1. Introduction

In this note we discuss the resonances of the three-dimensional Schrödinger operator $H_{\alpha,X}$ with point interactions of constant strength $\alpha \in \mathbb{R}$ supported on the discrete set $X = \{x_n\}_{n=1}^N \subset \mathbb{R}^3$, $N \geq 2$. The corresponding Hamiltonian $H_{\alpha,X}$ is associated with the formal differential expression

$$ -\Delta + \alpha \sum_{n=1}^N \delta(x - x_n) \quad \text{on} \quad \mathbb{R}^3, \tag{1.1} $$

where $\delta(\cdot)$ stands for the point $\delta$-distribution in $\mathbb{R}^3$. The Hamiltonian $H_{\alpha,X}$ can be rigorously defined as a self-adjoint extension of a certain symmetric operator in the Hilbert space $L^2(\mathbb{R}^3)$; cf. Sect. 3 for details. Resonances of $H_{\alpha,X}$ were discussed in the monograph [1] and in several more recent publications, e.g. [2–4], see also the review [5] and the references therein.

Our ultimate goal is to obtain the asymptotic distribution for the resonances of $H_{\alpha,X}$. To this aim, we define the size of $X$ by

$$ V_X := \max_{\pi \in \Pi_N} \sum_{n=1}^N |x_n - x_{\pi(n)}|, \tag{1.2} $$

where $\Pi_N$ is the family of all the permutations of the set $\{1,2,\ldots,N\}$. A graph-theoretic interpretation of the value $V_X$ through so-called irreducible pseudo-orbits is given in Remark 4.2. This definition of the size is motivated by the condition on resonances for $H_{\alpha,X}$ given in Sect. 4.1. As the main result of this note, we prove that the number $N_{\alpha,X}(R)$ of the resonances of $H_{\alpha,X}$ lying inside the disc $\{z \in \mathbb{C} : |z| < R\}$ and with multiplicities taken into account behaves asymptotically linear

$$ N_{\alpha,X}(R) = \frac{W_X}{\pi} R + O(1), \quad R \to \infty, \tag{1.3} $$

where the constant $W_X \in [0,V_X]$ does not depend on $\alpha$ and can be viewed as the effective size of $X$. The constant $W_X$ can be computed by an implicit formula. It is not at all clear whether a simple explicit formula for $W_X$ in terms of $X$ can be found.

In the proof of (1.3) we use that the resonance condition for $H_{\alpha,X}$ acquires the form of an exponential polynomial, which can be obtained by a direct computation or alternatively using the pseudo-orbit expansion as explained in Sect. 4.3. Recall that an exponential polynomial is a sum of finitely many terms, each of which is a product of a rational function and an exponential; cf. the review paper [6] and the monographs [7, 8]. In order to obtain the asymptotics (1.3) we employ a classical result on the distribution of zeros of exponential polynomials, recalled in Sect. 2 for the convenience of the reader.

A configuration of points $X$ for which $W_X = V_X$ is said to be of Weyl-type. We show that for any $N \geq 2$ there exist the Weyl-type configurations consisting of $N$ points. For two and three points ($N \leq 3$), in fact, any configuration is of Weyl-type, as shown in Sect. 5.1. On the other hand, we present in Sect. 5.2 an example of a non-Weyl configuration for $N = 4$, for which strict inequality $W_X < V_X$ holds. We expect that such configurations can also be constructed for any $N > 4$. One can trace an analogy with non-Weyl quantum graphs studied in [9, 10]. Non-uniqueness of the permutation at which the maximum in (1.2) is attained, is a necessary condition for a configuration of points $X$ to be non-Weyl.

Exact geometric characterization of non-Weyl-type point configurations remains an open question. Besides that a physical interpretation of this mathematical observation still needs to be clarified.

It is worth pointing out that $N_{\alpha,X}(R)$ is asymptotically linear similarly as the counting function for resonances of the one-dimensional Schrödinger operator $-\Delta + V$ with a potential $V \in C_0^\infty(\mathbb{R}, \mathbb{R})$; see [11]. The exact asymptotics of the counting function for resonances of the three-dimensional Schrödinger operator $-\Delta + V$ with a potential $V \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ is known only in some special cases, but “generic” potentials this counting function behaves as $\sim R^3$, thus being not asymptotically linear; see [12] for details.
2. Exponential polynomials

In this section we introduce exponential polynomials and recall a classical result on the asymptotic distribution of their zeros. This result was first obtained by Pólya [13] and later improved by many authors, including Schmengler [14] and Moreno [15]. We refer the reader to the review [6] by Langer and to the monographs [7, 8].

Definition 2.1: An exponential polynomial $F: \mathbb{C} \to \mathbb{C}$ is a function of the form

$$F(z) = \sum_{m=1}^{M} \nu_m z^m A_m(z) e^{iz \sigma_m}, \quad (2.1)$$

where $\nu_m \in \mathbb{R}$, $m = 1, 2, \ldots, M$, $A_m(z)$ are rational functions in $z$ not vanishing identically, and the constants $\sigma_m \in \mathbb{R}$ are ordered increasingly ($\sigma_{\min} := \sigma_1 < \sigma_2 < \ldots < \sigma_M := \sigma_{\max}$).

For example, for the exponential polynomial $F(z) = \frac{z+1}{z-1} e^z + \frac{z^2+1}{z^2+1} e^{2iz}$, we have $M = 2$, $\nu_1 = 1$, $\nu_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 2$, $A_1(z) = \frac{z+1}{z-1}$, $A_2(z) = \frac{z^2+1}{z^2+1}$.

The zero set of an exponential polynomial $F$ is defined by

$$\mathcal{Z}_F := \{ z \in \mathbb{C} : F(z) = 0 \}. \quad (2.2)$$

For any $z \in \mathcal{Z}_F$ we define its multiplicity $m_F(z) \in \mathbb{N}$ as the algebraic multiplicity of the root $z$ of the function (2.1). Moreover, we introduce the counting function for an exponential polynomial $F$ by

$$N_F(R) = \sum_{z \in \mathcal{Z}_F \cap \mathbb{D}_R} m_F(z),$$

where $\mathbb{D}_R := \{ z \in \mathbb{C} : |z| < R \}$ is the disc in the complex plane centered at the origin and having the radius $R > 0$. Thus, the value $N_F(R)$ equals the number of zeros of $F$ counted with multiplicities and lying inside $\mathbb{D}_R$. Now we have all the tools at our disposal to state the result on the asymptotics of $N_F(R)$, proven in ([6], Thm. 6), see also ([9], Thm. 3.1).

Theorem 2.1: Let $F$ be an exponential polynomial as in (2.1) such that

$$\lim_{z \to \infty} A_m(z) = a_m \in \mathbb{C} \setminus \{0\}, \quad \forall m = 1, 2, \ldots, M.$$  

Then the counting function for $F$ asymptotically behaves as

$$N_F(R) = \frac{\sigma_{\max} - \sigma_{\min}}{\pi} R + O(1), \quad R \to \infty.$$  

3. Rigorous definition of $H_{\alpha,X}$

The Schrödinger operator $H_{\alpha,X}$ associated with the formal differential expression (1.1) can be rigorously defined as a self-adjoint extension in $L^2(\mathbb{R}^3)$ of the closed, densely defined, symmetric operator

$$S_X u := -u_x,$$

$$\text{dom } S_X := \{ u \in H^2(\mathbb{R}^3) : u|_{\mathbb{R}^3} = 0 \}, \quad (3.1)$$

where the vector $u|_{\mathbb{R}^3} = (u(x_1), u(x_2), \ldots, u(x_N))^T \in \mathbb{C}^N$ is well-defined by the Sobolev embedding theorem ([16], Thm. 3.26). The self-adjoint extensions of $S_X$ with $N = 1$ have been first analyzed in the seminal paper [17]. For $N > 1$ the symmetric operator $S_X$ possesses a rich family of self-adjoint extensions, not all of which correspond to point interactions. The self-adjoint extensions of $S_X$ corresponding to point interactions are investigated in detail in the monographs [1, 18], see also the references therein. Several alternative ways for parameterizing of all the self-adjoint extensions of $S_X$ can be found in a more recent literature; see e.g. [19–21]. Below we follow the strategy of [19] and use some of notations therein. According to ([19], Prop. 4.1), the adjoint of $S_X$ can be characterized as follows:

$$\text{dom } S_X^* = \{ u \in u_0 + \sum_{n=1}^{N} \left( \xi_0 e^{-r_n} + \xi_n e^{-\gamma r_n} \right) : u_0 \in \text{dom } S_X, \xi_0, \xi_1 \in \mathbb{C}^N \},$$

$$S_X^* u = -\Delta u_0 - \sum_{n=1}^{N} \left( \xi_0 e^{-r_n} + \xi_n \left( e^{-r_n} - 2 e^{-\gamma r_n} \right) \right),$$

where $r_n: \mathbb{R}^3 \to \mathbb{R}_+$, $r_n(x) := |x - x_n|$ for all $n = 1, 2, \ldots, N$ and $\zeta_0 = (\zeta_0 n)_{n=1}^N$, $\zeta_1 = (\zeta_1 n)_{n=1}^N$. Next, we introduce the mappings $I_0, I_1: \text{dom } S_X^* \to \mathbb{C}^N$ by

$$I_0 u := 4\pi \zeta_0$$

and

$$I_1 u := \left( \lim_{x \to x_n} \left( u(x) - \frac{\zeta_0 n}{r_n} \right) \right)_{n=1}^N.$$

Eventually, the operator $H_{\alpha,X}$ is defined as the restriction of $S_X^*$

$$H_{\alpha,X} u := S_X^* u,$$

$$\text{dom } H_{\alpha,X} := \{ u \in \text{dom } S_X^* : I_1 u = \alpha I_0 u \}, \quad (3.3)$$

cf. ([19], Rem. 4.3). Finally, by ([19], Prop. 4.2], the operator $H_{\alpha,X}$ is self-adjoint in $L^2(\mathbb{R}^3)$. Note also that the operator $H_{\alpha,X}$ is the same as the one considered in ([1], Chap. II.1). We remark that the usual self-adjoint free Laplacian in $L^2(\mathbb{R}^3)$ formally corresponds to the case $\alpha = \infty$.

4. Resonances of $H_{\alpha,X}$

The main aim of this section is to prove asymptotics of resonances given in (1.3). Apart from that we provide a condition on resonances through the pseudo-orbit expansion, which is of independent interest and which leads to an interpretation of the constant $V_X$ in the graph theory.

4.1. A condition on resonances for $H_{\alpha,X}$

First, we recall the definition of resonances for $H_{\alpha,X}$ borrowed from ([1], Sec. II.1.1). This definition provides at the same time a way to find them. To this aim we introduce the function

$$F_{\alpha,X}(\kappa) := \det \left[ \left( \alpha - \frac{i \kappa}{4\pi} \right) \delta_{nn'} - \tilde{G}_\alpha(x_n - x_{n'}) \right]_{n,n'=1}^{N,N},$$

where $\tilde{G}_\alpha(x_n - x_{n'})$ are the entries of the Green function $G_\alpha(x_n - x_{n'})$. The determinant $F_{\alpha,X}(\kappa)$ is a well-defined function in the complex plane and its zeros are the resonances of the operator $H_{\alpha,X}$. Moreover, $F_{\alpha,X}(\kappa)$ is a meromorphic function of $\kappa$ with simple poles at the resonances. The residues of $F_{\alpha,X}(\kappa)$ at the poles are equal to the norm of the resonant states associated with the corresponding resonance. Finally, the number of zeros of $F_{\alpha,X}(\kappa)$ in the upper half-plane is equal to the number of resonances in the lower half-plane.
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where \( \delta_{nn'} \) is the Kronecker symbol and \( \tilde{G}_\kappa(\cdot) \) is given by

\[
\tilde{G}_\kappa(x) := \begin{cases} 
0, & x = 0, \\
e^{i|x|}/(4\pi|x|), & x \neq 0. \end{cases}
\]

We say that \( \kappa_0 \in \mathbb{C} \) with \( \text{Im} \kappa_0 \leq 0 \) is a resonance of \( \mathcal{H}_{\alpha,X} \) if

\[
\mathcal{F}_{\alpha,X}(\kappa_0) = 0,
\]

holds. The multiplicity of the resonance \( \kappa_0 \) equals the multiplicity of the zero of \( \mathcal{F}_{\alpha,X}(\cdot) \) at \( \kappa = \kappa_0 \). In our convention true resonances and negative eigenvalues of \( \mathcal{H}_{\alpha,X} \) correspond to \( \text{Im} \kappa_0 < 0 \) and \( \text{Im} \kappa_0 > 0 \), respectively. According to ([1], Thm. II.1.1.4) the number of negative eigenvalues of \( \mathcal{H}_{\alpha,X} \) is finite and in the end it does not contribute to the asymptotics of the counting function for resonances of \( \mathcal{H}_{\alpha,X} \). A connection between the above definition of the resonances for \( \mathcal{H}_{\alpha,X} \) and a more fundamental definition through the poles of the analytic continuation of the resolvent for \( \mathcal{H}_{\alpha,X} \) can be justified through the Krein formula in ([1], III.1.1, Thm 1.1.1).

It is not difficult to see using standard formula for the determinant of a matrix that \( \mathcal{F}_{\alpha,X}(\cdot) \) is an exponential polynomial as in Definition 2.1 with the coefficients dependent on \( \alpha \) and on the set \( X \).

4.2. Asymptotics of the number of resonances

Recall the definition of the counting function for resonances of \( \mathcal{H}_{\alpha,X} \).

**Definition 4.1:** We define the counting function \( N_{\alpha,X}(R) \) as the number of resonances of \( \mathcal{H}_{\alpha,X} \) with multiplicities lying inside the disc \( D_R \).

Now, we have all the tools to provide a proof for the asymptotics of resonances (1.3) stated in the introduction.

**Theorem 4.1:** The counting function for resonances of \( \mathcal{H}_{\alpha,X} \) asymptotically behaves as

\[
N_{\alpha,X}(R) = \frac{W_X}{\pi} R + O(1), \quad R \to \infty,
\]

with a constant \( W_X \in [0, V_X] \), where \( V_X \) is the size of \( X \) defined in (1.2). In addition, \( W_X \) is independent of \( \alpha \).

**Proof:** The argument relies on the resonance condition (4.2). Note that the element of the matrix under the determinant in (4.1) located in the \( n \)-th row and the \( n' \)-th column is a product of a polynomial in \( \kappa \) and the exponential \( \exp(\pm in \kappa) \) with \( \pm in \kappa = |x_n - x_{n'}| \). Hence, expanding \( \mathcal{F}_{\alpha,X} \) by means of a standard formula for the determinant, we get that each single term in \( \mathcal{F}_{\alpha,X} \) is a product of a polynomial in \( \kappa \) and the exponential \( \exp(\pm i\sum_{n=1}^{N} \ell_{n}(\pi) \), where \( \pi \in \Pi_N \) is a permutation of the set \( \{1, 2, \ldots, N\} \).

The term with the lowest multiple of \( \pm \kappa \) in the exponential is \( (\alpha - \frac{i\pi}{2\kappa})^N \), i.e. there is no exponential at all and hence \( \sigma_{\text{min}} = 0 \). The largest possible multiple of \( \kappa \) in the exponentials of \( \mathcal{F}_{\alpha,X} \) is \( V_X \). Hence, we get \( \sigma_{\text{max}} \leq V_X \). The equality \( \sigma_{\text{max}} = V_X \) is not always satisfied. If the polynomial coefficient by \( \exp(\pm iV_X\kappa) \) vanishes, we have strict inequality \( \sigma_{\text{max}} < V_X \). Finally, Theorem 2.1 yields

\[
N_{\alpha,X}(R) = N_{\mathcal{F}_{\alpha,X}}(R) = \frac{W_X}{\pi} R + O(1), \quad R \to \infty,
\]

with some \( W_X \in [0, V_X] \).

The term with the largest multiple of \( \pm \kappa \) in the exponential can be represented as a product \( P(\alpha - \frac{i\pi}{2\kappa}) \exp(\pm i\sigma_{\text{max}}) \), where \( P \) is a polynomial with real coefficients of degree \( < N \). For simple algebraic reasons, if this term does not identically vanish as a function of \( \kappa \) for some \( \alpha = \alpha_0 \in \mathbb{R} \), then it does not identically vanish in the same sense for all \( \alpha \in \mathbb{R} \). Hence, we obtain by Theorem 2.1 that \( W_X \) is independent of \( \alpha \).

The argument in Theorem 4.1 suggests the following implicit formula for the constant \( W_X \)

\[
W_X = \inf \left\{ w \in [0, \infty): \lim_{t \to \infty} e^{-w|F_{\alpha,X}(-it)|} = 0 \right\},
\]

where \( F_{\alpha,X}(\cdot) \) is as in (4.1).

**Remark 4.1:** The proof of Theorem 4.1 gives slightly more, namely the case \( W_X < V_X \) can occur only if the maximum in the definition (1.2) of the size \( V_X \) of \( X \) is attained at more than one permutation, as otherwise cancellation of the principal term in the exponential polynomial \( F_{\alpha,X} \) can not occur.

4.3. Pseudo-orbit expansion for the resonance condition

The resonance condition (4.2) can be alternatively expressed by contributions of the irreducible pseudo-orbits similarly as for quantum graphs [22–24]. This expression is just yet another way how to write the determinant. However, in some cases one can easier find the terms of the determinant by studying pseudo-orbits on the corresponding directed graph and, eventually, verify their cancellations.

Consider a complete metric graph \( G \) having \( N \) vertices identified with the respective points in the set \( X \) and connected by \( \frac{N(N-1)}{2} \) edges of lengths \( \ell_{nn'} = |x_n - x_{n'}| \). To this graph we associate its oriented \( G' \) counterpart, which is obtained from \( G \) by replacing each edge \( e \) of \( G \) (\( e \) is the edge between the points with indices \( n \) and \( n' \)) by two oriented bonds \( b, \tilde{b} \) of lengths \( |b| = |	ilde{b}| = \ell_{nn'} \). The orientation of the bonds is opposite; \( b \) goes from \( x_n \) to \( x_{n'} \), whereas \( \tilde{b} \) goes from \( x_{n'} \) to \( x_n \).

**Definition 4.2:** With the graph \( G' \) we associate the following concepts.

(a) A periodic orbit \( \gamma \) in the graph \( G' \) is a closed path, which begins and ends at the same vertex, we label it by the oriented bonds, which it subsequently visits \( \gamma = (b_1, b_2, \ldots, b_n) \).

(b) A pseudo-orbit \( \tilde{\gamma} \) is a collection of periodic orbits \( \tilde{\gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \). The number of periodic orbits contained in the pseudo-orbit \( \tilde{\gamma} \) will be denoted by \( |\tilde{\gamma}|_0 \in \mathbb{N}_0 \).

(c) An irreducible pseudo-orbit \( \tilde{\gamma} \) is a pseudo-orbit which does not contain any bond more than once. Furthermore, we define

\[
B_{\tilde{\gamma}}(\kappa) = \prod_{b_j \in \tilde{\gamma}} \left( \frac{e^{i\kappa|b_j|}}{4\pi|b_j|} \right).
\]
For $|\gamma|_0 = 0$ we set $B_\gamma := 1$. We denote by $\mathcal{O}_m$ the set of all irreducible pseudo-orbits in $G'$ containing exactly $m \in \mathbb{N}_0$ bonds. Note that the total length of $\gamma$ is given by $\sum_{b_j \in \gamma} |b_j|$.

Note that any permutation $\pi \in H_N$ can be represented as a product of disjoint cycles ([25], Sec. 3.1)

$$\pi = (v_1, v_2, \ldots, v_{n_1})(v_{n_1+1}, \ldots, v_{n_1+n_2}) \ldots (v_{n_1+\ldots+n_{m(\gamma)}-1}, \ldots, v_{n_1+\ldots+n_{m(\gamma)}}),$$

where $m(\gamma)$ is the number of them, $n_j = n_j(\gamma)$ is the length of the $j$-th cycle, and $n$ is the number of cycles in $\pi$ of length one. In this notation, each parenthesis denotes one cycle and e.g. for a cycle $(v_1, v_2, \ldots, v_n)$ it holds that $\pi(v_1) = v_2$, $\pi(v_2) = v_3$, $\ldots$, $\pi(v_n) = v_1$.

The permutations $H_N$ are in one-to-one correspondence with irreducible pseudo-orbits in Definition 4.1 through the decomposition into cycles; cf. ([22], Sec. 3). Namely, an irreducible pseudo-orbit $\gamma = \pi(\gamma)$ consists of periodic orbits, each of which is a cycle of $\pi$ in its decomposition, satisfying $n_j(\pi) > 1$.

With these definitions in hands, we can state the following proposition, whose proof is inspired by the proof of ([22], Thm. 1).

**Proposition 4.1:** The resonance condition $F_{\alpha, N}(\kappa) = 0$ in (1.2) can be alternatively written as

$$\sum_{\pi \in H_N} \text{sign } \pi \prod_{n=1}^{N} \left( \alpha - \frac{i\kappa}{4\pi} \right) \delta_{n(\pi)} - \tilde{G}_\kappa(x_n - x_{\pi(n)}) = (-1)^N \sum_{n=0}^{N} \sum_{\pi \in \mathcal{O}_n} (-1)^{|\gamma|_0} B_{\gamma}(\kappa) \left( \frac{i\kappa}{4\pi} - \alpha \right)^{N-n} = 0. $$

**Proof.** Expanding the determinant in the definition of $F_{\alpha, N}$ we get

$$F_{\alpha, N}(\kappa) = \sum_{\pi \in H_N} \text{sign } \pi \prod_{n=1}^{N} \left( \alpha - \frac{i\kappa}{4\pi} \right) \delta_{n(\pi)} - \tilde{G}_\kappa(x_n - x_{\pi(n)}).$$

According to ([26], Sec 4.1), we have $\pi = (-1)^{N+n(\pi)}$. Substituting this formula for sign $\pi$ into (4.4), making use of the correspondence between irreducible periodic orbits and permutations, the formula $m(\gamma) = n(\pi) + |\gamma(\pi)|_0$, and performing some simple rearrangements, we find

$$F_{\alpha, N}(\kappa) = \sum_{n=0}^{N} \sum_{\pi \in \mathcal{O}_n} \text{sign } \pi \prod_{n=1}^{N} \left( \alpha - \frac{i\kappa}{4\pi} \right) \delta_{n(\pi)} - \tilde{G}_\kappa(x_n - x_{\pi(n)}) = \sum_{n=0}^{N} \sum_{\pi \in \mathcal{O}_n} (-1)^{N+n(\pi)} \left( \frac{i\kappa}{4\pi} - \alpha \right)^{N-n} = \sum_{n=0}^{N} \sum_{\pi \in \mathcal{O}_n} (-1)^{|\gamma|_0} B_{\gamma}(\kappa) \left( \frac{i\kappa}{4\pi} - \alpha \right)^{N-n}.$$
For $N = 2m$, $m \in \mathbb{N}$, we choose the configuration $X = \{x_n\}_{n=1}^{2m}$ as follows. First, we fix arbitrary distinct point $x_1, x_2, \ldots, x_m$ on the unit sphere $S^2 \subset \mathbb{R}^3$, so that none of them is diametrically opposite to the other. Secondly, we select the point $x_m+k$, $k = 1, \ldots, m$ to be diametrically opposite to $x_k$. For simple geometric reasons, we have $V_X = 4m$ and this maximum is attained at the unique permutation $\pi$ having the following decomposition into cycles $\pi = (1, m+1)(2, m+2) \ldots (m, 2m)$. In view of Remark 4.1, we conclude that $W_X = V_X$.

For $N = 2m+1$, $m \in \mathbb{N}$, we choose the configuration $X = \{x_n\}_{n=1}^{2m+1}$, as follows. First, we distribute the points $\{x_n\}_{n=1}^{2m}$ on $S^2$ as in the case of even $N$. Second, we put the point $x_{2m+1}$ into the center of $S^2$. If a permutation $\pi \in \Pi_{2m+1}$ does not contain the cycle $(2m+1)$, then we have $v_X(\pi) \leq 4m$ and the case of equality occurs only for the permutations

\[
\pi_1 = (1, m+1)(2, m+2) \ldots (m-1, 2m-1)(2m, 2m+1),
\]

\[
\pi_2 = (1, m+1)(2, m+2) \ldots (m-1, 2m-1)(m, 2m+1),
\]

\[
\pi_3 = (1, m+1)(2, m+2) \ldots (m-2, 2m-2)(m-1, 2m-1)(m, 2m),
\]

\[
\pi_4 = (1, m+1)(2, m+2) \ldots (m-2, 2m-2)(m-1, 2m-1)(2m, 2m+1),
\]

\[
\pi_5 = (2, m+1)(m, 2m+1),
\]

\[
\pi_6 = (2, m+1)(m, 2m+1),
\]

\[
\pi_7 = (1, m+1)(2, m+2) \ldots (m, 2m)(m, 2m+1).
\]

If a permutation $\pi \in \Pi_{2m+1}$ contains the cycle $(2m+1)$, then we again have $v_X(\pi) \leq 4m$ and the case of equality happens for the unique permutation $\pi_{2m+1} = (1, m+1)(2, m+2) \ldots (m, 2m)(2m, 2m+1)$. Hence, we obtain that $V_X = 4m$. Moreover, the exponential polynomial $F_{a,X}$ in (4.1) can be written as

\[
F_{a,X}(\kappa) = (-1)^{\nu_0}4m + 4\alpha \kappa - \kappa e^{-i\nu_0} + g_0(\kappa)
\]

\[
+ \sum_{l=1}^{L} g_l(\kappa)e^{i\nu_l},
\]

where $\sigma_l \in (0, 4m)$ and $g_0, g_l$ are polynomials, $l = 1, 2, \ldots, L$. Finally, by Theorem 2.1 we get $W_X = V_X = 4m$.

5.2. An example of a non-Weyl-type configuration

Eventually, we provide an example of a configuration of points $X = \{x_n\}_{n=1}^{4}$ for which $W_X < V_X$ in Theorem 4.1, since there will be a significant cancellation of some terms.

For $a, b, c > 0$, we consider a configuration of points $X = \{x_{n}\}_{n=1}^{4}$, where

\[
x_1 = (0, 0, 0)^T, \quad x_2 = (a, -b, 0)^T,
\]

\[
x_3 = (a, b, 0)^T, \quad x_4 = (c, 0, 0)^T.
\]

![Fig. 5.1. Discrete set $X = \{x_n\}_{n=1}^{4}$ related to example in Section 5.2.](image)

\[
x_3 = (a, b, 0)^T, \quad x_4 = (c, 0, 0)^T;
\]

see Figure 5.1. Notice that

\[
\ell_{12} = \sqrt{a^2 + b^2}, \quad \ell_{23} = 2b,
\]

\[
\ell_{34} = \sqrt{(a - c)^2 + b^2}, \quad \ell_{14} = c.
\]

Let us assume that $b$ and $c$ are sufficiently small in comparison to $a$, being more precise $2b < \sqrt{a^2 + b^2} < (a - c)^2 + b^2$. Let us first write down the general resonance condition (4.2) for four points,

\[
c_0 - c_0^2(c_2 + c_4) + c_0^2 + c_1^2 + c_2^2 + c_3^2 + c_4^2 + 2c_0(2c_1c_2c_4 + c_1c_3c_5 + c_2c_3c_4 + c_4c_5c_6 + c_1^2 + c_2^2 + c_3^2 + c_4^2 - 2c_1c_2c_3c_5 + c_1c_4c_5 + c_2c_4c_5 + c_3c_4c_5) = 0,
\]

where

\[
c_0 = a - \frac{i\kappa}{4\pi}, \quad c_1 = -\frac{e^{i\kappa\ell_{12}}}{4\pi\ell_{12}}, \quad c_2 = \frac{e^{i\kappa\ell_{13}}}{4\pi\ell_{13}},
\]

\[
c_3 = -\frac{e^{i\kappa\ell_{23}}}{4\pi\ell_{23}}, \quad c_4 = -\frac{e^{i\kappa\ell_{24}}}{4\pi\ell_{24}},
\]

\[
c_6 = -\frac{e^{i\kappa\ell_{34}}}{4\pi\ell_{34}}.
\]

In our special case we have

\[
\ell_{12} = \ell_{14}, \quad \ell_{34} = \ell_{24}
\]

and

\[
\ell_{12} + \ell_{34} > \ell_{14} + \ell_{23}.
\]

Moreover, using (5.1) we get

\[
\ell_{12} + \ell_{23} + \ell_{34} + \ell_{14} = 2b + c + \sqrt{a^2 + b^2}
\]

\[
+ \sqrt{(a - c)^2 + b^2} < 2\sqrt{a^2 + b^2} + \sqrt{(a - c)^2 + b^2}.
\]

The elements of the group $\Pi_4$ can be decomposed into disjoint cycles as:

\[
\pi_1 = (1)(2)(3)(4), \quad \pi_2 = (3, 4)(1)(2), \quad \pi_3 = (2, 3)(1)(4),
\]

\[
\pi_4 = (2, 3, 4)(1), \quad \pi_5 = (2, 4, 3)(1), \quad \pi_6 = (2, 4)(1)(3),
\]

\[
\pi_7 = (1, 2)(3)(4), \quad \pi_8 = (1, 2)(3, 4), \quad \pi_9 = (1, 2, 3)(4),
\]

\[
\pi_{10} = (1, 2, 3, 4), \quad \pi_{11} = (1, 2, 4, 3), \quad \pi_{12} = (1, 2, 4)(3),
\]

\[
\pi_{13} = (1, 3, 2)(4), \quad \pi_{14} = (1, 3, 4, 2), \quad \pi_{15} = (1, 3)(2)(4),
\]

\[
\pi_{16} = (1, 3, 4)(2), \quad \pi_{17} = (1, 3)(2, 4), \quad \pi_{18} = (1, 3, 2, 4),
\]

\[
\pi_{19} = (1, 4, 3, 2), \quad \pi_{20} = (1, 4, 2)(3), \quad \pi_{21} = (1, 4, 3)(2),
\]

\[
\pi_{22} = (1, 4)(2)(3), \quad \pi_{23} = (1, 4, 2, 3), \quad \pi_{24} = (1, 4)(2, 3).
\]

Using the above decompositions of permutations and (5.2), (5.3) we find

\[
v_X(\pi_8) = v_X(\pi_{11}) = v_X(\pi_{14}) = v_X(\pi_{17}) > v_X(\pi_{10}) =
\]
\[ v_X(\pi_{18}) = v_X(\pi_{19}) = v_X(\pi_{23}) > \ldots > v_X(\pi_1) = 0. \]

Hence, \( V_X = v_X(\pi_8) = v_X(\pi_{11}) = v_X(\pi_{14}) = v_X(\pi_{17}) \) and in view of (5.2) the leading term corresponding to \( \exp(i\kappa V_X) \) in the resonance condition (4.2) cancels

\[
\frac{e^{2i\kappa(\ell_{12}+\ell_{23})}}{(4\pi)^4 \ell_{12}^2 \ell_{23}^2} + \frac{e^{2i\kappa(\ell_{13}+\ell_{24})}}{(4\pi)^4 \ell_{13}^2 \ell_{24}^2} = 0.
\]

However, the succeeding term in the condition (4.2) corresponding to the exponent \( \exp(i\kappa v_X(\pi_{10})) \) does not cancel

\[
\frac{2}{(4\pi)^4} \left( \frac{e^{i\kappa(\ell_{12}+\ell_{23}+\ell_{34}+\ell_{14})}}{\ell_{12}\ell_{23}\ell_{34}\ell_{14}} + \frac{e^{i\kappa(\ell_{13}+\ell_{24}+\ell_{23}+\ell_{14})}}{\ell_{13}\ell_{23}\ell_{24}\ell_{14}} \right) \neq 0.
\]

Finally, we end up with

\[ W_X = v_X(\pi_{10}) = v_X(\pi_{18}) = v_X(\pi_{19}) = v_X(\pi_{23}) < V_X. \]

6. Conclusions

In this note, we considered the three-dimensional Schrödinger operator \( \mathcal{H}_{\alpha,X} \) with finitely many point interactions of equal strength \( \alpha \in \mathbb{R} \) supported on the discrete set \( X \). As the main result, we obtained that the resonance counting function for \( \mathcal{H}_{\alpha,X} \) behaves asymptotically linear. The constant coefficient standing by the linear term in this asymptotics can be seen as the effective size of \( X \).

The obtained law of distribution for resonances is very much different from the behaviour of the resonance counting function for the three-dimensional Schrödinger operator with a regular potential \([12]\). On the other hand, it resembles the corresponding law for one-dimensional Schrödinger operators with regular potentials \([11]\) and for quantum graphs \([9, 10]\).

We associated a complete directed weighted graph \( G' \) with the configuration of points \( X \) in a natural way. The effective size of \( X \) can be estimated from above by the actual size of \( X \) defined as the maximal possible total length of an irreducible pseudo-orbit in the graph \( G' \). An implicit formula for finding the effective size of \( X \) was given. We also provided examples showing sharpness of our upper bound on the effective size of \( X \). Point configurations for which the effective size of \( X \) is strictly smaller than its actual size can be seen as analogues of ‘non-Weyl’ quantum graphs \([9, 10]\).

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