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Asymptotics of Resonances Induced by Point Interactions

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We consider the resonances of the self-adjoint three-dimensional Schrödinger operator with point interactions of constant strength supported on the set $X = \{x_n\}_{n=1}^N$. The size of X is defined by $V_X = \max_{\pi\in\Pi_N} \sum_{n=1}^N |x_n - x_{\pi(n)}|$, where Π_N is the family of all the permutations of the set $\{1, 2, \dots, N\}$. We prove that the number of resonances counted with multiplicities and lying inside the disc of radius R behaves asymptotically linear $\frac{W_X}{\pi} R + \mathcal{O}(1)$ as $R \to \infty$, where the constant $W_X \in [0, V_X]$ can be seen as the effective size of X. Moreover, we show that there exist a configuration of any number of points such that $W_X = V_X$. Finally, we construct an example for $N = 4$ with $W_X < V_X$, which can be viewed as an analogue of a quantum graph with non-Weyl asymptotics of resonances.

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1. Introduction

In this note we discuss the resonances of the three-dimensional Schrödinger operator $\mathcal{H}_{\alpha,X}$ with point interactions of constant strength $\alpha \in \mathbb{R}$ supported on the discrete set $X = \{x_n\}_{n=1}^N \subset \mathbb{R}^3$, $N \geq 2$. The corresponding Hamiltonian $\mathcal{H}_{\alpha,X}$ is associated with the formal differential expression

$$-\Delta + \alpha \sum_{n=1}^N \delta(x - x_n) \text{ on } \mathbb{R}^3, \tag{1.1}$$

where $\delta(\cdot)$ stands for the point δ-distribution in $\mathbb{R}^3$. The Hamiltonian $\mathcal{H}_{\alpha,X}$ can be rigorously defined as a self-adjoint extension of a certain symmetric operator in the Hilbert space $L^2(\mathbb{R}^3)$; cf. Sect. 3 for details. Resonances of $\mathcal{H}_{\alpha,X}$ were discussed in the monograph [1] and in several more recent publications, e.g. [2–4], see also the review [5] and the references therein.

Our ultimate goal is to obtain the asymptotic distribution for the resonances of $\mathcal{H}_{\alpha,X}$. To this aim, we define the *size* of X by

$$V_X := \max_{\pi\in\Pi_N} \sum_{n=1}^N |x_n - x_{\pi(n)}|, \tag{1.2}$$

where Π_N is the family of all the permutations of the set $\{1, 2, \dots, N\}$. A graph-theoretic interpretation of the value V_X through so-called irreducible pseudo-orbits is given in Remark 4.2. This definition of the size is motivated by the condition on resonances for $\mathcal{H}_{\alpha,X}$ given in Sect. 4.1. As the main result of this note, we prove that the number $\mathcal{N}_{\alpha,X}(R)$ of the resonances of $\mathcal{H}_{\alpha,X}$ lying inside the disc $\{z \in \mathbb{C}\colon |z| < R\}$ and with multiplicities taken into account behaves asymptotically linear

$$\mathcal{N}_{\alpha,X}(R) = \frac{W_X}{\pi} R + \mathcal{O}(1), \quad R \to \infty, \tag{1.3}$$

where the constant $W_X \in [0, V_X]$ does not depend on α and can be viewed as the effective size of X. The constant W_X can be computed by an implicit formula. It is not at all clear whether a simple explicit formula for W_X in terms of X can be found.

In the proof of (1.3) we use that the resonance condition for $\mathcal{H}_{\alpha,X}$ acquires the form of an exponential polynomial, which can be obtained by a direct computation or alternatively using the pseudo-orbit expansion as explained in Sect. 4.3. Recall that an exponential polynomial is a sum of finitely many terms, each of which is a product of a rational function and an exponential; cf. the review paper [6] and the monographs [7, 8]. In order to obtain the asymptotics (1.3) we employ a classical result on the distribution of zeros of exponential polynomials, recalled in Sect. 2 for the convenience of the reader.

A configuration of points X for which $W_X = V_X$ is said to be of *Weyl-type*. We show that for any $N \geq 2$ there exist the Weyl-type configurations consisting of N points. For two and three points ($N \leq 3$), in fact, any configuration is of Weyl-type, as shown in Sect. 5.1. On the other hand, we present in Sect. 5.2 an example of a non-Weyl configuration for $N = 4$, for which strict inequality $W_X < V_X$ holds. We expect that such configurations can also be constructed for any $N > 4$. One can trace an analogy with non-Weyl quantum graphs studied in [9, 10]. Non-uniqueness of the permutation at which the maximum in (1.2) is attained, is a necessary condition for a configuration of points X to be non-Weyl. Exact geometric characterization of non-Weyl-type point configurations remains an open question. Besides that a physical interpretation of this mathematical observation still needs to be clarified.

It is worth pointing out that $\mathcal{N}_{\alpha,X}(R)$ is *asymptotically linear* similarly as the counting function for resonances of the one-dimensional Schrödinger operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ with a potential $V \in C_0^\infty(\mathbb{R};\mathbb{R})$; see [11]. The exact asymptotics of the counting function for resonances of the three-dimensional Schrödinger operator $-\Delta + V$ with a potential $V \in C_0^\infty(\mathbb{R}^3;\mathbb{R})$ is known only in some special cases, but for "generic" potentials this counting function behaves as $\sim R^3$, thus being *not asymptotically linear*; see [12] for details.

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2. Exponential polynomials

In this section we introduce exponential polynomials and recall a classical result on the asymptotic distribution of their zeros. This result was first obtained by Pólya [13] and later improved by many authors, including Schwengeler [14] and Moreno [15]. We refer the reader to the review [6] by Langer and to the monographs [7, 8].

Definition 2.1: An exponential polynomial $F\colon \mathbb{C} \to \mathbb{C}$ is a function of the form

$$F(z) = \sum_{m=1}^{M} z^{\nu_m} A_m(z) \mathrm{e}^{\mathrm{i}\, z\sigma_m}, \tag{2.1}$$

where $\nu_m \in \mathbb{R}$, $m = 1, 2, \ldots, M$, $A_m(z)$ are rational functions in z not vanishing identically, and the constants $\sigma_m \in \mathbb{R}$ are ordered increasingly ($\sigma_{\min} := \sigma_1 < \sigma_2 < \ldots < \sigma_M =: \sigma_{\max}$).

For example, for the exponential polynomial

$$F(z) = z\frac{z+\mathrm{i}}{z-\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\, z} + z^2\frac{z^2+\mathrm{i}}{z^2+1}\,\mathrm{e}^{2\,\mathrm{i}\, z},$$

we have $M = 2$, $\nu_1 = 1$, $\nu_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 2$, $A_1(z) = \frac{z+\mathrm{i}}{z-\mathrm{i}}$, $A_2(z) = \frac{z^2+\mathrm{i}}{z^2+1}$.

The zero set of an exponential polynomial F is defined by

$$\mathcal{Z}_F := \{z \in \mathbb{C} \colon F(z) = 0\}. \tag{2.2}$$

For any $z \in \mathcal{Z}_F$ we define its multiplicity $\mathsf{m}_F(z) \in \mathbb{N}$ as the algebraic multiplicity of the root z of the function (2.1). Moreover, we introduce the counting function for an exponential polynomial F by

$$\mathcal{N}_F(R) = \sum_{z\in\mathcal{Z}_F\cap\mathcal{D}_R} \mathsf{m}_F(z),$$

where $\mathcal{D}_R := \{z \in \mathbb{C} \colon |z| < R\}$ is the disc in the complex plane centered at the origin and having the radius $R > 0$. Thus, the value $\mathcal{N}_F(R)$ equals the number of zeros of F counted with multiplicities and lying inside $\mathcal{D}_R$. Now we have all the tools at our disposal to state the result on the asymptotics of $\mathcal{N}_F(R)$, proven in ([6], Thm. 6), see also ([9], Thm. 3.1).

Theorem 2.1: Let F be an exponential polynomial as in (2.1) such that

$$\lim_{z\to\infty} A_m(z) = a_m \in \mathbb{C} \setminus \{0\}, \quad \forall m = 1, 2, \ldots, M.$$

Then the counting function for F asymptotically behaves as

$$\mathcal{N}_F(R) = \frac{\sigma_{\max} - \sigma_{\min}}{\pi} R + \mathcal{O}(1), \quad R \to \infty.$$

3. Rigorous definition of $\mathcal{H}_{\alpha,X}$

The Schrödinger operator $\mathcal{H}_{\alpha,X}$ associated with the formal differential expression (1.1) can be rigorously defined as a self-adjoint extension in $L^2(\mathbb{R}^3)$ of the closed, densely defined, symmetric operator

$$\mathcal{S}_X u := -\Delta u,$$
$$\mathrm{dom}\,\mathcal{S}_X := \{u \in H^2(\mathbb{R}^3)\colon u|_X = 0\}, \tag{3.1}$$

where the vector $u|_X = (u(x_1), u(x_2), \ldots, u(x_N))^\top \in \mathbb{C}^N$ is well-defined by the Sobolev embedding theorem ([16], Thm. 3.26). The self-adjoint extensions of $\mathcal{S}_X$ with $N = 1$ have been first analyzed in the seminal paper [17]. For $N > 1$ the symmetric operator $\mathcal{S}_X$ possesses a rich family of self-adjoint extensions, not all of which correspond to point interactions. The self-adjoint extensions of $\mathcal{S}_X$ corresponding to point interactions are investigated in detail in the monographs [1, 18], see also the references therein. Several alternative ways for parameterizing of all the self-adjoint extensions of $\mathcal{S}_X$ can be found in a more recent literature; see *e.g.* [19–21]. Below we follow the strategy of [19] and use some of notations therein. According to ([19], Prop. 4.1), the adjoint of $\mathcal{S}_X$ can be characterized as follows:

$$\mathrm{dom}\,\mathcal{S}_X^* = \Big\{u = u_0 + \sum_{n=1}^{N}\left(\xi_{0n}\frac{\mathrm{e}^{-r_n}}{r_n} + \xi_{1n}\mathrm{e}^{-r_n}\right) : u_0 \in \mathrm{dom}\,\mathcal{S}_X, \xi_0, \xi_1 \in \mathbb{C}^N\Big\},$$

$$\mathcal{S}_X^* u = -\Delta u_0 - \sum_{n=1}^{N}\left(\xi_{0n}\frac{\mathrm{e}^{-r_n}}{r_n} + \xi_{1n}\left(\mathrm{e}^{-r_n} - \frac{2\,\mathrm{e}^{-r_n}}{r_n}\right)\right),$$

where $r_n\colon \mathbb{R}^3 \to \mathbb{R}_+$, $r_n(x) := |x - x_n|$ for all $n = 1, 2, \ldots, N$ and $\xi_0 = \{\xi_{0n}\}_{n=1}^N$, $\xi_1 = \{\xi_{1n}\}_{n=1}^N$. Next, we introduce the mappings $\mathit{\Gamma}_0, \mathit{\Gamma}_1\colon \mathrm{dom}\,\mathcal{S}_X^* \to \mathbb{C}^N$ by

$$\mathit{\Gamma}_0 u := 4\pi\xi_0$$

and

$$\mathit{\Gamma}_1 u := \left\{\lim_{x\to x_n}\left(u(x) - \frac{\xi_{0n}}{r_n}\right)\right\}_{n=1}^{N}. \tag{3.2}$$

Eventually, the operator $\mathcal{H}_{\alpha,X}$ is defined as the restriction of $\mathcal{S}_X^*$

$$\mathcal{H}_{\alpha,X} u := \mathcal{S}_X^* u,$$
$$\mathrm{dom}\,\mathcal{H}_{\alpha,X} := \{u \in \mathrm{dom}\,\mathcal{S}_X^*\colon \mathit{\Gamma}_1 u = \alpha\mathit{\Gamma}_0 u\}, \tag{3.3}$$

cf. ([19], Rem. 4.3). Finally, by ([19], Prop. 4.2], the operator $\mathcal{H}_{\alpha,X}$ is self-adjoint in $L^2(\mathbb{R}^3)$. Note also that the operator $\mathcal{H}_{\alpha,X}$ is the same as the one considered in ([1], Chap. II.1). We remark that the usual self-adjoint free Laplacian in $L^2(\mathbb{R}^3)$ formally corresponds to the case $\alpha = \infty$.

4. Resonances of $\mathcal{H}_{\alpha,X}$

The main aim of this section is to prove asymptotics of resonances given in (1.3). Apart from that we provide a condition on resonances through the pseudo-orbit expansion, which is of independent interest and which leads to an interpretation of the constant V_X in the graph theory.

4.1. A condition on resonances for $\mathcal{H}_{\alpha,X}$

First, we recall the definition of resonances for $\mathcal{H}_{\alpha,X}$ borrowed from ([1], Sec. II.1.1). This definition provides at the same time a way to find them. To this aim we introduce the function

$$F_{\alpha,X}(\kappa) := \det\left[\left\{\left(\alpha - \frac{\mathrm{i}\kappa}{4\pi}\right)\delta_{nn'} - \widetilde{G}_\kappa(x_n - x_{n'})\right\}_{n,n'=1}^{N,N}\right], \tag{4.1}$$

where $\delta_{nn'}$ is the Kronecker symbol and $\widetilde{G}_\kappa(\cdot)$ is given by

$$\widetilde{G}_\kappa(x) := \begin{cases} 0, & x = 0, \\ e^{i\kappa|x|}/(4\pi|x|), & x \neq 0. \end{cases}$$

We say that $\kappa_0 \in \mathbb{C}$ with $\mathrm{Im}\kappa_0 \leq 0$ is a resonance of $\mathcal{H}_{\alpha,X}$ if

$$F_{\alpha,X}(\kappa_0) = 0, \tag{4.2}$$

holds. The multiplicity of the resonance κ_0 equals the multiplicity of the zero of $F_{\alpha,X}(\cdot)$ at $\kappa = \kappa_0$. In our convention true resonances and negative eigenvalues of $\mathcal{H}_{\alpha,X}$ correspond to $\mathrm{Im}\kappa_0 < 0$ and $\mathrm{Im}\kappa_0 > 0$, respectively. According to ([1], Thm. II.1.1.4) the number of negative eigenvalues of $\mathcal{H}_{\alpha,X}$ is finite and in the end it does not contribute to the asymptotics of the counting function for resonances of $\mathcal{H}_{\alpha,X}$. A connection between the above definition of the resonances for $\mathcal{H}_{\alpha,X}$ and a more fundamental definition through the poles of the analytic continuation of the resolvent for $\mathcal{H}_{\alpha,X}$ can be justified through the Krein formula in ([1], §II.1.1, Thm 1.1.1).

It is not difficult to see using standard formula for the determinant of a matrix that $F_{\alpha,X}$ is an exponential polynomial as in Definition 2.1 with the coefficients dependent on α and on the set X.

4.2. Asymptotics of the number of resonances

Recall the definition of the counting function for resonances of $\mathcal{H}_{\alpha,X}$.

Definition 4.1: We define the *counting function* $\mathcal{N}_{\alpha,X}(R)$ as the number of resonances of $\mathcal{H}_{\alpha,X}$ with multiplicities lying inside the disc $\mathcal{D}_R$.

Now, we have all the tools to provide a proof for the asymptotics of resonances (1.3) stated in the introduction.

Theorem 4.1: The counting function for resonances of $\mathcal{H}_{\alpha,X}$ asymptotically behaves as

$$\mathcal{N}_{\alpha,X}(R) = \frac{W_X}{\pi}R + \mathcal{O}(1), \quad R \to \infty, \tag{4.3}$$

with a constant $W_X \in [0, V_X]$, where V_X is the size of X defined in (1.2). In addition, W_X is independent of α.

Proof. The argument relies on the resonance condition (4.2). Note that the element of the matrix under the determinant in (4.1) located in the n-th row and the n'-th column is a product of a polynomial in κ and the exponential $\exp(i\kappa\ell_{nn'})$ with $\ell_{nn'} = |x_n - x_{n'}|$. Hence, expanding $F_{\alpha,X}$ by means of a standard formula for the determinant, we get that each single term in $F_{\alpha,X}$ is a product of a polynomial in κ and the exponential $\exp(i\kappa \sum_{n=1}^N \ell_{n\pi(n)})$, where $\pi \in \mathit{\Pi}_N$ is a permutation of the set $\{1, 2, \dots, N\}$.

The term with the lowest multiple of $i\kappa$ in the exponential is $\left(\alpha - \frac{i\kappa}{4\pi}\right)^N$, *i.e.* there is no exponential at all and hence $\sigma_{\min} = 0$. The largest possible multiple of $i\kappa$ in the exponentials of $F_{\alpha,X}$ is V_X. Hence, we get $\sigma_{\max} \leq V_X$. The equality $\sigma_{\max} = V_X$ is not always satisfied. If the polynomial coefficient by $\exp(i\kappa V_X)$ vanishes, we have strict inequality $\sigma_{\max} < V_X$. Finally, Theorem 2.1 yields

$$\mathcal{N}_{\alpha,X}(R) = \mathcal{N}_{F_{\alpha,X}}(R) = \frac{W_X}{\pi}R + \mathcal{O}(1), \quad R \to \infty,$$

with some $W_X \in [0, V_X]$.

The term with the largest multiple of $i\kappa$ in the exponent can be represented as a product $P\left(\alpha - \frac{i\kappa}{4\pi}\right)\exp(i\kappa\sigma_{\max})$, where P is a polynomial with real coefficients of degree $< N$. For simple algebraic reasons, if this term does not identically vanish as a function of κ for some $\alpha = \alpha_0 \in \mathbb{R}$, then it does not identically vanish in the same sense for all $\alpha \in \mathbb{R}$. Hence, we obtain by Theorem 2.1 that W_X is independent of α. □

The argument in Theorem 4.1 suggests the following implicit formula for the constant W_X

$$W_X = \inf\left\{w \in [0, \infty) \colon \lim_{t\to\infty} e^{-wt}|F_{\alpha,X}(-it)| = 0\right\},$$

where $F_{\alpha,X}(\cdot)$ is as in (4.1).

Remark 4.1: The proof of Theorem 4.1 gives slightly more, namely the case $W_X < V_X$ can occur only if the maximum in the definition (1.2) of the size V_X of X is attained at more than one permutation, as otherwise cancellation of the principal term in the exponential polynomial $F_{\alpha,X}$ can not occur.

4.3. Pseudo-orbit expansion for the resonance condition

The resonance condition (4.2) can be alternatively expressed by contributions of the irreducible pseudo-orbits similarly as for quantum graphs [22–24]. This expression is just yet another way how to write the determinant. However, in some cases one can easier find the terms of the determinant by studying pseudo-orbits on the corresponding directed graph and, eventually, verify their cancellations.

Consider a complete metric graph G having N vertices identified with the respective points in the set X and connected by $\frac{N(N-1)}{2}$ edges of lengths $\ell_{nn'} = |x_n - x_{n'}|$. To this graph we associate its oriented G' counterpart, which is obtained from G by replacing each edge e of G (e is the edge between the points with indices n and n') by two oriented bonds b, $\widehat{b}$ of lengths $|b| = |\widehat{b}| = \ell_{nn'}$. The orientation of the bonds is opposite; b goes from x_n to $x_{n'}$, whereas $\widehat{b}$ goes from $x_{n'}$ to x_n.

Definition 4.2: With the graph G' we associate the following concepts.

(a) A *periodic orbit* γ in the graph G' is a closed path, which begins and ends at the same vertex, we label it by the oriented bonds, which it subsequently visits $\gamma = (b_1, b_2, \dots, b_n)$.

(b) A *pseudo-orbit* $\widetilde{\gamma}$ is a collection of periodic orbits $\widetilde{\gamma} = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$. The number of periodic orbits contained in the pseudo-orbit $\widetilde{\gamma}$ will be denoted by $|\widetilde{\gamma}|_o \in \mathbb{N}_0$.

(c) An *irreducible pseudo-orbit* $\bar{\gamma}$ is a pseudo-orbit which does not contain any bond more than once. Furthermore, we define

$$B_{\bar{\gamma}}(\kappa) = \prod_{b_j \in \bar{\gamma}} \left(-\frac{e^{i\kappa|b_j|}}{4\pi|b_j|}\right).$$

□

For $|\overline{\gamma}|_{\rm o} = 0$ we set $B_{\overline{\gamma}} := 1$. We denote by $\bar{\mathcal{O}}_m$ the set of all irreducible pseudo-orbits in G' containing exactly $m \in \mathbb{N}_0$ bonds. Note that the total length of $\overline{\gamma}$ is given by $\sum_{b_j\in\overline{\gamma}} |b_j|$.

Note that any permutation $\pi \in \Pi_N$ can be represented as a product of disjoint cycles ([25], Sec. 3.1)

$$\pi = (v_1, v_2, \dots, v_{n_1})(v_{n_1+1}, \dots, v_{n_1+n_2}) \dots (v_{n_1+\ldots+n_{m(\pi)-1}+1}, \dots, v_{n_1+\ldots+n_{m(\pi)}}),$$

where $m(\pi)$ is the number of them, $n_j = n_j(\pi)$ is the length of the $j^{\rm th}$-cycle, and $n(\pi)$ is the number of cycles in π of length one. In this notation, each parenthesis denotes one cycle and *e.g.* for a cycle $(v_1, v_2, \dots, v_{n_1})$ it holds that $\pi(v_1) = v_2$, $\pi(v_2) = v_3$, $\dots$, $\pi(v_{n_1}) = v_1$. The permutations Π_N are in one-to-one correspondence with irreducible pseudo-orbits in Definition 4.2 through the decomposition into cycles; *cf.* ([22], Sec. 3). Namely, an irreducible pseudo-orbit $\overline{\gamma} = \overline{\gamma}(\pi)$ consists of periodic orbits, each of which is a cycle of π in its decomposition, satisfying $n_j(\pi) > 1$.

With these definitions in hands, we can state the following proposition, whose proof is inspired by the proof of ([22], Thm. 1).

Proposition 4.1: The resonance condition $F_{\alpha,X}(\kappa) = 0$ in (1.2) can be alternatively written as

$$\sum_{\pi\in\Pi_N} {\rm sign}\ \pi \prod_{n=1}^{N} \left(\left(\alpha - \frac{{\rm i}\kappa}{4\pi}\right) \delta_{n\pi(n)} - \widetilde{G}_\kappa(x_n - x_{\pi(n)}) \right) = (-1)^N \sum_{n=0}^{N} \sum_{\overline{\gamma}\in\bar{\mathcal{O}}_n} (-1)^{|\overline{\gamma}|_{\rm o}} B_{\overline{\gamma}}(\kappa) \left(\frac{{\rm i}\kappa}{4\pi} - \alpha\right)^{N-n} = 0.$$

Proof. Expanding the determinant in the definition of $F_{\alpha,X}$ we get

$$F_{\alpha,X}(\kappa) = \sum_{\pi\in\Pi_N} {\rm sign}\ \pi \times \prod_{n=1}^{N} \left(\left(\alpha - \frac{{\rm i}\kappa}{4\pi}\right) \delta_{n\pi(n)} - \widetilde{G}_\kappa(x_n - x_{\pi(n)}) \right). \quad (4.4)$$

According to ([26], Sec 4.1), we have ${\rm sign}\, \pi = (-1)^{N+m(\pi)}$. Substituting this formula for ${\rm sign}\, \pi$ into (4.4), making use of the correspondence between irreducible periodic orbits and permutations, the formula $m(\pi) = n(\pi) + |\overline{\gamma}(\pi)|_{\rm o}$, and performing some simple rearrangements, we find

$$F_{\alpha,X}(\kappa) = \sum_{n=0}^{N} \sum_{\substack{\pi\,\in\,\Pi_N \\ n(\pi)\,=\,N\,-\,n}} {\rm sign}\ \pi \times \prod_{s=1}^{N} \left(\left(\alpha - \frac{{\rm i}\kappa}{4\pi}\right) \delta_{s\pi(s)} - \widetilde{G}_\kappa(x_s - x_{\pi(s)}) \right) =$$

$$\sum_{n=0}^{N} \sum_{\substack{\pi\,\in\,\Pi_N \\ n(\pi)\,=\,N\,-\,n}} (-1)^{N+n(\pi)} (-1)^{|\overline{\gamma}(\pi)|_{\rm o}} B_{\overline{\gamma}(\pi)}(\kappa) \left(\alpha - \frac{{\rm i}\kappa}{4\pi}\right)^{N-n} =$$

$$(-1)^N \sum_{n=0}^{N} \sum_{\overline{\gamma}\in\mathcal{O}_n} (-1)^{|\overline{\gamma}|_{\rm o}} B_{\overline{\gamma}}(\kappa) \left(\frac{{\rm i}\kappa}{4\pi} - \alpha\right)^{N-n}.$$

□

Remark 4.2: In view of Proposition 4.1, the value V_X in (1.2) can be interpreted as the maximal possible total length of an irreducible pseudo-orbit in the graph G'.

5. Point configurations of Weyl- and non-Weyl-types

Recall that a configuration of points is said to be of Weyl-type if $W_X = V_X$ and of non-Weyl-type if $W_X < V_X$. In this section we provide examples for both types of point configurations and discuss related questions. For the sake of convenience, for a configuration of points $X = \{x_n\}_{n=1}^N$ and a permutation $\pi \in \Pi_N$ we define

$$\mathsf{v}_X(\pi) := \sum_{n=1}^{N} |x_n - x_{\pi(n)}|.$$

5.1. *Weyl-type configurations*

First, we show that for low number of points non-Weyl configurations do not exist.

Proposition 5.1: For $N = 2, 3$, $W_X = V_X$ holds for any $X = \{x_n\}_{n=1}^N$.

Proof. For $N = 2$, we have $V_X = 2\ell_{12}$. From (4.1) and (4.2) we obtain the resonance condition

$$\left(\frac{{\rm i}\kappa}{4\pi} - \alpha\right)^2 - \frac{{\rm e}^{2{\rm i}\kappa\ell_{12}}}{(4\pi\ell_{12})^2} = 0.$$

Obviously, the coefficient at ${\rm e}^{{\rm i}\kappa V_X}$ does not identically vanish and the claim follows from Theorem 2.1.

Let $N = 3$. Without loss of generality we assume that $\ell_{12} \geq \ell_{23} \geq \ell_{13}$. By triangle inequality we have $\ell_{12} + \ell_{23} + \ell_{13} \geq 2\ell_{12}$. The equality is attained only if all three points belong to a straight line. Hence, we have $V_X = \ell_{12} + \ell_{23} + \ell_{13}$, which is attained at the cyclic shift, having the decomposition $\pi = (1, 2, 3)$. From (4.2) we obtain the resonance condition

$$\left(\frac{{\rm i}\kappa}{4\pi} - \alpha\right)^3 - \left(\frac{{\rm i}\kappa}{4\pi} - \alpha\right) f(\kappa) + g(\kappa) = 0,$$

where

$$f(\kappa) := \frac{1}{(4\pi)^2} \left(\frac{{\rm e}^{2{\rm i}\kappa\ell_{12}}}{(\ell_{12})^2} + \frac{{\rm e}^{2{\rm i}\kappa\ell_{23}}}{(\ell_{23})^2} + \frac{{\rm e}^{2{\rm i}\kappa\ell_{13}}}{(\ell_{13})^2} \right),$$

$$g(\kappa) := \frac{2{\rm e}^{{\rm i}\kappa(\ell_{12}+\ell_{23}+\ell_{13})}}{(4\pi)^3 \ell_{12}\ell_{23}\ell_{13}}.$$

For simple algebraic reasons, in both cases $\ell_{12} + \ell_{23} + \ell_{13} > 2\ell_{12}$ and $\ell_{12} + \ell_{23} + \ell_{13} = 2\ell_{12}$ the coefficient at ${\rm e}^{{\rm i}\kappa V_X}$ does not vanish identically and the claim also follows from Theorem 2.1. □

Next, we show that Weyl-type configurations are not something specific for low number of points and they can be constructed for any number of them.

Theorem 5.1: For any $N \geq 2$ there exist a configuration of points $X = \{x_n\}_{n=1}^N$ such that $W_X = V_X$.

Proof. We provide two different constructions for the cases of even and odd number of points in the set X.

For $N = 2m$, $m \in \mathbb{N}$, we choose the configuration $X = \{x_n\}_{n=1}^{2m}$ as follows. First, we fix arbitrary distinct point $x_1, x_2, \ldots, x_m$ on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, so that none of them is diametrically opposite to the other. Second, we select the point $x_{m+k} \in \mathbb{S}^2$, $k = 1, \ldots, m$ to be diametrically opposite to x_k. For simple geometric reasons, we have $V_X = 4m$ and this maximum is attained at the unique permutation π having the following decomposition into cycles $\pi = (1, m+1)(2, m+2)\ldots(m, 2m)$. In view of Remark 4.1, we conclude that $W_X = V_X$.

For $N = 2m + 1$, $m \in \mathbb{N}$, we choose the configuration $X = \{x_n\}_{n=1}^{2m+1}$, as follows. First, we distribute the points $\{x_n\}_{n=1}^{2m}$ on $\mathbb{S}^2$ as in the case of even N. Second, we put the point x_{2m+1} into the center of $\mathbb{S}^2$. If a permutation $\pi \in \Pi_{2m+1}$ does not contain the cycle $(2m+1)$, then we have $\mathsf{v}_X(\pi) \leq 4m$ and the case of equality occurs only for the permutations

$$\begin{aligned}
&\pi_1 = (1, m+1)(2, m+2)\ldots \\
&\quad (m-1, 2m-1)(m, 2m, 2m+1), \\
&\pi_2 = (1, m+1)(2, m+2)\ldots \\
&\quad (m-1, 2m-1)(m, 2m+1, 2m), \\
&\pi_3 = (1, m+1)(2, m+2)\ldots \\
&\quad (m-2, 2m-2)(m-1, 2m-1, 2m+1)(m, 2m), \\
&\pi_4 = (1, m+1)(2, m+2)\ldots \\
&\quad (m-2, 2m-2)(m-1, 2m+1, 2m-1)(m, 2m), \\
&\ldots\ldots \\
&\pi_{2m-1} = (2, m+2)\ldots \\
&\quad (m-1, 2m-1)(m, 2m)(1, m+1, 2m+1), \\
&\pi_{2m} = (2, m+2)\ldots \\
&\quad (m-1, 2m-1)(m, 2m)(1, 2m+1, m+1).
\end{aligned}$$

If a permutation $\pi \in \Pi_{2m+1}$ contains the cycle $(2m+1)$, then we again have $\mathsf{v}_X(\pi) \leq 4m$ and the case of equality happens for the unique permutation

$$\pi_{2m+1} = (1, m+1)(2, m+2)\ldots(m, 2m)(2m+1).$$

Hence, we obtain that $V_X = 4m$. Moreover, the exponential polynomial $F_{\alpha,X}$ in (4.1) can be written as

$$\begin{aligned}
F_{\alpha,X}(\kappa) &= (-1)^m \frac{4m + 4\pi\alpha - \mathrm{i}\kappa}{2^{2m}(4\pi)^{2m+1}} \mathrm{e}^{\mathrm{i}(4m)\kappa} + g_0(\kappa) \\
&\quad + \sum_{l=1}^{L} g_l(\kappa) \mathrm{e}^{\mathrm{i}\sigma_l\kappa},
\end{aligned}$$

where $\sigma_l \in (0, 4m)$ and g_0, g_l are polynomials, $l = 1, 2, \ldots, L$. Finally, by Theorem 2.1 we get $W_X = V_X = 4m$. □

5.2. An example of a non-Weyl-type configuration

Eventually, we provide an example of a configuration of points $X = \{x_n\}_{n=1}^4$ for which $W_X < V_X$ in Theorem 4.1, since there will be a significant cancellation of some terms.

For $a, b, c > 0$, we consider a configuration of points $X = \{x_n\}_{n=1}^4$, where

$$x_1 = (0,0,0)^\top, \quad x_2 = (a, -b, 0)^\top,$$

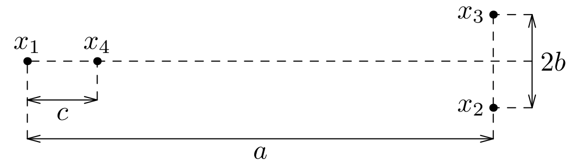

Fig. 5.1. Discrete set $X = \{x_n\}_{n=1}^4$ related to example in Section 5.2.

$$x_3 = (a, b, 0)^\top, \quad x_4 = (c, 0, 0)^\top;$$

see Figure 5.1. Notice that

$$\begin{aligned}
&\ell_{12} = \sqrt{a^2 + b^2}, \quad \ell_{23} = 2b, \\
&\ell_{34} = \sqrt{(a-c)^2 + b^2}, \quad \ell_{14} = c.
\end{aligned} \tag{5.1}$$

Let us assume that b and c are sufficiently small in comparison to a, being more precise $2b + c < \sqrt{a^2 + b^2} + \sqrt{(a-c)^2 + b^2}$. Let us first write down the general resonance condition (4.2) for four points.

$$\begin{aligned}
&c_0^4 - c_0^2(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2) + 2c_0(c_1c_2c_4 \\
&\quad + c_1c_3c_5 + c_2c_3c_6 + c_4c_5c_6) + c_1^2c_6^2 + c_2^2c_5^2 \\
&\quad + c_3^2c_4^2 - 2(c_1c_2c_5c_6 + c_1c_3c_4c_6 + c_2c_3c_4c_5) = 0,
\end{aligned}$$

where

$$\begin{aligned}
&c_0 = \alpha - \frac{\mathrm{i}\kappa}{4\pi}, \quad c_1 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{12}}}{4\pi\ell_{12}}, \quad c_2 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{13}}}{4\pi\ell_{13}}, \\
&c_3 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{14}}}{4\pi\ell_{14}}, \quad c_4 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{23}}}{4\pi\ell_{23}}, \quad c_5 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{24}}}{4\pi\ell_{24}}, \\
&c_6 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{34}}}{4\pi\ell_{34}}.
\end{aligned}$$

In our special case we have

$$\ell_{12} = \ell_{13}, \quad \ell_{34} = \ell_{24}$$

and

$$\ell_{12} + \ell_{34} > \ell_{14} + \ell_{23}. \tag{5.2}$$

Moreover, using (5.1) we get

$$\begin{aligned}
&\ell_{12} + \ell_{23} + \ell_{34} + \ell_{14} = 2b + c + \sqrt{a^2 + b^2} \\
&\quad + \sqrt{(a-c)^2 + b^2} < 2\sqrt{a^2 + b^2} + 2\sqrt{(a-c)^2 + b^2} = \\
&\ell_{12} + \ell_{34} + \ell_{13} + \ell_{24}.
\end{aligned} \tag{5.3}$$

The elements of the group Π_4 can be decomposed into disjoint cycles as:

$$\begin{aligned}
&\pi_1 = (1)(2)(3)(4), \ \pi_2 = (3,4)(1)(2), \ \pi_3 = (2,3)(1)(4), \\
&\pi_4 = (2,3,4)(1), \quad \pi_5 = (2,4,3)(1), \quad \pi_6 = (2,4)(1)(3), \\
&\pi_7 = (1,2)(3)(4), \quad \pi_8 = (1,2)(3,4), \quad \pi_9 = (1,2,3)(4), \\
&\pi_{10} = (1,2,3,4), \quad \pi_{11} = (1,2,4,3), \quad \pi_{12} = (1,2,4)(3), \\
&\pi_{13} = (1,3,2)(4), \ \pi_{14} = (1,3,4,2), \ \pi_{15} = (1,3)(2)(4), \\
&\pi_{16} = (1,3,4)(2), \ \pi_{17} = (1,3)(2,4), \ \pi_{18} = (1,3,2,4), \\
&\pi_{19} = (1,4,3,2), \quad \pi_{20} = (1,4,2)(3), \ \pi_{21} = (1,4,3)(2), \\
&\pi_{22} = (1,4)(2)(3), \ \pi_{23} = (1,4,2,3), \quad \pi_{24} = (1,4)(2,3).
\end{aligned}$$

Using the above decompositions of permutations and (5.2), (5.3) we find

$$\mathsf{v}_X(\pi_8) = \mathsf{v}_X(\pi_{11}) = \mathsf{v}_X(\pi_{14}) = \mathsf{v}_X(\pi_{17}) > \mathsf{v}_X(\pi_{10}) =$$

$$\mathsf{v}_X(\pi_{18}) = \mathsf{v}_X(\pi_{19}) = \mathsf{v}_X(\pi_{23}) > \ldots > \mathsf{v}_X(\pi_1) = 0.$$

Hence, $V_X = \mathsf{v}_X(\pi_8) = \mathsf{v}_X(\pi_{11}) = \mathsf{v}_X(\pi_{14}) = \mathsf{v}_X(\pi_{17})$ and in view of (5.2) the leading term corresponding to $\exp(\mathrm{i}\kappa V_X)$ in the resonance condition (4.2) cancels

$$\frac{\mathrm{e}^{2\mathrm{i}\kappa(\ell_{12}+\ell_{34})}}{(4\pi)^4\ell_{12}^2\ell_{34}^2} + \frac{\mathrm{e}^{2\mathrm{i}\kappa(\ell_{13}+\ell_{24})}}{(4\pi)^4\ell_{13}^2\ell_{24}^2} - \frac{2\,\mathrm{e}^{\mathrm{i}\kappa(\ell_{12}+\ell_{34}+\ell_{13}+\ell_{24})}}{(4\pi)^4\ell_{12}\ell_{34}\ell_{13}\ell_{24}} = 0.$$

However, the succeeding term in the condition (4.2) corresponding to the exponent $\exp(\mathrm{i}\kappa\mathsf{v}_X(\pi_{10}))$ does not cancel

$$-\frac{2}{(4\pi)^4}\left(\frac{\mathrm{e}^{\mathrm{i}\kappa(\ell_{12}+\ell_{23}+\ell_{34}+\ell_{14})}}{\ell_{12}\ell_{23}\ell_{34}\ell_{14}} + \frac{\mathrm{e}^{\mathrm{i}\kappa(\ell_{13}+\ell_{23}+\ell_{24}+\ell_{14})}}{\ell_{13}\ell_{23}\ell_{24}\ell_{14}}\right) \neq 0.$$

Finally, we end up with

$$W_X = \mathsf{v}_X(\pi_{10}) = \mathsf{v}_X(\pi_{18}) = \mathsf{v}_X(\pi_{19}) = \mathsf{v}_X(\pi_{23}) < V_X.$$

6. Conclusions

In this note, we considered the three-dimensional Schrödinger operator $\mathcal{H}_{\alpha,X}$ with finitely many point interactions of equal strength $\alpha \in \mathbb{R}$ supported on the discrete set X. As the main result, we obtained that the resonance counting function for $\mathcal{H}_{\alpha,X}$ behaves asymptotically linear. The constant coefficient standing by the linear term in this asymptotics can be seen as the effective size of X.

The obtained law of distribution for resonances is very much different from the behaviour of the resonance counting function for the three-dimensional Schrödinger operator with a regular potential [12]. On the other hand, it resembles the corresponding law for one-dimensional Schrödinger operators with regular potentials [11] and for quantum graphs [9, 10].

We associated a complete directed weighted graph G' with the configuration of points X in a natural way. The effective size of X can be estimated from above by the actual size of X defined as the maximal possible total length of an irreducible pseudo-orbit in the graph G'. An implicit formula for finding the effective size of X was given. We also provided examples showing sharpness of our upper bound on the effective size of X. Point configurations for which the effective size of X is strictly smaller than its actual size can be seen as analogues of 'non-Weyl' quantum graphs [9, 10]. The physical experiment which could illustrate mathematical results found in this paper still awaits realization; one of the possibilities could be to use microwave cavities (see e.g. [27]).

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