.08	107.89	101.04	95.83	91.85	88.83
$\lambda = 0.8$	ω_1	25.18	18.18	15.17	13.43	12.29	11.49	10.90	10.46	10.12	9.87
	ω_2	100.33	72.22	60.10	53.13	48.59	45.43	43.15	41.50	40.30	39.48
	ω_3	225.53	162.19	134.89	119.18	108.96	101.86	96.78	93.12	90.53	88.83
$\lambda = 1$	ω_1	22.77	16.63	14.03	12.56	11.63	10.99	10.54	10.23	10.01	9.87
	ω_2	90.75	66.05	55.59	49.69	45.96	43.45	41.73	40.58	39.86	39.48
	ω_3	203.99	148.34	124.76	111.47	103.04	97.41	93.59	91.06	89.53	88.83

TABLE IV

Center deflection of simply supported nanobeam subjected to uniform load \bar{f} ($L = 10$ nm, $E = 169$ GPa, $\bar{f} = 10$, $\lambda = l/L$)

L/h	Length scale	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$ (classical theory)
100	$\lambda = 0.2$	0.474	1.011	1.622	2.318	3.114	4.027	5.074	6.276	7.657	9.243
	$\lambda = 0.4$	0.885	1.761	2.635	3.513	4.404	5.313	6.246	7.209	8.207	9.243
	$\lambda = 0.6$	1.274	2.435	3.499	4.481	5.394	6.249	7.054	7.818	8.546	9.243
	$\lambda = 0.8$	1.651	3.065	4.280	5.325	6.229	7.011	7.690	8.281	8.796	9.243
	$\lambda = 1$	2.018	3.664	5.003	6.088	6.964	7.665	8.223	8.659	8.994	9.243
75	$\lambda = 0.2$	0.150	0.320	0.513	0.733	0.985	1.274	1.605	1.986	2.423	2.924
	$\lambda = 0.4$	0.280	0.557	0.834	1.112	1.394	1.681	1.976	2.281	2.597	2.924
	$\lambda = 0.6$	0.403	0.770	1.107	1.418	1.707	1.977	2.232	2.474	2.704	2.924
	$\lambda = 0.8$	0.522	0.970	1.354	1.685	1.971	2.218	2.433	2.620	2.783	2.924
	$\lambda = 1$	0.639	1.159	1.583	1.926	2.203	2.425	2.602	2.740	2.846	2.924
50	$\lambda = 0.2$	0.030	0.063	0.101	0.145	0.195	0.252	0.317	0.392	0.479	0.578
	$\lambda = 0.4$	0.055	0.110	0.165	0.220	0.275	0.332	0.390	0.451	0.513	0.578
	$\lambda = 0.6$	0.080	0.152	0.219	0.280	0.337	0.391	0.441	0.489	0.534	0.578
	$\lambda = 0.8$	0.103	0.192	0.267	0.333	0.389	0.438	0.481	0.518	0.550	0.578
	$\lambda = 1$	0.126	0.229	0.313	0.381	0.435	0.479	0.514	0.541	0.562	0.578
30	$\lambda = 0.2$	0.004	0.008	0.013	0.019	0.025	0.033	0.041	0.051	0.062	0.075
	$\lambda = 0.4$	0.007	0.014	0.021	0.028	0.036	0.043	0.051	0.058	0.066	0.075
	$\lambda = 0.6$	0.010	0.020	0.028	0.036	0.044	0.051	0.057	0.063	0.069	0.075
	$\lambda = 0.8$	0.013	0.025	0.035	0.043	0.050	0.057	0.062	0.067	0.071	0.075
	$\lambda = 1$	0.016	0.030	0.041	0.049	0.056	0.062	0.067	0.070	0.073	0.075

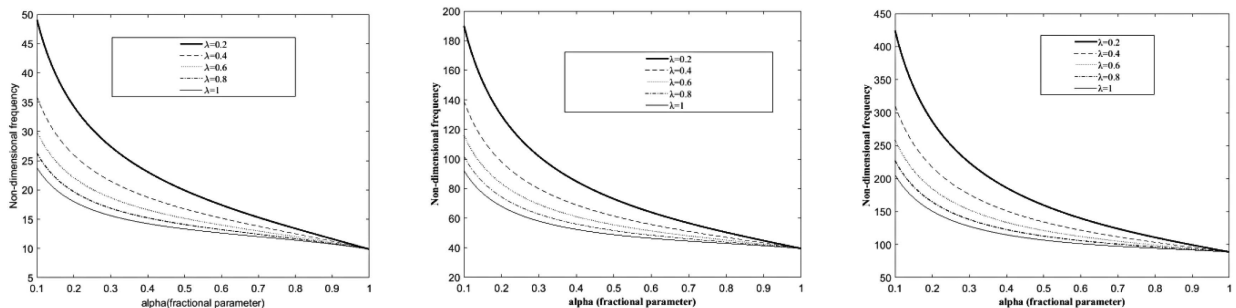


Fig. 5. The effects of the fractional parameter (α) and the length scale parameter (λ) on the non-dimensional frequency of the fractional Euler–Bernoulli simply supported nanobeam. From left: (1) The first non-dimensional frequency (ω_1), (2) the second non-dimensional frequency (ω_2), (3) the third non-dimensional frequency (ω_3).

Non-dimensional critical buckling load ($\bar{p}_{cr} = p_{cr}(\frac{L^2}{EI})$) of simply supported nanobeam ($L = 10$ nm, $b = h = 0.1L, \lambda = l/L$)

TABLE V

Length scale	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$ (classical theory)
$\lambda = 0.1$	417.461	176.797	99.961	63.669	43.316	30.742	22.476	16.802	12.784	9.870
$\lambda = 0.2$	223.712	101.543	61.533	42.006	30.629	23.298	18.256	14.627	11.927	9.870
$\lambda = 0.3$	155.313	73.414	46.328	32.935	25.009	19.810	16.165	13.488	11.454	9.870
$\lambda = 0.4$	119.884	58.321	37.878	27.713	21.658	17.657	14.829	12.734	11.129	9.870
$\lambda = 0.5$	98.072	48.786	32.401	24.241	19.372	16.149	13.868	12.178	10.883	9.870
$\lambda = 0.6$	83.230	42.165	28.518	21.729	17.684	15.013	13.130	11.742	10.687	9.870
$\lambda = 0.7$	72.448	37.273	25.601	19.809	16.372	14.116	12.537	11.385	10.523	9.870
$\lambda = 0.8$	64.244	33.497	23.317	18.284	15.315	13.381	12.045	11.085	10.383	9.870
$\lambda = 0.9$	57.783	30.485	21.471	17.036	14.439	12.766	11.626	10.827	10.262	9.870
$\lambda = 1$	52.555	28.020	19.945	15.993	13.698	12.239	11.265	10.601	10.154	9.870

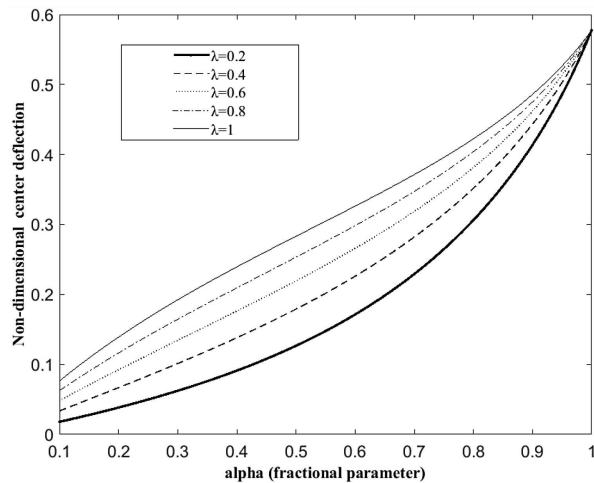


Fig. 6. The effects of the fractional parameter α and the length scale parameter λ on the non-dimensional center deflection of the fractional Euler–Bernoulli simply supported nanobeam ($L/h = 50$).

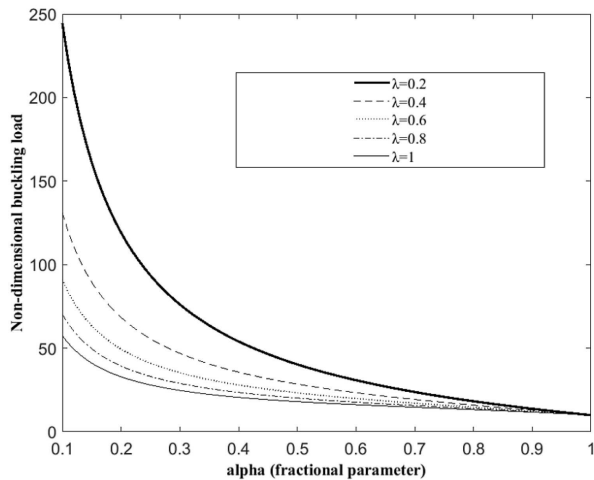


Fig. 7. The effects of the fractional parameter α and the length scale parameter λ on the non-dimensional buckling load of the fractional Euler–Bernoulli simply supported nanobeam ($L/h = 100$).

Finally, in Table V the non-dimensional buckling load has been shown based on FEBBT and EBT when $L/h = 100$. The non-dimensional buckling loads calculated by the EBT are smaller than those one calculated by the FEBBT. From Table V, decrease of α from 1 to 0.1 causes the decrease in the non-dimensional buckling load and when α is constant the increase of λ causes decrease of buckling load. In Fig. 7, the non-dimensional buckling load versus α for different values of λ has been shown. As it can be seen, the variation of the non-dimensional buckling load increases as the value of λ decreases.

4. Conclusion

The general form of the strain and the fractional deformation gradient has been presented based on the modified CFDD. Its formulation has been studied based on two concepts: (1) the geometrical — by using fractional Taylor series expansion method and (2) the continuum mechanics' view.

The presented FEBBT is based on this general form of strain. The FEBBT two additional free parameters appear: the fractional parameter to control the order of the displacement in constitutive relation; and the length scale parameter to consider the size effects in micron and sub-micron scales. To show that that FEBBT can be more efficient than classical EBT the vibration of micro C-C beams under axial force was studied and it was shown that the errors of the calculated non-dimensional frequencies compared to the experimental evidence is smaller for the FEBBT compared to the results obtained by EBT.

Finally, the effects of the fractional parameter and the length scale parameter were studied on the free vibration, bending and buckling of simply supported nanobeam. It appeared that when the length scale parameter is constant, decrease of the fractional parameter from 1 to 0.1 increases both the natural frequencies, the buckling load, and decreases the maximum deflection of nanobeam. On the other hand, when the fractional parameter is constant, the increase of the non-dimensional length scale from 0 to 1 decreases both the natural frequency and the buckling load, and increases center deflection of the nanobeam.

Acknowledgments

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Appendix A

Conformable fractional derivatives for multi-variables function:

Assume the function $f(x, t)$, we have

$$f_x(x, y) = \frac{df(x, y)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h},$$

$$f_y(x, y) = \frac{df(x, y)}{dy} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}. \quad (\text{A1})$$

Based on modified CFDD we have

$$f_x^\alpha(x, y) = \frac{d^\alpha f(x, y)}{dx^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(x + \varepsilon x^{([\alpha]-\alpha)}, y) - f^{([\alpha]-1)}(x, y)}{\varepsilon},$$

$$f_y^\alpha(x, y) = \frac{d^\alpha f(x, y)}{dy^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(x, y + \varepsilon y^{([\alpha]-1)}) - f^{([\alpha]-1)}(x, y)}{\varepsilon}. \quad (\text{A2})$$

If $0 < \alpha \leq 1$, let $h = \varepsilon |x|^{\alpha-1}$, $h = \varepsilon |y|^{\alpha-1}$ then Eq. (A2) is

$$f_x^\alpha(x, y) = \frac{d^\alpha f(x, y)}{dx^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(x + \varepsilon |x|^{([\alpha]-1)}, y) - f^{([\alpha]-1)}(x, y)}{\varepsilon} =$$

$$|x|^{1-\alpha} \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = |x|^{1-\alpha} \frac{df(x, y)}{dx},$$

$$f_y^\alpha(x, y) = \frac{d^\alpha f(x, y)}{dy^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(x, y + \varepsilon |y|^{([\alpha]-1)}) - f^{([\alpha]-1)}(x, y)}{\varepsilon} =$$

$$|y|^{1-\alpha} \lim_{h \rightarrow 0} \frac{f(x, y) - f(x, y+h)}{h} = |y|^{1-\alpha} \frac{df(x, y)}{dy}.$$

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