The Temperature Effect on the RLC_q Circuit

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Starting from the motion equation that corresponds to a RLC_q circuit with source, we discuss the q-deformed internal energy of the circuit by using a fluctuation dissipation theorem. Also, we study the q-deformed heat capacity and q-deformed entropy.

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1. Introduction

An important class of generalized coherent states of harmonic oscillator is provided by the q-deformed coherent states of q-deformed harmonic oscillator. That is related to deformations of the canonical commutation relation or, equivalently, to deformed boson operators. These states play the important role in many branches of physics as quantum optics. A series of articles [1–3] have studied some interpretations and physical properties of independent and time-dependent q-deformed coherent states of the independent and time-dependent q-deformed harmonic oscillator. References [4–13] treat quantum effects and quantum fluctuation of a damped harmonic oscillator, mesoscopic RLC circuit and radiation reaction, by using fluctuation dissipation theorem and correlation (or autocorrelation) function. Also, Refs. [14–20] developed some properties of q-deformed coherent state of q-deformed oscillator, and thus obtain some significant results.

To study the temperature effect on the RLC_q circuit, we propose to study the q-deformed internal energy of RLC_q circuit, using the time-dependent correlation function and the fluctuation-dissipation theorem. After that, we study the q-deformed heat capacity and q-deformed entropy.

The paper is organized as follows. In Sect. 2 we study the time dependent q-deformed harmonic oscillator. The q-deformed mechanical and electrical oscillations are studied in Sect. 3.

2. The time dependent q-deformed harmonic oscillator

The algebraic symmetry of the time dependent q-deformed harmonic oscillator, is defined in terms of q-deformed annihilation and creation operators, $a_q(t)$ and $a_q^+(t)$, as

$$a_q(t)a_q^+(t) - qa_q^+(t)a_q(t) = \phi(N(t)), \tag{1}$$

where $\phi(N(t)) = 1$ for the "M-type" (Maths) q-deformed bosons, and $\phi(N(t)) = q^{-N(t)}$ for the "P-type" (Physics) q-deformed bosons, knowing that $N(t) = a^+(t)\, a(t)$.

In this paper, we consider the cases where the deformation parameter q is real. The basic q-deformed number is then defined as the "asymmetric q-deformed number" $[n]_q = \frac{1-q^n}{1-q}$ for the "M-type", and as the "symmetric q-deformed number" $[n]_q = \frac{q^{-n}-q^n}{q^{-1}-q}$ for the "P-type".

In both cases (M-type and P-type) we recover the natural numbers (and natural bosons) as for $q \to 1$, we have $[n]_q \to n$.

The Fock states, at time t, are spanned by the orthornormalized eigenstates $\{|n,t\rangle, n = 0,1,2,3,\ldots\}$.

First, we define the vacuum state, at time t, as the state which is annihilated by the annihilation operator, at time t:

$$a_q(t)\,|0,t\rangle = 0. \tag{2}$$

Then, we act on this state using the creation operator, at time t:

$$|n,t\rangle = \frac{1}{\sqrt{[n]_q!}}\left(a_q^+(t)\right)^n |0,t\rangle, \tag{3}$$

with the q-factorial defined by

$$[n]_q! = [n]_q\,[n-1]_q\,[n-2]_q\ldots[1]_q,\ \ [0]_q! = 1. \tag{4}$$

The actions of $a_q(t)$, $a_q^+(t)$ and $N(t)$ are given by

$$a_q(t)\,|n,t\rangle = \sqrt{[n]_q}\,|n-1,t\rangle,$$
$$a_q^+(t)\,|n,t\rangle = \sqrt{[n+1]_q}\,|n+1,t\rangle,$$
$$N(t)\,|n,t\rangle = n\,|n,t\rangle. \tag{5}$$

We also have the following algebraic equalities:

$$a_q(t)\,a_q^+(t) = [N(t)+1]_q,$$
$$a_q^+(t)\,a_q(t) = [N(t)]_q, \tag{6}$$

To analyze the dynamics of the time dependent q-deformed harmonic oscillator, we use the time independent q-deformed position (X_q) and the momentum (P_q)

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operators related to the time dependent q-deformed boson operators $a_q(t)$ and $a_q^+(t)$ as follows:

$$a_q(t) = \frac{1}{\sqrt{2}}\sqrt{\frac{m(t)\omega(t)}{\hbar}}X_q + \mathrm{i}\sqrt{\frac{1}{\hbar m(t)\omega(t)}}P_q, \tag{7}$$

$$a_q^+(t) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m(t)\omega(t)}{\hbar}}X_q - \mathrm{i}\sqrt{\frac{1}{\hbar m(t)\omega(t)}}P_q\right), \tag{8}$$

where $\hbar = \frac{h}{2\pi}$, h is the Planck constant, $m(t) = m\mathrm{e}^{2\beta t}$ (m is the initial mass of the oscillator, $\beta > 0$ is a damping constant) and $\omega(t) = \omega = \left(\frac{k}{m}\right)^{\frac{1}{2}}$ (k is an elastic coefficient) are, respectively, the time dependent mass of the oscillating system and the time dependent frequency.

Then Eqs. (7), (8) become

$$a_q(t) \equiv a_q(\beta,t) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\mathrm{e}^{\beta t}X_q + \mathrm{i}\sqrt{\frac{1}{\hbar m\omega}}\mathrm{e}^{-\beta t}P_q\right), \tag{9}$$

$$a_q^+(t) \equiv a_q^+(\beta,t) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\mathrm{e}^{\beta t}X_q - \mathrm{i}\sqrt{\frac{1}{\hbar m\omega}}\mathrm{e}^{-\beta t}P_q\right). \tag{10}$$

The dynamics of the time dependent q-deformed harmonic oscillator is governed by the (q-deformed) Hamiltonian $H_q(t)$, that is constructed in analogy with the Hamiltonian of the harmonic oscillator

$$H_q(t) = \frac{P_q^2}{2m(t)} + \frac{1}{2}m(t)\,\omega^2(t)X_q^2. \tag{11}$$

Using the definitions of $m(t)$ and $\omega(t)$, one gets

$$H_q(t) = \frac{\mathrm{e}^{-2\beta t}P_q^2}{2m} + \frac{1}{2}m\omega^2\,\mathrm{e}^{2\beta t}X_q^2. \tag{12}$$

The time dependent q-deformed coherent state $|z,\beta,t\rangle_q$ is governed by the spectrum E_{nq} of the time independent q-deformed harmonic oscillator

$$|z,\beta,t\rangle_q = \mathcal{N}_q\left(|z|^2\right)\sum_{n=0}^{\infty}\frac{z^n\,\mathrm{e}^{\frac{-iE_{nq}t}{\hbar}}}{\sqrt{[n]_q!}}|n,t\rangle = \mathcal{N}_q\left(|z|^2\right)\sum_{n=0}^{\infty}\frac{z^n\,\mathrm{e}^{\frac{-iE_{nq}t}{\hbar}}\left(a_q^+(\beta,t)\right)^n}{[n]_q!}|0,t\rangle, \tag{13}$$

where $z = |z|\,\mathrm{e}^{\mathrm{i}\varphi}$ and the normalization constant $\mathcal{N}_q\left(|z|^2\right)$ is given by the relation

$$\mathcal{N}_q\left(|z|^2\right) = \left(\sum_{n=0}^{\infty}\frac{|z|^{2n}}{[n]_q!}\right)^{-\frac{1}{2}} = \left(e_q^{|z|^2}\right)^{-\frac{1}{2}}, \tag{14}$$

and the eigenvalues E_{nq} of the time independent q-deformed harmonic oscillator are given by

$$E_{nq} = \frac{\hbar\omega}{2}\left([n+1]_q + [n]_q\right). \tag{15}$$

By construction, the time dependent q-deformed coherent states $|z,\beta,t\rangle_q$ are right eigenstates of the lowering operator $a_q(t)$:

$$a_q(t)\,|z,\beta,t\rangle_q = z\,|z,\beta,t\rangle_q\,. \tag{16}$$

3. q-deformed mechanical and electrical oscillations

3.1. q-deformed mechanical oscillations

The mean value of the time independent q-deformed position operator on the time dependent q-deformed harmonic oscillator states is:

$$x_q(t) = {}_q\langle z,\beta,t|\,X_q\,|z,\beta,t\rangle_q = 2\sqrt{\frac{\hbar}{2m\omega}}\,\mathrm{e}^{-\beta t}\left(e_q^{|z|^2}\right)^{-1}\sum_{n=0}^{\infty}\frac{|z|^{2n+1}}{[n]_q!} \times\cos\left(\frac{\omega t}{2}([n+2]_q - [n]_q) - \varphi\right). \tag{17}$$

This position function can be seen as a particular solution of the following differential equation defining a q-deformed damped and forced time-dependent harmonic oscillator:

$$\frac{\mathrm{d}^2x_q(t)}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}x_q(t)}{\mathrm{d}t} + \omega_q^2x_q(t) = g_q(t), \tag{18}$$

where

$$\omega_q^2 = \omega^2\frac{(1+q)^2}{4} \tag{19}$$

is the frequency (the resonance frequency) of the oscillations, and

$$g_q(t) = 2\sqrt{\frac{\hbar}{2m\omega}}\,\mathrm{e}^{-\beta t}\left(e_q^{|z|^2}\right)^{-1}\sum_{n=0}^{\infty}\frac{|z|^{2n+1}}{[n]_q!}\left(\omega_q^2 - \frac{\omega^2}{4}([n+2]_q - [n]_q)^2 - \beta^2\right) \times\cos\left(\frac{\omega t}{2}([n+2]_q - [n]_q) - \varphi\right) \tag{20}$$

is the external force of angular frequency $\frac{\omega}{2}([n+2]_q - [n]_q)$.

This allows to interpret the q-deformed harmonic oscillators defined in (1) as being the quantized versions of a classical damped and forced oscillator described by the classical differential equation (18) with the proper choice of the box function $[\ ]_q$.

On the other hand, and as expected, in the case $q \to 1$ the differential equation (18) becomes the usual differential equation of a damped and forced oscillator

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega^2x(t) = g(t).$$

3.2. q-deformed electrical oscillations

Using the analogy between q-deformed mechanical and electrical phenomena ($x_q(t) \to Q_q(t)$, $m \to L$, $k \to \frac{1}{C}$, $2\beta \to \frac{R}{L}$), Eq. (17) becomes

$$Q_q(t) = 2\sqrt{\frac{\hbar}{2L\omega}}\,\mathrm{e}^{\frac{-Rt}{2L}}\left(e_q^{|z|^2}\right)^{-1}\sum_{n=0}^{\infty}\frac{|z|^{2n+1}}{[n]_q!}\cos\left(\frac{\omega t}{2}([n+2]_q - [n]_q) - \varphi\right), \tag{21}$$

where $Q_q(t)$ is the electric charge in the circuit, L, R, and C stand for inductance, resistance, and capacity, respectively. As in (18), this is a particular solution of a damped and forced differential equation for an RLC_q circuit with a power source

$$\frac{\mathrm{d}^2Q_q(t)}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}Q_q(t)}{\mathrm{d}t}+\omega_q^2Q_q(t)=\epsilon_q(t)=\frac{1}{L}\widehat{\epsilon}_q(t)\,, \tag{22}$$

where

$$\epsilon_q(t)=2\sqrt{\frac{\hbar}{2L\omega}}\,\mathrm{e}^{\frac{-Rt}{2L}}\left(e_q^{|z|^2}\right)^{-1}\sum_{n=0}^{\infty}\frac{|z|^{2n+1}}{[n]_q!}\left(\omega_q^2-\frac{\omega^2}{4}([n+2]_q-[n]_q)^2-\left(\frac{R}{2L}\right)^2\right)\times\cos\left(\frac{\omega t}{2}([n+2]_q-[n]_q)-\varphi\right) \tag{23}$$

is the electromotive force (source of the RLC_q electric circuit).

It is worth noting that because the natural frequency got modified and becomes q-dependent (as in (19)), the capacitance is q-dependent, too

$$\omega_q^2=\frac{1}{LC_q}=\omega^2\frac{(1+q)^2}{4}, \tag{24}$$

$$C_q=C\frac{4}{(1+q)^2}. \tag{25}$$

For $q\to1$ we have $\omega_q\to\omega$ and $C_q\to C$; so one can state that the effect of "q-deforming" the harmonic oscillator (1) is to modify (deform) the resonance frequency (24), the capacitance (25) and the power source (23).

From (22) and (21), we see that the variation of the electric charges is accompanied by the following four sorts of energy changes [2, 4, 5]:

(i) The capacity energy,

$$E_{Cq}(t)=\frac{(Q_q(t))^2}{2C_q}. \tag{26}$$

(ii) The inductance energy,

$$E_{Lq}(t)=\frac{1}{2}L\left(\frac{\mathrm{d}Q_q(t)}{\mathrm{d}t}\right)^2. \tag{27}$$

(iii) The loss of energy caused by the resistance,

$$E_{Rq}(t)=\int_0^t R\left(\frac{\mathrm{d}Q_q(t')}{\mathrm{d}t'}\right)^2\mathrm{d}t'. \tag{28}$$

(iv) The energy supplied by the source $E_{gq}(t)$,

$$E_{gq}(t)=\int_0^t\epsilon_q(t')\frac{\mathrm{d}Q_q(t')}{\mathrm{d}t'}\mathrm{d}t'. \tag{29}$$

Therefore, the total energy change of the system can be written as

$$E_{Tq}(t)=E_{Cq}(t)+E_{Lq}(t)+E_{Rq}(t)=E_{gq}(t)\,. \tag{30}$$

Accordingly, the variation of the energy can be obtained through the equation

$$\Delta E_q(t)=\frac{(Q_q(t))^2}{2C_q}+RQ_q(t)\frac{\mathrm{d}Q_q(t)}{\mathrm{d}t}+\frac{L}{2}\left(\frac{\mathrm{d}Q_q(t)}{\mathrm{d}t}\right)^2-\epsilon_q(t)Q_q(t)\,. \tag{31}$$

The thermal expectation value of the energy change in the circuit is

$$\langle\Delta E_q(t)\rangle=\frac{1}{2C_q}\left\langle(Q_q(t))^2\right\rangle+R\left\langle Q_q(t)\frac{\mathrm{d}Q_q(t)}{\mathrm{d}t}\right\rangle+\frac{L}{2}\left\langle\left(\frac{\mathrm{d}Q_q(t)}{\mathrm{d}t}\right)^2\right\rangle-\epsilon_q(t)\langle Q_q(t)\rangle\,. \tag{32}$$

To calculate $\langle\Delta E_q(t)\rangle$, we introduce a time-dependent correlation function $\psi_q(t-t')$:

$$\psi_q(t-t')=\frac{1}{2}\langle Q_q(t)Q_q(t')+Q_q(t')Q_q(t)\rangle\,. \tag{33}$$

From Eq. (33), we know that

$$\left\langle(Q_q(t))^2\right\rangle=\psi_q(t-t')\,|_{t=t'}, \tag{34}$$

$$\left\langle\left(\frac{\mathrm{d}Q_q(t)}{\mathrm{d}t}\right)^2\right\rangle=\frac{\partial^2\psi_q(t-t')}{\partial t^2}\,|_{t=t'}, \tag{35}$$

$$\left\langle Q_q(t)\frac{\mathrm{d}Q_q(t)}{\mathrm{d}t}\right\rangle=\frac{\partial\psi_q(t-t')}{\partial t}\,|_{t=t'}\,. \tag{36}$$

We assume that the RLC_q circuit is in equilibrium when the power becomes zero, so

$$\langle Q_q(t)\rangle=0. \tag{37}$$

The fluctuation-dissipation theorem gives [4, 7]:

$$\psi_q(t-t')=\frac{\hbar}{\pi}\int_0^{+\infty}\mathrm{d}\Omega_q\coth\left(\frac{\hbar\Omega_q}{2k_\mathrm{B}T}\right)\times\mathrm{Im}\alpha(\Omega_q)\,\mathrm{e}^{\mathrm{i}\,\Omega_q(t-t')}, \tag{38}$$

where k_B is the Boltzmann constant, T is the absolute temperature, and

$$\Omega_q^2=\omega_q^2-\frac{R^2}{4L^2}=\omega_q^2-\beta^2, \tag{39}$$

as definition, we add that $\alpha(\Omega_q)$ is called the generalized susceptibility in the RLC_q circuit,

$$\alpha(\Omega_q)=\frac{1}{-L\Omega_q^2+\frac{1}{C_q}+iR\Omega_q}. \tag{40}$$

Substituting Eq. (38) into Eqs. (34)–(37), we obtain the q-deformed internal energy $U_q(T)$:

$$U_q=\langle\Delta E_q(t)\rangle=\frac{\hbar}{2\pi}\int_0^{+\infty}\mathrm{d}\Omega_q(-L\Omega_q^2+\frac{1}{C_q}+2\mathrm{i}R\Omega_q)\times\coth\left(\frac{\hbar\Omega_q}{2k_\mathrm{B}T}\right)\mathrm{Im}\alpha(\Omega_q). \tag{41}$$

U_q as a function of a absolute temperature T is sketched in Fig. 1.

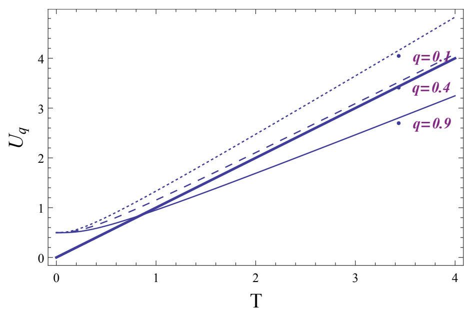

Fig. 1. U_q as a function of T with $\hbar=\omega=L=C=k_\mathrm{B}=1$, $\beta=0.5$ and $z=1$ for M-type.

In Fig. 1, we display the average energy of a RLC_q circuit in thermal equilibrium, as a function of temperature for $q = 0.1$, $q = 0.4$ and $q = 0.9$. The bisectrix (thick line) is merely a reference line, to clarify the asymptotic behavior, which is important to understand RLC_q circuit, because many things that we see around us can be modeled as RLC_q circuit or collections of RLC_q circuit.

We can learn a lot from this figure. We start by considering the limiting cases. At high temperatures, the q-deformed internal energy of the RLC_q circuit is proportional to T, which is the classical result, as expected. Meanwhile, at low temperatures, the q-deformed internal energy is asymptotically U_{0q} at $T = 0$, which is the celebrated zero-point energy associated with quantum fluctuations.

In addition, this result indicates that the movement of the charge in the circuit at high temperature is classical since the quantum fluctuation is dominated by the classical thermodynamic fluctuation. Therefore at low temperature, the movement of the charge in the circuit is purely a quantum effect and the origin of the quantum phenomenon can be attributed to the fluctuations of zero point vibrations of the charge.

The increase of the deformation parameter q and close to the limit value $q = 1$ (undeformed case) favors the decrease of the q-deformed internal energy (Fig. 1).

We also calculate the q-deformed heat capacity of RLC_q circuit as

$$C_{Vq} = \left(\frac{\partial U_q}{\partial T}\right)_V . \tag{42}$$

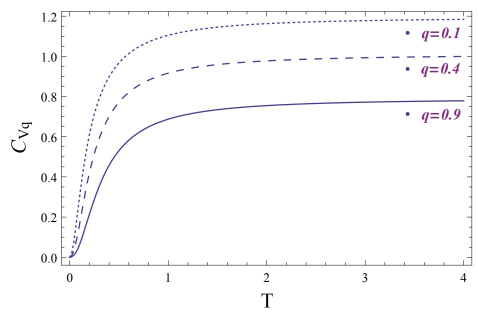

Fig. 2. C_{Vq} as a function of T with $\hbar = \omega = L = C = k_B = 1$, $\beta = 0.5$ and $z = 1$ for M-type.

In the q-deformed heat capacity C_{Vq} as a function of the temperature T for the RLC_q circuit (Fig. 2), we notice that as the temperature increases, the q-deformed heart capacity approaches to a constant value. It reduces to the classical rule at high temperatures.

Similarly, the q-deformed entropy is calculated as

$$S_q = \int_0^T \frac{\mathrm{d}U_q}{T}. \tag{43}$$

In the q-deformed entropy S_q as a function of the temperature T for the RLC_q ciruit (Fig. 3), we notice that when $T \to 0$, we obtain $S_q \to 0$ obeying the third law of thermodynamics. Also, when T increases, S_q also in-

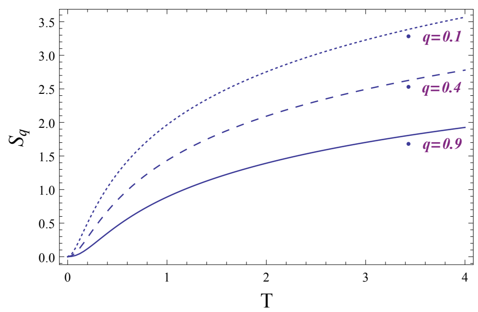

Fig. 3. S_q as a function of T with $\hbar = \omega = L = C = k_B = 1$, $\beta = 0.5$ and $z = 1$ for M-type.

creases in agreement with the second law of thermodynamics.

The integrals U_q, C_{Vq} and S_q cannot be evaluated analytically. An adequate numerical procedure based on the IMT-Legendre quadrature in conjunction with a change of variables of the integration interval is detailed in the appendix [21].

4. Conclusion

In this paper, we have studied the q-deformed internal energy of RLC_q circuit, the q-deformed heat capacity and q-deformed entropy of RLC_q circuit. In addition, we used the time-dependent correlation function and the fluctuation-dissipation theorem for RLC_q circuit. Consequently, we have found at high temperature, the energy U_q is proportional to T, which agrees with the classical result. Moreover, at low temperature, the energy U_q is asymptotically just U_{0q} at $T = 0$, which is the celebrated zero-point energy associated with quantum fluctuations. The result heat capacity approaches to the classical result for high temperatures and goes to zero for vanishing temperature. The entropies of both systems also obey to the second law of thermodynamics as well as the third law of thermodynamics.

The increasing behaviour of the deformation parameter q and close to the limit value $q = 1$ (undeformed case) favors the decrease of the q-deformed internal energy (Fig. 1), the q-deformed heat capacity (Fig. 2) and the q-deformed entropy (Fig. 3).

Appendix: Numerical evaluation of integrals

Numerical evaluation procedure of the integral U_q is described here. The susbstitution $\Omega_q = \exp\left(1 - \frac{1}{t}\right)$ changes the interval $0 \le \Omega_q < \infty$ into the interval $0 \le t \le 1$ so that

$$\int_0^{+\infty} F(\Omega_q)\,\mathrm{d}\Omega_q = \int_0^1 F\left(\exp\left(1 - \tfrac{1}{t}\right)\right) \frac{\exp(1 - \frac{1}{t})}{t^2}\,\mathrm{d}t,$$

where $F(\Omega_q) = \frac{\hbar}{2\pi}(-L\Omega_q^2 + \frac{1}{C_q} + 2\,\mathrm{i}\,R\Omega_q)\,\mathrm{Im}(\alpha(\Omega_q))$, $\alpha(\Omega_q) = \frac{1}{-L\Omega_q^2 + \frac{1}{C_q} + \mathrm{i}\,R\Omega_q}$. Then, we apply the IMT [21] transformation which is based upon the idea of transforming the independent variable in such a way that all the derivatives of the new integrand vanish at both end points of the integration interval. This has the effect of removing the singularity at the end point $t = 0$. Let

$$\phi_0(t) = \exp\left(-\frac{1}{t} - \frac{1}{1-t}\right),$$
$$\psi_0(x) = \frac{1}{K}\int_0^x \phi_0(t)\,\mathrm{d}t,$$
$$K = \int_0^1 \phi_0(t)\,\mathrm{d}t \simeq 0.00702985840.$$

The function $\psi_0(x)$ is monotonously increasing, performing a one-one transformation of [0, 1] onto itself. Consequently,

$$\int_0^{+\infty} F(\Omega_q)\,\mathrm{d}\Omega_q = \frac{1}{K}\int_0^1 F\left(\exp\left(1-\tfrac{1}{\psi_0(t)}\right)\right)\frac{\exp(1-\frac{1}{\psi_0(t)})}{\psi_0^2(t)}\phi_0(t)\,\mathrm{d}t.$$

Applying the Gauss–Legendre quadrature to this integral, we obtain the following expression:

$$\int_0^{+\infty} F(\Omega_q)\,\mathrm{d}\Omega_q \simeq \frac{1}{2K}\sum_{i=1}^{n-1} F\left(\exp\left(1-\frac{1}{\psi_0(\frac{x_i+1}{2})}\right)\right) \times w_i \frac{\exp\left(1-\frac{1}{\psi_0(\frac{x_i+1}{2})}\right)}{\psi_0^2(\frac{x_i+1}{2})}\phi_0(\frac{x_i+1}{2}),$$

where $w_i = \frac{2}{(1-x_i^2)[P_n'(x_i)]^2}$, and x_i are n zeros of the n-th-degree Legendre polynomial $P_n(x)$.

To calculate the isochoric thermal capacity C_{Vq}, we proceeded differently. As suggested by Squire [22], we split the range $[0, +\infty)$ into two intervals $[0, \beta_0]$ and $[\beta_0, +\infty)$ and set $t = \Omega_q/\beta_0$ in the first interval, $t = \beta_0/\Omega_q$ in the second. This gives

$$\int_0^{+\infty} G(\Omega_q)\,\mathrm{d}\Omega_q = \beta_0\int_0^1\left[G(\beta_0 t) + \frac{1}{t^2}G(\beta_0/t)\right]\mathrm{d}t,$$

where $G = \frac{\Omega_q\hbar^2\,\mathrm{Im}(\alpha(\Omega_q))\left(\frac{1}{C_q} - L\Omega_q^2 + 2\,\mathrm{i}\,R\Omega_q\right)\mathrm{csch}^2\left(\frac{\Omega_q\hbar}{2k_BT}\right)}{4\pi k_B T^2}$, for C_{Vq}. The β_0 is chosen equal to the value 800. The integral over the interval [0, 1] is evaluated efficiently by the IMT-Legendre rule

$$\int_0^{+\infty} G(\Omega_q)\,\mathrm{d}\Omega_q = \frac{\beta_0}{2K}\sum_{i=1}^{n} w_i\phi_0\left(\frac{x_i+1}{2}\right) \times\left[G\left(\beta_0\psi_0\left(\frac{x_i+1}{2}\right)\right) + \frac{G\left(\beta_0/\psi_0\left(\frac{x_i+1}{2}\right)\right)}{\psi_0^2\left(\frac{x_i+1}{2}\right)}\right].$$

Finally, the application of the IMT-Legendre quadrature allows us to obtain the numerical value of the q-deformed entropy S_q as follows:

$$S_q = \frac{1}{2K}\sum_{i=1}^{n} w_i\phi_0\left(\frac{x_i+1}{2}\right)\frac{C_{Vq}(T\psi_0\left(\frac{x_i+1}{2}\right))}{\psi_0\left(\frac{x_i+1}{2}\right)}.$$

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