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Torsion in Cohomology Groups of Configuration Spaces

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An important and surprising discovery in physics in the last fifty years is that if quantum particles are constrained to move in two rather than three dimensions, they can in principle exhibit new forms of quantum statistics, called anyons. Although anyons were initially only a theoretical concept, they quickly proved to be useful in explaining one of the most significant discoveries of condensed matter physics in the last century, i.e. fractional quantum Hall effect. Recently, it was shown that particles constrained to move on a graph can exhibit even more exotic forms of quantum statistics, depending on the topology of the graph. In this paper we discuss what possible new quantum signatures of topology may arise when one takes into account more complex topological information, called higher (co)homology groups, which may also be associated with graph configuration spaces. In particular we focus on the significance of a torsion component.

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1. Introduction

Since the quantum field theoretic proofs of the *Spin-Statistics Theorem* by Fierz [1], Pauli [2], and Schwinger [3] (see also [4]), the major challenge in mathematical physics was to derive this fundamental principle within the framework of ordinary quantum mechanics [5–8]. The inevitable step in such a derivation needs to explain the existence of only bosons and fermions in the three-dimensional space. As was noticed by Souriau [9] and then later rederived in a more systematic way by Leinaas and Myrheim in 1977 [10] (see also [11, 12]) this can be understood using topological properties of configuration spaces $C_n(\mathbb{R}^m)$. In fact, the authors of [10] not only gave topological arguments supporting the existence of bosons and fermions in $\mathbb{R}^3$ but also discovered that for particles restricted to move in two dimensions a new form of quantum statistics, called *anyons*, can appear. Several years later, anyons turned out to be crucial concept in condense matter physics [13].

Recently the idea of Leinaas and Myrheim has been explored in the setting of graphs [14]. In particular, the authors of [14] showed that particles constrained to move on a graph can exhibit even more exotic forms of anyons. In fact, this previous work was based on the simplest topological information about graph configuration spaces, i.e. the first homology group. The main goal of this paper is to discuss possible implications of more complex topological properties of these spaces, viz. higher (co)homology groups. Although the description of these groups attracted much attention in the mathematical community in recent years [15–22], only the first homology group has been fully characterised. Describing the remaining groups is an open problem. The authors of [14, 23] proposed a method for calculation of homology groups that combines tools from algebraic topology and graph theory with a set of combinatorial relations derived from the analysis of certain small canonical graphs. The main advantage of this approach is that it allows to easily predict the results of many complex calculations. The method recovers all the known results concerning (co)homology groups of graph configuration spaces [14, 23]. Its application to higher (co)homology groups, however, still requires considerable development.

2. Configuration spaces

We will consider two types of configuration spaces of n points living in a topological space X. If the considered points are distinguishable the configuration space will be denoted by $F_n(X)$ and for indistinguishable by $C_n(X)$. They are defined as follows:

$$F_n(X) := X^{\times n} - \Delta_n,\ C_n(X) := (X^{\times n} - \Delta_n)/S_n,$$

where $\Delta_n = \{(x_1, \ldots, x_n) \in X^{\times n} :\ \exists_{i\neq j}\ x_i = x_j\}$ will be called the diagonal and S_n is the permutation group acting naturally on $X^{\times n}$. Thus, configurations in which one or more points coincide are excluded. The notion of configuration space of n points appears naturally in some important contemporary problems of quantum physics and engineering. Strikingly, problems as different as description of quantum statistics for indistinguishable particles or collision-free motion planning for robots [21] can be attacked in a similar way if considered from configuration spaces perspective. Of particular importance for the description of quantum statistics for indistinguishable particles is topology of the underlying configuration space. The central role is played by the fundamental group $\pi_1(C_n(X))$ also called the braid group of space X and denoted by B_nX.

In this paper, we will be interested mostly in the case when $X = \Gamma$ is a graph viewed as a one-dimensional CW-complex. Although Γ has a structure of CW complex the corresponding configuration spaces $F_n(\Gamma)$ and $C_n(\Gamma)$ are not CW complexes. The examples of spaces $F_n(\Gamma)$ and $C_n(\Gamma)$ for $n = 2$ are given in Fig. 1.

3. Quantum statistics

When quantum mechanics is applied to particles moving in a three-dimensional world, the fact that they

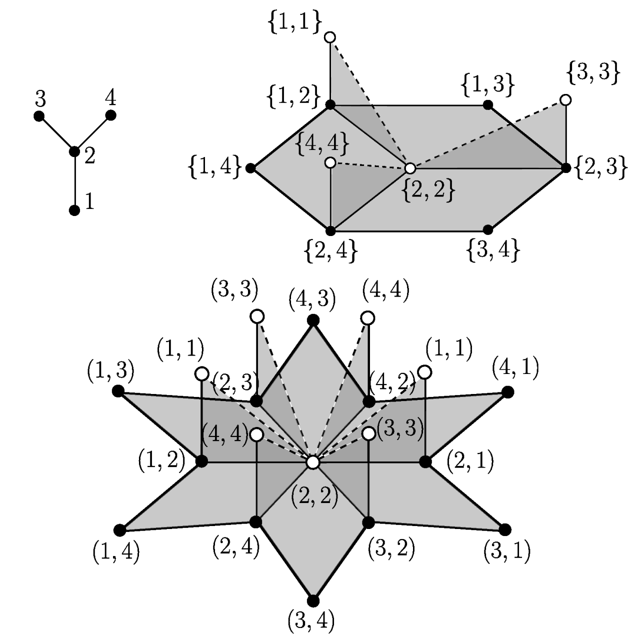

Fig. 1. Configuration spaces $C_2(Y)$ and $F_2(Y)$ for the Y-graph. Dashed lines denote Δ_2.

are indistinguishable vastly restricts their possible collective behaviour to just two types, namely *fermions* and *bosons* (and which of these types applies is determined by quantum spin). The two alternatives taken together — fermions or bosons — are called *quantum statistics*. The authors of [10] made the crucial observation that if particles are meant to be indistinguishable on the quantum level they should be so classically. Treating particles as indistinguishable already on the classical level means that we quantise $C_n(X)$ rather than $X^{\times n}$. Importantly, the space $C_n(X)$ is topologically different from $X^{\times n}$, as the diagonal Δ_n, which describes collisions between particles, is excluded. Note also that loops in $C_n(X)$ describe all possible continuous particle exchanges, as configurations that differ by permutations of particles, are identified in $C_n(X)$. If $\pi_1(C_n(X))$ is nontrivial, there is an additional freedom stemming from nontrivial zero-curvature connections in defining momentum operators that satisfy canonical commutation relations. A process of particle exchange corresponds to the parallel transport of the wave function along the closed loop in $C_n(X)$. For scalar particles, the parallel transport along a loop can result with a multiplication by a phase factor which is a manifestation of $\pi_1(C_n(X))$ being nontrivial. The phase factors add in a way that is consistent with the composition of loops, i.e. if γ_1 and γ_2 are two loops sharing the same base point, then

$$\hat{T}_{\gamma_1+\gamma_2}\Psi = e^{i\phi(\gamma_1)+i\phi(\gamma_2)}\Psi, \quad \hat{T}_\gamma \Psi = \Psi$$

for a contractible loop, where $\hat{T}_\gamma$ is the parallel transport (holonomy) operator along a loop $\gamma \subset C_n(X)$. Abelian quantum statistics is described by abelianization of the fundamental group, which is the first homology group over integers $H_1(C_n(X),\mathbb{Z})$.

Quantum statistics in $\mathbb{R}^n$, $n \geq 2$. When $X = \mathbb{R}^3$, we have $H_1(C_n(\mathbb{R}^3)) = \mathbb{Z}_2$ [11, 12], which means that in $\mathbb{R}^3$ there are only two possible abelian statistics. The fermionic representation yields exchange phase $\phi = \pi$, and the bosonic representation corresponds to $\phi = 0$. For $X = \mathbb{R}^2$, $H_1(C_n(\mathbb{R}^2)) = \mathbb{Z}$, hence there are infinitely many abelian statistics (see Fig. 2 below for explanation in case of two particles). Particles that have this kind of statistics are called *anyons* and their existence was confirmed experimentally in fractional quantum Hall effect [13].

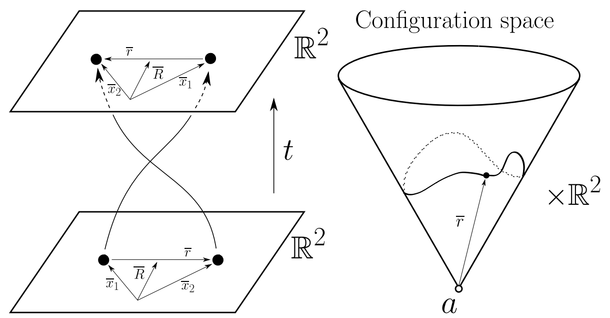

Fig. 2. Derivation of $H_1(C_2(\Gamma),\mathbb{Z}) = \mathbb{Z}$. Consider two particles in $\mathbb{R}^2$. Let $\overline{x}_1$ and $\overline{x}_2$ be positions of particles, $\overline{R}$ — the center of mass coordinate, and $\overline{r}$ — the relative position coordinate. The space-time diagram (top figure) shows exchange of particles. The action of permutation $(1,2) \in S_2$ on $(\overline{r},\overline{R})$ gives $(-\overline{r},\overline{R})$. Thus the configuration spaces $C_2(\mathbb{R}^2) = \left(\mathbb{R}^2 \times \mathbb{R}^2 - \Delta_2\right)/S_2$ in coordinates $(\overline{r},\overline{R})$ (see bottom figure) reduces to the Cartesian product of a cone without apex a and $\mathbb{R}^2$ ($\Delta_2 = a \times \mathbb{R}^2$). The exchange process described by the space-time diagram is represented by a closed loop in the configuration space. This loop is not contractile and it can be given any phase factor $e^{i\phi}$.

3.1. Quantum statistics on graphs and the first homology group, $H_1(C_n(X),\mathbb{Z})$

One cannot extend this approach to $\mathbb{R}$, since there is no possibility to exchange particles on a line without collisions. However, exchanges of particles can be made possible by placing them on a one-dimensional network, i.e. on a graph. The approach taken in [14] enabled identification of the key topological determinants of the quantum statistics on graphs.

The key topological determinants of the quantum statistics on graphs

$$H_1(C_n(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A.$$

- A determines quantum statistics. $\beta_1(\Gamma)$ is the number of independent cycles in Γ, which may be associated with one-particle Aharonov–Bohm phases, and does not contribute to quantum statistics.

- For 1-connected graphs, the number of phases depends on the number of particles, i.e. A depends on n.
- For 2-connected graphs, quantum statistics stabilizes with respect to the number of particles; i.e., $H_1(C_n(\Gamma)) = H_1(C_2(\Gamma))$.
- For 3-connected graphs (which are also 2-connected), $A = \mathbb{Z}_2$ for nonplanar graphs, i.e. the usual bosonic/fermionic statistics is the only possibility, whereas $A = \mathbb{Z}$ for planar graphs, which therefore support a single anyon phase.
- From the quantum statistics perspective 3-connected graphs mimic $\mathbb{R}^2$ when they are planar and $\mathbb{R}^3$ when not.

3.2. Torsion of $H^2(C_n(\Gamma), \mathbb{Z})$ and Bose/Fermi statistics

It is known that different topological phases of matter correspond to nonisomorphic complex vector bundles over particularly defined spaces [24]. In Fig. 3 we show the construction of the complex line bundle describing two fermions on $\mathbb{R}^3$.

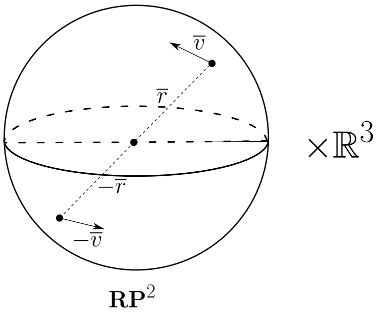

Fig. 3. Complex line bundle describing two fermions in $\mathbb{R}^3$. The configuration space is $C_2(\mathbb{R}^3) \cong \mathbf{RP^2} \times \mathbb{R}^\mathbf{3}$. To see this, denote by $\overline{x}_1$ and $\overline{x}_2$ the positions of particles, $\overline{R} := \overline{x}_1 + \overline{x}_2$ — the center of mass coordinate, and $\overline{r} := \overline{x}_2 - \overline{x}_1$ — the relative position coordinate. The action of permutation group S_2 on $(\overline{r}, \overline{R})$ gives $(-\overline{r}, \overline{R})$. Hence, the $\overline{R}$-coordinate factorises in $C_2(\mathbb{R}^3)$. The second factor is $\mathbb{R}^3 - \overline{0}$, where we identify $\overline{r} \sim -\overline{r}$. This is the relation defining real projective plane $\mathbf{RP^2}$. To define the fermionic line bundle, we view $\mathbf{RP^2}$ as sphere S^2 with every two antipodal points identified. The complex line bundle L is constructed by considering the tangent bundle on S^2 with relation $(\overline{r}, \overline{v}) \sim (-\overline{r}, -\overline{v})$, where $\overline{r} \in S^2$, $\overline{v} \in T_{\overline{r}}S^2$. Any section of L is by definition a fermionic wavefunction $\Psi : C_2(\mathbb{R}^3) \to L$. The exchange symmetry $\Psi(\overline{r}, \overline{R}) = -\Psi(-\overline{r}, \overline{R})$ is automatically satisfied.

In the setting of graph configuration spaces one can interpret different isomorphism classes of bundles over $C_n(\Gamma)$ as different species of particles. In the ideal situation every species should be characterised by some unique quantum properties. It turns out that for 3-connected nonplanar graphs the Bose and Fermi statistics are those quantum features that distinguish between two nonisomorphic classes of vector bundles. To see this we recall that isomorphism classes of complex line bundles are classified completely by the second integral cohomology group $H^2(C_n(\Gamma), \mathbb{Z})$. That is, there is a bijection between the isomorphism classes of complex line bundles over $C_2(\Gamma)$ and the elements of $H^2(C_n(\Gamma), \mathbb{Z})$ which associates to a line bundle a topological invariant called the first *Chern class*. For higher dimensional bundles, we need to consider higher cohomologies $H^{2k}(C_n(\Gamma), \mathbb{Z})$. Moreover, we typically do not have a bijection and we can only say that bundles with different Chern classes are nonisomorphic but not vice versa. From our perspective, particularly important is the torsion $H^2(C_n(\Gamma), \mathbb{Z})$ as bundles corresponding to torsion Chern classes can admit flat connections. Knowing $H_1(C_n(\Gamma), \mathbb{Z})$ we can use universal coefficient theorem [25] to show that the torsion of $H^2(C_n(\Gamma))$ is exactly the torsion of $H_1(C_n(\Gamma))$. Thus for 3-connected nonplanar graphs the torsion of $H^2(C_n(\Gamma), \mathbb{Z})$ is exactly $\mathbb{Z}_2$. That means there are two nonisomorphic vector bundles that can admit flat connection — one for bosons and one for fermions — and they correspond to totally different collective properties of particles.

4. Torsion of $H^{2k}(C_n(\Gamma), \mathbb{Z})$

In Sect. 3.2 we have indicated that by looking at the torsion component of $H^2(C_n(\Gamma), \mathbb{Z})$ one can learn that bosons and fermions in fact correspond to two nonisomorphic complex line bundles over $C_n(\Gamma)$. One can explore this idea further using higher cohomology groups, in particular by finding their torsion components. Groups $H^{2k}(C_n(\Gamma), \mathbb{Z})$ contain information about complex vector bundles over $C_n(\Gamma)$ known as the Chern classes [26]. The Chern classes are cohomology classes associated to vector bundles. They measure how a vector bundle is twisted, or nontrivial. Vector bundles over a fixed base space are in general classified by homotopy classes of maps into $G_n = Gr_n(\mathbb{R}^\infty)$, where G_n is called the *classifying space* for n-dimensional vector bundles. For an n-dimensional vector bundle $E \to C_n(\Gamma)$ to be trivial, for example, is equivalent to its classifying map $f : C_n(\Gamma) \to G_n$ being nullhomotopic. However, it can be quite difficult to determine whether this is the case or not. Much more accessible is the weaker question of whether f induces a nontrivial map on homology or cohomology, and this is precisely what the Chern classes measure. For compact manifolds the Chern classes can be given nice formulation in terms of a connection 1-form. For a vector bundle $E \to M$ over a manifold M let A be a connection 1-form. The Chern classes of E are constructed from the curvature form DA and, after the choice of section $\psi : M \to E$, are elements of even De Rham cohomology groups $H^{2k}(M, \mathbb{R})$. These groups are torsion free and correspond to the free part of integral cohomology groups $H^{2k}(M, \mathbb{Z})$. Note that if the connection is flat, i.e. curvature is zero, the Chern classes in the de Rham cohomology vanish. Nevertheless, this does not mean the vector bundle is trivial. It is, however, close to being trivial, in the sense that there is a finite covering $p : M' \to M$ such that the flat vector bundle p^*E pulled over to M' is trivial. We conjecture that for configuration spaces $C_n(X)$ this covering should be a map $p : F_n(X) \to C_n(X)$, i.e. it should place some constrains on the structure of the wave function defined on $F_n(X)$.

In order to be able to distinguish between this kind of bundles one needs to use secondary invariants theory developed by Cheeger and Simons [27]. For knowing what are the possibilities it is enough, however, to first look at the torsions of $H^{2k}(M, \mathbb{Z})$. If one is interested with complex line bundles (that physically correspond to scalar wave functions) one needs to look at the torsion of $H^2(M, \mathbb{Z})$ only. One should explore these ideas when M is replaced by the graph configuration space $C_n(\Gamma)$. We already know that for $H^2(C_n(\Gamma), \mathbb{Z})$, we have two nonisomorphic bundles that correspond to two different quantum statistics. We believe that it should be possible to discover the quantum mechanical meaning of other bundles corresponding to torsions in $H^{2k}(C_n(\Gamma), \mathbb{Z})$.

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