Special issue of the 3rd International Conference on Computational and Experimental Science and Engineering (ICCESEN 2016)

Characterizations for the Dual Split Quaternionic Curves

A. Tuna Aksoy[a,*] and A.C. Çöken[b]

[a]Süleyman Demirel University, Department of Mathematics, Isparta, Turkey

[b]Akdeniz University, Department of Mathematics, Antalya, Turkey

We define harmonic curves and inclined curves for dual split quaternionic curves. And then, we give some characterizations for dual split quaternionic inclined curves by means of the harmonic curvatures.

DOI: 10.12693/APhysPolA.132.905

PACS/topics: 02.40.k, 02.40.Hw

1. Introduction

The quaternions were first described by Hamilton in 1843. In 1987, the Serret-Frenet formulae for quaternionic curves in E^3 and E^4 were given by Bharathi and Nagaraj [1], and then, Serret-Frenet formulas for a dual quaternionic curve D^3 and D^4 was defined by Sivridağ et al. [2]. Inclined curves and characterization of quaternionic Lorentz manifolds were given by Karadağ [3] and split quaternions were identified with semi-Euclidean space E_2^4, while the vector part of split quaternions was identified with Minkowski 3-space by Inoguchi [4]. After that, Çöken and Tuna in their study [5] have given inclined curves, harmonic curvatures and some characterizations for a quaternionic curve in the semi-Euclidean spaces E_2^4. In 2009, formulas for dual-split quaternionic curves were obtained by Çöken et al. [6].

In this study with the help of Frenet formulas, we give harmonic curvatures and inclined curves for the dual split quaternionic curves. Also, the characterizations are given in terms of harmonic curvatures of curves.

2. Preliminaries

A dual number has the form $a+\xi a^*$, where a and a^* are real numbers and $\xi=(0,1)$ is the dual unit, having property $\xi^2=0$. The set of all dual numbers forms a commutative ring over the real number field denoted by Veldkamp [7].

D^3 dual vector space (ID-module) can be written as $D^3=\{(A_1,A_2,A_3): A_1,A_2,A_3\in D\}$. Similarly, D^4 dual vector space can be written as $D^4=\{(A_1,A_2,A_3,A_4): A_1,A_2,A_3,A_4\in D\}$. The same definitions of inner-product, norm and cross-product hold for D^4. The Lorentzian inner-product of two dual vectors $A=a+\xi a^*$ and $B=b+\xi b^*$, $ab\in\mathbb{R}_1^3$ is given as $<A,B>=<a,b>+\xi(<a^*,b>+<a,b^*>)$ with the signature $(-,+,+)$ in $\mathbb{R}_1^3$. The ID-module D^3 with the Lorentzian inner-product is named as semi-dual space D_1^3. On the other hand, a semi-Euclidean inner-product of two dual vectors in D^4, $A=a+\xi a^*$ and $B=b+\xi b^*$; $a,b\in\mathbb{R}_2^4$, can be defined as

$$<A,B>=<a,b>+\xi(<a^*,b>+<a,b^*>),$$

with the signature $(-,-,+,+)$ in $\mathbb{R}_2^4$. The dual space D^4 with semi-Euclidean inner product is named as semi-dual space D_2^4 or dual-split quaternion [8, 9].

A split quaternion q is an expression of the form $q=ae_1+be_2+ce_3+d$ where a, b, c and d are real numbers, and e_1, e_2, e_3 are split quaternionic units which satisfy the non-commutative multiplication rules $Q_\nu=\{q|q=ae_1+be_2+ce_3+d;\ a,b,c,d\in\mathbb{R}$, $e_1,e_2,e_3\in\mathbb{R}^3_{\nu(\nu=1,2)}$, $h_\nu(e_i,e_i)=-\varepsilon(e_i)$, $1\le i\le 3\}$, where $e_i\times e_i=-\varepsilon(e_i)$, $1\le i\le 3$, $e_i\times e_j=\varepsilon(e_i)\varepsilon(e_j)e_k$ in $\mathbb{R}_1^3$, $e_i\times e_j=-\varepsilon(e_i)\varepsilon(e_j)e_k$ in $\mathbb{R}_2^4$ [5].

A dual split quaternion Q is written as $Q=Ae_1+Be_2+Ce_3+D$. As a consequence of this definition, a dual split quaternion Q can also be written as $Q=q+\xi q^*$, $\xi^2=0$, where $q=ae_1+be_2+ce_3+d$ and $q^*=a^*e_1+b^*e_2+c^*e_3+d^*$ are, respectively, real and dual split quaternion components. Let p and p^* be two semi-real quaternions.

We define the semi-dual quaternion by $P=p+\xi p^*$, and denote the set of semi-dual quaternions by Q_{D_ν} with an index $\nu=1,2$ such, that $Q_{D_\nu}=\{(P|P=Ae_1+Be_2+Ce_3+D;\ A,B,C,D\in ID,\ e_1,e_2,e_3\in\mathbb{R}_1^3,\ H_\nu(e_i,e_i)=-\varepsilon(e_i),\ 1\le i\le 3)\}$.

The multiplication of two dual quaternions P and Q is defined by: $P\times Q=p\times q+\xi(p{\times}q^*+p^*\times q)$, where $P=p+\xi p^*$ and $Q=q+\xi q^*$ and $\times$ shows the quaternion multiplication. It is clear that $P\times Q=S_PS_Q+S_PV_Q+S_QV_P-<V_P,V_Q>+V_P\Lambda V_Q$, where $<,>$ is the inner-product and Λ is the cross-product on ID_1^3. The conjugate of $P=S_P+V_P$ is denoted by $\alpha P=S_P-V_P$.

For every $P,Q\in Q_{D_\nu}$, we define the symmetric dual-valued bilinear form $H_\nu: Q_{D_\nu}\times Q_{D_\nu}\to ID$ by

$$H(P,Q)=\frac{1}{2}[\varepsilon(P)\varepsilon(\alpha Q)(P\times\alpha Q)+\varepsilon(Q)\varepsilon(\alpha P)(Q\times\alpha P)],\ \text{for } D_1^3,$$

$$H(P,Q)=-\frac{1}{2}[\varepsilon(P)\varepsilon(\alpha Q)(P\times\alpha Q)+\varepsilon(Q)\varepsilon(\alpha P)(Q\times\alpha P)],\ \text{for } D_2^4.$$

*corresponding author; e-mail: abidebytr@yahoo.com

The following results may be obtained:

1) For $\forall\, P,Q$ of Q_{D_ν}, we have $H_\nu(P,Q) = h_\nu(p,q) + \xi\,[h_\nu(p,q^*) + h_\nu(p^*,q)]$, where h is the symmetric real-valued bilinear form.

2) If $P = Ae_1 + Be_2 + Ce_3 + D$, then $H_\nu(P,P) = -A^2 - B^2 + C^2 + D^2$.

3) $\forall\, P,Q \in Q_{D_\nu}$ scalar part and vector part of P are $S_P = \frac{1}{2}(P + \alpha P)$, $V_P = \frac{1}{2}(P - \alpha P)$.

The concept of a dual spatial quaternion will be used throughout our work. P is called a semi-dual spatial quaternion whenever $P + \alpha P = 0$. It is a semi-dual temporal quaternion whenever $P - \alpha P = 0$. Let P and Q be two semi-dual spatial quaternions. If $H_\nu(P,Q) = 0$, then P and Q are H_ν-ortogonal [6].

Theorem 2.1. $\beta : I \to Q_{D_1^3}$ regular semi-dual quaternionic curve is given by arc-length parameter s. Let $\{T(s), N_1(s), N_2(s)\}$ be the Frenet trihedron in the point $\beta(s)$ of the curve β, where $K(s)$ and $R(s)$ are curvatures. Then Frenet apparatus are [6]

$$T'(s) = \varepsilon_{N_1} K(s) N_1(s),$$

$$N_1'(s) = \varepsilon_T[\varepsilon_T\varepsilon_{N_1} R N_2(s) - KT(s)],$$

$$N_2'(s) = -\varepsilon_{N_2} R N_1(s).$$

Theorem 2.2. $\bar\beta : I \to Q_{D_2^4}$ regular semi-dual quaternionic curve is given by arc-length parameter s. Let $\{\bar T(s), \bar N_1(s), \bar N_2(s), \bar N_3(s)\}$ be the Frenet trihedron in the point $\bar\beta(s)$ of the curve $\bar\beta$. Then Frenet apparatus are [6],

$$\bar T'(s) = \varepsilon_{\bar N_1} \bar K\, \bar N_1, \tag{1}$$

$$\bar N_1'(s) = \varepsilon_{N_1} K \bar N_2 - \varepsilon_T \varepsilon_{\bar N_1} \bar K \bar T, \tag{2}$$

$$\bar N_2'(s) = -\varepsilon_T K \bar N_1 + \varepsilon_{N_1}[\big(R - \varepsilon_T \varepsilon_{\bar T} \varepsilon_{\bar N_1} \bar K\,\big)]\bar N_3, \tag{3}$$

$$\bar N_3'(s) = -\varepsilon_{N2}[R - \varepsilon_T \varepsilon_{\bar T} \varepsilon_{\bar N_1} \bar K]\bar N_2. \tag{4}$$

Definition 2.3. $\beta : I \to Q_{ID_1^3}$ regular semi-dual quaternionic curve is given by arc-length parameter s, such that u is a constant and unit vector in $Q_{ID_1^3}$ for $\forall s \in I$, let $H(\beta'(s), u)$ be a constant defined by

$$H(\beta'(s), u) = \left\{ \begin{array}{c} \cos\Phi,\ \beta \text{ is spacelike curve} \\ -\cosh\Phi, \beta \text{ is timelike curve} \end{array} \right\} = \text{const}, \varphi \neq \frac{\pi}{2}. \tag{5}$$

Then curve β is called semi-dual spatial quaternionic inclined curve in $Q_{D_1^3}$

Definition 2.4. Let $\gamma : I \to \mathbb{R}_1^3$ be a semi-real quaternionic curve. Such that $\gamma(s) = \gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3$, $\beta : I \to Q_{D_1^3}$,

$$\beta(s) = \gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3 + \xi(\gamma_1^*(s)e_1 + \gamma_2^*(s)e_2 + \gamma_3^*(s)e_3)$$

or $\beta(s) = A(s)e_1 + B(s)e_2 + C(s)e_3$; $A(s), B(s), C(s) \in ID$, obtained from γ, such that $\{T(s), N_1(s), N_2(s)\}$ be the Frenet trihedron in the point $\beta(s)$ of the curve β and let $u = u_0 + \xi u_0^*$ be a unit and constant vector. Then, $\beta'(s) = T_0(s) + \xi T_0^*(s) = T(s)\,\alpha\beta'(s) = -T_0(s) - \xi T_0^*(s)$, $u = u_0 + \xi u_0^*$, $\alpha u = -u_0 - \xi u_0^*$ (u is semi-dual spatial quaternion). Thus, $u, T(s) \in Q_{ID_1^3}$

$$H(u,T) = h(u_0,T_0) + \xi[h(u_0,T_0^*) + h(u_0^* T_0)].$$

Definition 2.5. $\beta : I \to Q_{D_1^3}$ regular semi-dual quaternionic curve is given by arclength parameter s. Let $\{T(s), N_1(s), N_2(s)\}$ be the Frenet trihedron in the point $\beta(s)$ of the curve β and let $u = u_0 + \xi u_0^*$ be a unit and constant vector, such that angle $\Phi = \varphi + \xi\varphi^*$ is between $\beta'(s)$ and u. $H : I \to ID$,

$$H(N_2(s), u) = \bar H(s) H(T(s), u) = \left\{ \begin{array}{c} \bar H \cos\Phi, \beta \text{ is a spacelike curve} \\ -\bar H \cosh\Phi, \beta \text{ is a timelike curve} \end{array} \right\},\ \varphi \neq \frac{\pi}{2} \tag{6}$$

can be defined. Then function H is semi-dual harmonic curvature in the point $\beta(s)$ of curve β with respect to u.

Definition 2.6. $\bar\beta : I \to Q_{D_2^4}$ regular semi-dual quaternionic curve is given by arclength parameter s, such that $u = u_0 + \xi u_0^*$ is a unit and constant semi-dual spatial quaternion for every $s \in I$,

$$H\left(\bar\beta'(s), u\right) = \left\{ \begin{array}{c} \cos\Phi, \bar\beta \text{ is a spacelike curve} \\ -\cosh\Phi, \bar\beta \text{ is a timelike curve} \end{array} \right\} = \text{const},\ \varphi \neq \frac{\pi}{2}. \tag{7}$$

Then $\bar\beta$ is called semi-dual quaternionic inclined curve in semi-quaternion sets $Q_{D_2^4}$.

Definition 2.7. $\bar\beta : I \to Q_{D_2^4}$ regular semi-dual quaternionic curve is given by arc-length parameter s. Let $\{\bar T(s), \bar N_1(s), \bar N_2(s), \bar N_3(s)\}$ be the Frenet apparatus and let u be a unit and constant, such that angle $\Phi = \varphi + \xi\varphi^*$ is between $T(s)$ and u. Let $\bar H_i : I \to ID$ be functions defined by

$$H\left(\bar N_{i+1}, u\right) = \left\{ \begin{array}{c} \bar H_i \cos\Phi, \bar\beta \text{ is a spacelike curve} \\ -\bar H_i \cosh\Phi, \bar\beta \text{ is a timelike curve} \end{array} \right\},\ \varphi \neq \frac{\pi}{2}. \tag{8}$$

Then function $\bar H_i$ is called ith Harmonic curvature in the point $\bar\beta(s)$ of the $\bar\beta$ semi-dual quaternion curve with respect to u. Then, $\bar H_0$ is equal to zero.

3. Harmonic curvatures and characterizations in D_1^3

Theorem 3.1. Let $\beta : I \to Q_{D_1^3}$ be a semi-dual spatial quaternionic inclined curve given by arc-length parameter s. Curvatures at the point $\beta(s)$ of β are $K(s) = k(s) + \xi k^*(s)$, $R(s) = r(s) + \xi r^*(s)$ and in that case $\bar H$ is a harmonic curvature, it is

$$\bar H(s) = \frac{K(s)}{\varepsilon_T \varepsilon_{N_1} R(s)} \tag{9}$$

(see [3] for dual quaternionic curve).

Proof. Let $\Phi = \varphi + \xi\varphi^*$ be an angle between the u constant semi-dual vector and the tangent vectors of $\beta : I \to Q_{ID_1^3}$ semi-dual spatial quaternionic inclined line, such that $\{T(s), N_1(s), N_2(s)\}$ is Frenet trihedron in the point $\beta(s)$. We obtain, that $H(T(s), u) = \text{const}.$

Here, by differentiating with respect to s, we get $H(T'(s),u) = 0$. From Serret-Frenet formulas of β, we obtain that $H(N_1(s),u) = 0$. Here, by differentiating with respect to s, we obtain $H(N_1'(s),u) = 0$. Here, by using the Frenet formulas, $H(N_2(s),u) = \frac{\varepsilon_T K}{\varepsilon_T^2 \varepsilon_{N_1} R} H(T(s),u)$ is obtained. Thus, if Eq. (6) is used, Eq. (9) is found. Thus, harmonic curvature of semi-dual spatial quaternionic curve is obtained from the curvatures of a curve.

Theorem 3.2. Let $\beta : I \to Q_{D_1^3}$ be a semi-dual spatial quaternionic curve, given by arc-length parameter s. Let $\bar{H}(s)$ be harmonic curvature and $\{T(s), N_1(s), N_2(s)\}$ be the Frenet trihedron at the point $\beta(s)$. β is semi-dual quaternionic inclined curve if and only if $\bar{H}^2(s)$ is constant (see [3] for dual spatial quaternionic curve).

Proof. $(\Rightarrow)$ There is a unit semi-dual spatial quaternion u. Therefore, $H(\beta'(s),u) = 0$ is constant for β semi-dual spatial quaternionic inclined curve with respect to arc-length parameter s. If $\{T(s), N_1(s), N_2(s)\}$ is basis of semi-dual spatial quaternion in the point $\beta(s)$, semi dual quaternion u,

$$u = \varepsilon_T H(T(s),u)\,T(s) + \sum_{i=1}^{2} \varepsilon_{N_i} H(N_i(s),u) N_i(s) \tag{10}$$

is obtained. Hence, we have $\|u\|^2 = |H(u,u)| = |\varepsilon_u(u \times \alpha u)|$. With Eq. (6), by taking $\|T(s)\| = |\varepsilon_T|$, $\|N_1(s)\| = |\varepsilon_{N_1}|$, $\|N_2(s)\| = |\varepsilon_{N_2}|$ and $\|u\| = |\varepsilon_u| = 1$ into consideration,

$$\bar{H}^2(s) = \frac{\pm 1 - H^2(T(s),u)\varepsilon_T}{H^2(T(s),u)\varepsilon_{N2}} = \text{const}$$

is obtained.

$(\Leftarrow)$ In contrast, suppose that $\bar{H}^2(s) = a$ is constant for β semi-dual spatial quaternionic curve. Therefore, there is Φ angle such that $\frac{\pm 1 - H^2(T(s),u)\varepsilon_T}{H^2(T(s),u)\varepsilon_{N_2}} = a$. Thus, we define u semi-dual spatial quaternion, where

$$u = \varepsilon_T H(T(s),u)T(s) + \varepsilon_{N2}\bar{H}(s)H(T(s),u)N_2(s). \tag{11}$$

By differentiating Eq. (11) with respect to s

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \varepsilon_T H(T(s),u)K(s)N_1(s) - \varepsilon_{N2}^2 H(T(s),u)\frac{K(s)}{\varepsilon_T \varepsilon_{N_1} R(s)} N_1(s).$$

Here, using the Frenet formulas and by taking Eq. (9) into the consideration $\frac{\mathrm{d}u}{\mathrm{d}s} = 0$ is obtained. Thus u is a constant semi-dual spatial quaternion. Now, if it is given that u semi-dual spatial quaternion is a unit,

$$\|u\|^2 = |H(u,u)| = |\varepsilon_u (u \times \alpha u)| = \left|\varepsilon_u H^2(T(s),u)\left[\varepsilon_T + \bar{H}^2(s)\varepsilon_{N^2}\right]\right| = |\varepsilon_u(\pm 1)| = 1$$

is found. On the other hand, $H(T(s),u) = \text{const}$ is found. Therefore, β is a semi-dual spatial quaternionic inclined curve.

4. Harmonic curvatures and characterizations in D_2^4

Theorem 4.1. Let $\beta : I \to Q_{D_1^3}$ be semi-dual spatial quaternionic inclined curve. Such that $\beta(s) = A(s)e_1 + B(s)e_2 + C(s)e_3$; $A(s), B(s), C(s) \in ID$. $\bar{\beta}(s) = A(s)e_1 + B(s)e_2 + C(s)e_3 + D(s)$, $D(s) \in ID$ is obtained from β. Thus, semi-dual quaternionic curves $\bar{\beta}$ and β are semi-dual quaternionic inclined curves of the same axis (see [4] for dual quaternionic curves).

Proof. Let $\bar{\beta} : I \to Q_{D_2^4}$ be a semi-dual quaternionic curve given by arc-length parameter s. Let u be an unit and a constant semi-dual spatial quaternion, such that $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ is Frenet apparatus in the point $\bar{\beta}(s)$ of curve $\bar{\beta}$. As we know, $H(\bar{\beta}'(s),u) = H(\bar{T}(s),u)$, $\bar{T}(s) = D(s) + T(s, D(s)) = d_0 + \xi d_0^* \in ID$, $T(s) = T_0(s) + T_0^*(s)$, $u = u_0 + \xi u_0^*$ and $\alpha T(s) = D(s) - T(s)$. We obtained that

$$H(\bar{T}(s),u) = -\varepsilon_{\bar{T}}\varepsilon_{\alpha u} h(u_0, T_0) + \xi[h(u_0, T_0^*) + h(u_0^*, T_0)]\},$$

where h is a real semi-quaternionic inner product.

Theorem 4.2. Let $\bar{\beta} : I \to Q_{D_2^4}$ be a semi-dual quaternionic inclined curve given by arc-length parameter s. $k_i(s)$ are curvatures in the point $\bar{\beta}(s)$, $\bar{\delta}_i(s) = \frac{1}{k_i(s)}$, $1 \le i \le 3$, are curvature radii and $\bar{H}_j(s)$, $j = 1, 2$ are harmonic curvatures,

$$\bar{H}_1(s) = \frac{\text{1st curvature}}{\text{2nd curvature}}, \quad \text{and } \bar{H}_2(s) = \bar{H}_1'(s)\,\bar{\delta}_3(s), \tag{12}$$

(see [3] for dual quaternionic curves).

Proof. Let $\bar{\beta} : I \to Q_{D_2^4}$ be a regular semi-dual quaternionic curve. u is an unit and a constant semi-dual spatial quaternion. If $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ is Frenet apparatus in the point $\bar{\beta}(s)$, $H(\bar{T}(s),u) = \text{const}$ is written. By differentiating this equation with respect to s, we obtain that $H(\bar{T}',u) = 0$. Here, using Eq. (1), $H(\bar{N}_1,u) = 0$ is found. By differentiating $H(\bar{N}_1,u) = 0$ with respect to s, $H(\bar{N}_1'(s),u) = 0$ is obtained. Here, using Eq. (2),

$$H(\bar{N}_2,u) = \frac{\varepsilon_{\bar{N}_1}\varepsilon_T \bar{K}}{\varepsilon_{N_1} K} H(\bar{T}(s),u) \tag{13}$$

is found. Here, by taking Eqs. (7) and (8) into consideration, $\bar{H}_1 = \frac{\varepsilon_{\bar{T}}\varepsilon_{\bar{N}_1}\bar{K}}{\varepsilon_{N_1} K} = \frac{\text{1st curvature}}{\text{2nd curvature}}$ is obtained. Here, $\bar{K}$ is the first curvature of $\bar{\beta}$ curve in $Q_{D_2^4}$, $K(s)$ is both the first curvature of β curve in $Q_{D_1^3}$ and the second curvature of $\bar{\beta}$ curve in $Q_{D_2^4}$. On the other hand, if derivative of Eq. (13) with respect to s is taken,

$$H(\bar{N}_2',u) = \bar{H}_1' H(\bar{T}(s),u) \tag{14}$$

is found. Here, using Eq. (3)

$$-\varepsilon_T K H(\bar{N}_1,u) + \varepsilon_{N_1}[R - \varepsilon_T \varepsilon_{\bar{T}} \varepsilon_{\bar{N}_1}\bar{K}]H(\bar{N}_3,u) = -\bar{H}_1' \cosh\Phi \tag{15}$$

is found. If Eq. (8) is written in place of Eq. (15), $\bar{H}_2(s) = \bar{H}_1'(s)\bar{\delta}_3(s)$ is found. Thus, harmonic curvatures for semi-dual spatial quaternionic inclined curves are obtained from curvatures.

Theorem 4.3. Let $\bar{\beta} : I \to Q_{D_2^4}$ be semi dual quaternionic inclined curve, given by arc-length parameter s. If $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ is Frenet apparatus and harmonic curvature $H_i(s)$, $i = 1, 2$, (see [3] for dual quaternionic curve), $\bar{\beta}$ is a inclined line if and only if $\sum_{i=1}^{2} \varepsilon_{\bar{N}_{i+1}} H^2(\bar{N}_{i+1}(s), u)$ is a constant.

Proof ($\Rightarrow$): Suppose that $\bar{\beta} : I \to Q_{D_2^4}$ is an inclined line of semi-dual quaternionic curve. Thus, there is u unit and constant semi-dual quaternion such that $H(\bar{\beta}'(s), u) = \text{const}$, $\forall s \in I$ for $\bar{\beta}$ curve. Suppose that there is a basis of $\bar{\beta}$ curve in the point $\bar{\beta}(s)$. We define the semi-dual quaternion u as

$$u = \varepsilon_{\bar{T}} H(\bar{T}(s), u)\bar{T}(s) + \sum_{i=1}^{3} \varepsilon_{\bar{N}_i} H^2(\bar{N}_i(s), u)\bar{N}_i(s). \tag{16}$$

We have $\|u\|^2 = |H(u, u)| = |\varepsilon_u (u \times \alpha u)| = 1$. Because of $\|\bar{T}(s)\| = |\varepsilon_{\bar{T}}|$, $\|\bar{N}_1(s)\| = |\varepsilon_{\bar{N}_1}|$, $\|\bar{N}_2(s)\| = |\varepsilon_{\bar{N}_2}|$, $\|\bar{N}_3(s)\| = |\varepsilon_{\bar{N}_3}|$, $\|u\| = |\varepsilon_u| = 1$,

$$\sum_{i=1}^{2} \varepsilon_{\bar{N}_{i+1}} H^2(\bar{N}_{i+1}(s), u) = \pm 1 - \varepsilon_{\bar{T}} H^2(\bar{T}(s), u) = \text{const}$$

is obtained.

($\Leftarrow$) Suppose that $\sum_{i=1}^{2} \varepsilon_{\bar{N}_{i+1}} H^2(\bar{N}_{i+1}(s), u) = a$ is a constant for $\bar{\beta} : I \to Q_{ID_2^4}$ semi-dual quaternion curve. Therefore, there is a dual angle Φ, such that $\pm 1 - \varepsilon_{\bar{T}} H^2(\bar{T}(s), u) = a$. Thus, we defined that u is semi-dual spatial quaternion, such that

$$u = \varepsilon_{\bar{T}} H(\bar{T}(s), u)\bar{T}(s) + \sum_{i=2}^{3} \varepsilon_{\bar{N}_i} \bar{H}_{i-1}(s) H(\bar{T}(s), u)\bar{N}_i(s). \tag{17}$$

Here, we demonstrate that u is a constant. Thus, if derivative of Eq. (17) with respect to s is taken, and here, using Eq. (8) for $i = 1, 2$, $H(\bar{N}_2(s), u) = \bar{H}_1 H(\bar{T}(s), u)$ and $H(\bar{N}_3(s), u) = \bar{H}_2 H(\bar{T}(s), u)$ is obtained. If derivative of the last equation with respect to s is taken, $H(\bar{N}_3'(s), u) = \bar{H}_2' H(\bar{T}(s), u)$ is found. Here, using Eq. (4),

$$-\varepsilon_{N_2}(R - \varepsilon_T \varepsilon_{\bar{T}} \varepsilon_{\bar{N}_1} \bar{K}) H(\bar{N}_2, u) = \bar{H}_2' H(\bar{T}(s), u)$$

is obtained. Here, by considering the $H(\bar{N}_2(s), u) = \bar{H}_1 H(\bar{T}(s), u)$

$$\bar{H}_2'(s) = -\varepsilon_{N_2}(R - \varepsilon_T \varepsilon_{\bar{T}} \varepsilon_{\bar{N}_1} \bar{K}) \bar{H}_1(s) \tag{18}$$

is obtained. By taking the derivative of Eq. (17) and using the last equations, $\frac{\mathrm{d}u}{\mathrm{d}s} = 0$ is found. Thus, u is a constant. On the other hand, we demonstrate that u is an unit. That is, $\|u\|^2 = |H(u, u)| = |\varepsilon_u (u \times \alpha u)| = 1$ is obtained. Thus, $H(\bar{T}(s), u) = \text{const}$ is found. Therefore, $\bar{\beta}$ is an inclined curve.

Corollary 4.4. Derivative equations of harmonic curvatures obtained for semi-dual quaternionic curves by the aid of Eq. (18) and the Theorem (4.2.),

$$\begin{bmatrix} \bar{H}_1' \\ \bar{H}_2' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_{N_1} F \\ -\varepsilon_{\bar{N}_2} F & 0 \end{bmatrix} \begin{bmatrix} \bar{H}_1 \\ \bar{H}_2 \end{bmatrix}$$

is found in the matrix form, where $F = (R - \varepsilon_T \varepsilon_{\bar{T}} \varepsilon_{\bar{N}_1} \bar{K})$.

5. Conclusions

In this paper, we have studied the differential geometry of smooth curves in the semi-dual spaces D_1^3 and D_2^4. We gave new characterizations for dual quaternionic curves in the semi-dual spaces D_1^3 and D_2^4 using their harmonic curvature functions. These functions are

(i) $\beta : I \to Q_{D_1^3}$, which is a semi-dual quaternionic inclined curve if and only if $\bar{H}^2(s)$ is constant. Here $\bar{H}$ is harmonic curvature function of the curve β.

(ii) $\bar{\beta}$ is an inclined line if and only if $\sum_{i=1}^{2} \varepsilon_{\bar{N}_{i+1}} H^2(\bar{N}_{i+1}(s), u)$ is a constant. Here $\bar{H}_1$ and $\bar{H}_2$ are harmonic curvature functions of the curve $\bar{\beta}$.

Acknowledgments

This study was supported by the SDU Funding Agency under Grant 2059-D-09.

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