Characterizations for the Dual Split Quaternionic Curves

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We define harmonic curves and inclined curves for dual split quaternionic curves. And then, we give some characterizations for dual split quaternionic inclined curves by means of the harmonic curvatures.

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1. Introduction

The quaternions were first described by Hamilton in 1843. In 1987, the Serret-Frenet formulae for quaternionic curves in $E^3$ and $E^4$ were given by Bharathi and Nagaraj \cite{1}, and then, Serret-Frenet formulas for a dual quaternionic curve $D^3$ and $D^4$ was defined by Sirvizdağ et al. \cite{2}. Inclined curves and characterization of quaternionic Lorentz manifolds were given by Karadağ \cite{3} and split quaternions were identified with semi-Euclidean spaces. Similarly, we defined harmonic curves and some characterizations for a quaternionic curve in the semi-Euclidean spaces $E^2_1$. In 2009, formulas for dual-split quaternionic curves were obtained by Çöken et al. \cite{6}.

In this study with the help of Frenet formulas, we give characterizations for dual split quaternionic curves. And then, we give some characterizations for dual split quaternionic inclined curves by means of the harmonic curvatures.

2. Preliminaries

A dual number has the form $a + \xi a^*$, where $a$ and $a^*$ are real numbers and $\xi = (0, 1)$ is the dual unit, having property $\xi^2 = 0$. The set of all dual numbers forms a commutative ring over the real number field denoted by $\mathbb{D}$ \cite{7}.

$D^3$ dual vector space (ID-module) can be written as $D^3 = \{(A_1, A_2, A_3) : A_1, A_2, A_3 \in D\}$. Similarly, $D^4$ dual vector space can be written as $D^4 = \{(A_1, A_2, A_3, A_4) : A_1, A_2, A_3, A_4 \in D\}$. The same definitions of inner-product, norm and cross-product hold for $D^4$. The Lorentzian inner-product of two dual vectors $A = a + \xi a^*$ and $B = b + \xi b^*$, $ab \in D^3$ is given as $< A, B > = < a, b > + \xi (\xi a^*, b > + < a, b^*>)$ with the signature $(-, +, +)$ in $D^3$. The ID-module $D^4$ with the Lorentzian inner-product is named as semi-dual space $D^4$. On the other hand, a semi-Euclidean inner-product of two dual vectors in $D^4$, $A = a + \xi a^*$ and $B = b + \xi b^*$, $a, b \in \mathbb{R}^2$, can be defined as $< A, B > = < a, b > + \xi (\xi < a^*, b > + < a, b^* >)$, with the signature $(-, -, +, +)$ in $D^3$. The dual space $D^4$ with semi-Euclidean inner product is named as semi-dual space $D^4$ or dual-split quaternion \cite{8,9}.

A split quaternion $q$ is an expression of the form $q = ae_1 + be_2 + ce_3 + d$ where $a$, $b$, $c$ and $d$ are real numbers, and $e_1$, $e_2$, $e_3$ are split quaternion units which satisfy the non-commutative multiplication rules $Q_\nu = \{a, b, c, d : a, b, c, d \in D, e_1, e_2, e_3 \in D, h_\nu (e_i, e_j) = -\varepsilon (e_i) \varepsilon (e_j), 1 \leq i, j \leq 3\}$, where $e_i \times e_j = -\varepsilon (e_i) \varepsilon (e_j) e_k$ in $D^3$. As a consequence of this definition, a dual split quaternion $Q$ can also be written as $Q = q + \xi q^*$, $\xi^2 = 0$, where $q = ae_1 + be_2 + ce_3 + d^*$ and $q^* = a^* e_1 + b^* e_2 + c^* e_3 + d^*$ are, respectively, real and dual split quaternion components. Let $p$ and $p^*$ be two semi-real quaternions.

We define the semi-dual quaternion by $P = p + \xi p^*$, and denote the set of semi-dual quaternions by $Q_{D^4}$ with an index $a = 1, 2$ such that $Q_{D^4_{\nu}} = \{(p)P = p \in D^4_{\nu} + C e_3 + D, A, B, C, D \in ID, e_1, e_2, e_3 \in D^3, H_{\nu} (e_i, e_j) = -\varepsilon (e_i), 1 \leq i \leq 3\}$.

The multiplication of two dual quaternions $P$ and $Q$ is defined by: $P \times Q = p \times q + \xi (pq^* + p^* q)$, where $P = p + \xi p^*$ and $Q = q + \xi q^*$ and shows the quaternion multiplication. It is clear that $P \times Q = SPQ + SPVQ + S QVQ + V QVQ > + V P Q$, where $\langle, \rangle$ is the inner-product and $A$ is the cross-product on $D^3$. The conjugate of $P = SP + V P$ is denoted by $a \in D_+ = SV - V P$. For every $P, Q \in Q_\nu$, we define the symmetric dual-valued bilinear form $H_{\nu} : Q_{D^4} \times Q_{D^4} \to ID$ by

- $H(P, Q) = \frac{1}{2} \varepsilon (P) \varepsilon (Q) (P \times aQ) + \varepsilon (Q) \varepsilon (aP) (Q \times aP)$, for $D^4$
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Theorem 2.1. \( \beta : I \to Q_{D^3} \) regular semi-dual quaternionic curve is given by arc-length parameter \( s \). Let \( \{ T(s), N_1(s), N_2(s) \} \) be the Frenet trihedron in the point of the curve \( \beta(s) \), where \( K(s) \) and \( R(s) \) are curvatures. Then Frenet apparatus are [6]

\[
T'(s) = \varepsilon_{N_1} K N_1(s),
N'_1(s) = \varepsilon_T [\varepsilon_T \varepsilon_{N_1} R N_2(s) - KT(s)],
N'_2(s) = -\varepsilon_N R N_1(s).
\]

Theorem 2.2. \( \beta : I \to Q_{D^3} \) regular semi-dual quaternionic curve is given by arc-length parameter \( s \). Let \( \{ T(s), N_1(s), N_2(s), N_3(s) \} \) be the Frenet trihedron in the point of the curve \( \beta(s) \). Then Frenet apparatus are [6]

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N'_2(s) = -\varepsilon_N R N_1(s),
\]

Definition 2.3. \( \beta : I \to Q_{D^3} \) regular semi-dual quaternionic curve is given by arc-length parameter \( s \), such that \( u \) is a constant and unit vector in \( Q_{ID} \) for all \( s \in I \), let \( H(\beta'(s), u) \) be a constant defined by

\[
H(\beta'(s), u) = \begin{cases} 
\cos \Phi, \beta \text{ is spacelike curve} \\
-\cosh \Phi, \beta \text{ is timelike curve}
\end{cases}
\]

\[
\text{const, } \varphi \neq \frac{\pi}{2}
\]

Then \( H(\beta'(s), u) \) is a semi-dual spatial semi-uniform inclined curve in \( Q_{D^3} \).

Theorem 2.4. Let \( \gamma : I \to R^3_1 \) be a semi-real quaternionic curve. Such that \( \gamma(s) = \gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3 \), \( \beta : I \to Q_{D^3} \)

\[
\beta(s) = \gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3 + \xi(\gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3)
\]

or \( \beta(s) = A(s)e_1 + B(s)e_2 + C(s)e_3 \), \( A(s), B(s), C(s) \in ID \), obtained from \( \gamma \), such that \( \{ T(s), N_1(s), N_2(s) \} \) be the Frenet trihedron in the point of the curve \( \beta(s) \) and let \( u = u_0 + \xi u_0^* \) be a unit and constant vector. Then, \( \beta'(s) = T_0(s) + \xi T_0^*(s) = T(s) \alpha \beta'(s) = -T_0(s) - \xi T_0^*(s), u = u_0 + \xi u_0^* \), \( \alpha u = -u_0 - \xi u_0^* \) (u is semi-dual spatial quaternion). Thus, \( u, T(s) \in Q_{ID} \)

\[
H(u, T) = h(u_0, T_0) + \xi [h(u_0, T_0^*) + h(u_0^* T_0)].
\]

Definition 2.5. \( \beta : I \to Q_{D^3} \) regular semi-dual quaternionic curve is given by arc-length parameter \( s \). Let \( \{ T(s), N_1(s), N_2(s) \} \) be the Frenet trihedron in the point of the curve \( \beta(s) \), where \( \Phi = \varphi + \xi \varphi^* \) is between \( \beta(s) \) and \( u \). \( H : I \to ID \)

\[
H(N_2(s), u) = H(s) \text{H}(T(s), u) = \begin{cases} 
\cos \Phi, \beta \text{ is a spacelike curve} \\
-\cosh \Phi, \beta \text{ is a timelike curve}
\end{cases}, \varphi \neq \frac{\pi}{2}
\]

can be defined. Then function \( H \) is semi-dual harmonic curvature in the point \( \beta(s) \) of curve \( \beta \) with respect to \( u \).

Definition 2.6. \( \beta : I \to Q_{D^3} \) regular semi-dual quaternionic curve is given by arc-length parameter \( s \), such that \( u = u_0 + \xi u_0^* \) is a unit and constant semi-dual spatial quaternion for every \( s \in I \)

\[
H(\beta'(s), u) = \begin{cases} 
\cos \Phi, \beta \text{ is a spacelike curve} \\
-\cosh \Phi, \beta \text{ is a timelike curve}
\end{cases}
\]

\[
\text{const, } \varphi \neq \frac{\pi}{2}
\]

Then \( \beta \) is called semi-dual quaternionic inclined curve in semi-quaternion sets \( Q_{D^3} \).

Definition 2.7. \( \beta : I \to Q_{D^3} \) regular semi-dual quaternionic curve is given by arc-length parameter \( s \). Let \( \{ T(s), N_1(s), N_2(s), N_3(s) \} \) be the Frenet apparatus and let \( u \) be a unit and constant, such that angle \( \Phi = \varphi + \xi \varphi^* \) is between \( T(s) \) and \( u \). \( H \) is defined by

\[
H(N_{s+1}, u) = \begin{cases} 
\bar{H} \cos \Phi, \beta \text{ is a spacelike curve} \\
-\bar{H} \cosh \Phi, \beta \text{ is a timelike curve}
\end{cases}, \varphi \neq \frac{\pi}{2}
\]

Then function \( H \) is called \( i \)th Harmonic curvature in the point \( \beta(s) \) of the semi-dual quaternion curve with respect to \( u \). Then, \( \bar{H}_0 \) is equal to zero.

3. Harmonic curvatures and characterizations in \( D^3_1 \)

Theorem 3.1. Let \( \beta : I \to Q_{D^3} \) be a semi-dual spatial quaternionic inclined curve given by arc-length parameter \( s \). Curvatures at the point \( \beta(s) \) of \( \beta \) are \( K(s) = k(s) + \xi k^*(s) \), \( R(s) = r(s) + \xi r^*(s) \) and in that case \( H \) is a harmonic curvature, it is

\[
\bar{H}(s) = \frac{K(s)}{\varepsilon_T \varepsilon_{N_1}(R(s))}
\]

(see [3] for dual quaternionic curve).

Proof. Let \( \Phi = \varphi + \xi \varphi^* \) be an angle between the \( u \) constant semi-dual vector and tangent vectors of \( \beta : I \to Q_{ID} \) semi-dual spatial quaternionic inclined line, such that \( \{ T(s), N_1(s), N_2(s) \} \) is Frenet trihedron in the point \( \beta(s) \). We obtain, that \( H(T(s), u) = \text{const.} \)
Here, by differentiating with respect to $s$, we get $H(T'(s), u) = 0$. From Serret-Frenet formulas of $β$, we obtain that $H(N_1(s), u) = 0$. Here, by differentiating with respect to $s$, we obtain $H(N_1'(s), u) = 0$. Here, by using the Frenet formulas, $H(N_2(s), u) = \frac{ε_T}{ε_T + ε_N}H(T(s), u)$ is obtained. Thus, if Eq. (6) is used, Eq. (9) is found. Thus, harmonic curvature of semi-dual spatial quaternionic curve is obtained from the curvatures of a curve.

**Theorem 3.2.** Let $β : I \to D^3_2$ be a semi-dual spatial quaternionic curve, given by arc-length parameter $s$. Let $H(s)$ be harmonic curvature and $(T(s), N_1(s), N_2(s))$ be the Frenet trihedron at the point $β(s)$. $β$ is semi-dual quaternionic inclined curve if and only if $H^2(s)$ is constant (see [3] for dual spatial quaternionic curve).

**Proof.** ($\Rightarrow$) There is a unit semi-dual spatial quaternion $u$. Therefore, $H(β'(s), u) = 0$ is constant for $β$ semi-dual spatial quaternionic inclined curve with respect to arc-length parameter $s$. If $(T(s), N_1(s), N_2(s))$ is basis of semi-dual spatial quaternion in the point $β(s)$, semi dual quaternion $u$,

$$u = ε_T H(T(s), u) T(s) + \sum_{i=1}^{2} ε_N_i H(N_i(s), u) N_i(s)$$

is obtained. Hence, we have $∥u∥^2 = |H(u, u)| = |ε_u (u × au)|$. With Eq. (6), by taking $∥T(s)∥ = |ε_T|$, $∥N_1(s)∥ = |ε_N_1|$, $∥N_2(s)∥ = |ε_N_2|$ and $∥u∥ = |ε_u| = 1$ into consideration,

$$H^2(s) = \frac{1 - H^2(T(s), u)ε_T}{H^2(T(s), u)ε_N_2} = \text{const}$$

is obtained.

($\Leftarrow$) In contrast, suppose that $H^2(s) = a$ is constant for $β$ semi-dual spatial quaternionic curve. Therefore, there is an angle such that $\frac{1 - H^2(T(s), u)ε_T}{H^2(T(s), u)ε_N_2} = a$. Thus, we define $u$ semi-dual spatial quaternion, where

$$u = ε_T H(T(s), u) T(s) + ε_N_2 H(N_2(s), u) N_2(s),$$

By differentiating Eq. (11) with respect to $s$

$$\frac{du}{ds} = ε_T H(T(s), u) K(s) N_1(s) - ε_N_2 H(T(s), u) \frac{K(s)}{ε_T ε_N_1 R(s)} N_1(s).$$

Here, using the Frenet formulas and by taking Eq. (9) into the consideration $\frac{dv}{ds} = 0$ is obtained. Thus $u$ is a constant semi-dual spatial quaternion. Now, if it is given that $u$ semi-dual spatial quaternion is a unit, $∥u∥^2 = |H(u, u)| = |ε_u (u × au)| = |ε_u H^2(T(s), u)| = |ε_u + 1| = 1$ is found. On the other hand, $H(T(s), u) = \text{const}$ is found. Therefore, $β$ is a semi-dual spatial quaternion inclined curve.

4. Harmonic curvatures and characterizations in $D^3_2$

**Theorem 4.1.** Let $β : I \to D^3_2$ be semi-dual spatial quaternionic curve. Such that $β(s) = A(s)e_1 + B(s)e_2 + C(s)e_3; A(s), B(s), C(s) \in ID$. $β(s) = A(s)e_1 + B(s)e_2 + C(s)e_3 + D(s), D(s) \in ID$ is obtained from $β$. Thus, semi-dual quaternionic curvatures $β$ and $β$ are semi-dual quaternionic inclined curves of the same axis (see [4] for dual quaternionic curves).

**Proof.** Let $β : I \to D^3_2$ be a semi-dual quaternionic curve given by arc-length parameter $s$. Let $u$ be an unit and a constant semi-dual spatial quaternion, such that $(T(s), N_1(s), N_2(s), N_3(s))$ is Frenet apparatus in the point $β(s)$ of curve $β$. As we know, $H(β'(s), u) = H(T(s), u)$, $T(s) = D(s) + T(s)$, $D(s) = d_0 + ε_i d_0 ∈ ID$, $T(s) = T_0(s) + T_0'(s)$, $u = u_0 + ε_i u_0'. T(s)$ is found. We obtained that

$$H(T(s), u) = -ε_T ε_α h(u_0, T_0) + ξ[h(u_0, T_0) + h(u_0), T_0],$$

where $h$ is a real semi-quaternionic inner product.

**Theorem 4.2.** Let $β : I \to D^3_2$ be a semi-dual quaternionic inclined curve given by arc-length parameter $s$. $k_i(s)$ are curvatures in the point $β(s)$, $δ_i(s) = \frac{1}{k_i(s)}$. $1 ≤ i ≤ 3$, are curvature radii and $H_j(s), j = 1, 2$ are harmonic curvatures,

$$H_1(s) = \frac{1}{k_1(s)},$$

and $H_2(s) = H_1'(s) δ_3(s)$, (12)

(see [3] for dual quaternionic curves).

**Proof.** Let $β : I \to D^3_2$ be a regular semi-dual quaternionic curve. $u$ is an unit and a constant semi-dual spatial quaternion. If $(T(s), N_1(s), N_2(s), N_3(s))$ is Frenet apparatus in the point $β(s)$, $H(T(s), u) = \text{const}$ is written. By differentiating this equation with respect to $s$, we obtain that $H(T'(s), u) = 0$. Here, using Eq. (1), $H(N_1(s), u) = 0$ is found. By differentiating $H(N_1, u) = 0$ with respect to $s$, $H(N_1'(s), u) = 0$ is obtained. Here, using Eq. (2),

$$H(N_2(s), u) = \frac{ε_N_1 ε_T K}{ε_N_1 K} H(T(s), u)$$

(13) is found. Here, by taking Eqs. (7) and (8) into consideration, $H_1 = \frac{ε_T ε_N_1 K}{ε_N_1 K}$ is first curvature of $β$ curve in $D^3_2$, $K(s)$ is both the first curvature of $β$ curve in $D^3_1$ and the second curvature of $β$ curve in $D^3_2$. On the other hand, if derivative of Eq. (13) with respect to $s$ is taken,

$$H(N_2'(s), u) = H_1'(s) H(T(s), u)$$

(14) is found. Here, using Eq. (3),

$$-ε_T K H(N_1, u) + ε_N_1[R - ε_T ε_N_1 K] H(N_3, u) = -H_1' \cosh Φ$$

(15) is found. If Eq. (8) is written in place of Eq. (15), $H_2(s) = H_1'(s) δ_3$ is found. Thus, harmonic curvatures for semi-dual spatial quaternionic inclined curves are obtained from curvatures.
Theorem 4.3. Let $\tilde{\beta} : I \to Q_{D_2^1}$ be semi dual quaternionic inclined curve, given by arc-length parameter $s$. If $\{T(s), N_1(s), N_2(s), N_3(s)\}$ is Frenet apparatus and harmonic curvature $H_i(s), i = 1, 2$, (see [3] for dual quaternionic curve), $\tilde{\beta}$ is a inclined line if and only if
$$\sum_{i=1}^{3} \varepsilon N_i H^2(N_i + 1)(s, u) = \text{const.}$$

Proof ($\Rightarrow$): Suppose that $\tilde{\beta} : I \to Q_{D_2^1}$ is an inclined line of semi-dual quaternionic curve. Thus, there is $u$ unit and constant semi-dual quaternion such that $H(\tilde{\beta}'(s), u) = \text{const}, \forall s \in I$ for $\beta$ curve. Suppose that there is a basis of $\beta$ curve in the point $\beta(s)$. We define the semi-dual quaternion $u$ as
$$u = e_{\bar{u}} H(T(s), u)T(s)$$
$$+ \sum_{i=1}^{3} \varepsilon N_i H^2(N_i + 1)(s, u)N_i(s).$$ (16)

We have $\|u\|^2 = |H(u, u)| = |\varepsilon u (u \times cu)| = 1$. Because of $\|\varepsilon T(s)\| = |\varepsilon T|$, $\|N_1(s)\| = |\varepsilon N_1|$, $\|N_2(s)\| = |\varepsilon N_2|$, $\|N_3(s)\| = |\varepsilon N_3|$, $\|u\| = |\varepsilon u| = 1$,
$$\sum_{i=1}^{3} \varepsilon N_i H^2(N_i + 1)(s, u) =$$
$$\pm1 - e_{\bar{u}} H^2(T(s), u) = \text{const}$$
is obtained.

($\Leftarrow$) Suppose that $\sum_{i=1}^{3} \varepsilon N_i H^2(N_i + 1)(s, u) = a$ is a constant for $\tilde{\beta} : I \to Q_{D_2^1}$ semi-dual quaternion curve. Therefore, there is a dual angle $\Phi$, such that $\pm1 - e_{\bar{u}} H^2(T(s), u) = a$. Thus, we defined that $u$ is a semi-dual spatial quaternion, such that
$$u = e_{\bar{u}} H(T(s), u)T(s)$$
$$+ \sum_{i=1}^{3} \varepsilon N_i H_{i-1}(s)H(T(s), u)N_i(s).$$ (17)

Here, we demonstrate that $u$ is a constant. Thus, if derivative of Eq. (17) with respect to $s$ is taken and here, using Eq. (8) for $i = 1, 2$, $H(N_2(s), u) = H_1 H(T(s), u)$ and $H(N_3(s), u) = H_2 H(T(s), u)$ is obtained. If derivative of the last equation with respect to $s$ is taken, $H(N_1(s), u) = H_3 H(T(s), u)$ is found. Here, using Eq. (4),
$$-\varepsilon N_2(R - e_{\bar{T}} e_{\bar{u}} N_1 K) H(N_2, u) = H_2 H(T(s), u)$$
is obtained. Here, by considering the $H(N_2, u) = H_1 H(T(s), u)$
$$H_2(s) = -\varepsilon N_2(R - e_{\bar{T}} e_{\bar{u}} N_1 K) H_1(s)$$ (18)
is obtained. By taking the derivative of Eq. (17) and using the last equations, $\frac{du}{ds} = 0$ is found. Thus, $u$ is a constant. On the other hand, we demonstrate that $u$ is an unit. That is, $\|u\|^2 = |H(u, u)| = |\varepsilon u (u \times cu)| = 1$ is obtained. Thus, $H(T(s), u) = \text{const}$ is found. Therefore, $\tilde{\beta}$ is an inclined curve.

Corollary 4.4. Derivative equations of harmonic curvatures obtained for semi-dual quaternionic curves by the aid of Eq. (18) and the Theorem (4.2),
$$\begin{bmatrix}
H_1' \\
H_2'
\end{bmatrix} =
\begin{bmatrix}
0 & \varepsilon N_1 F' \\
-\varepsilon N_1 F & 0
\end{bmatrix}
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix}$$
is found in the matrix form, where $F = (R - e_{\bar{T}} e_{\bar{u}} N_1 K)$.

5. Conclusions

In this paper, we have studied the differential geometry of smooth curves in the semi-dual spaces $D_1^2$ and $D_2^2$. We gave new characterizations for dual quaternionic curves in the semi-dual spaces $D_1^2$ and $D_2^2$ using their harmonic curvature functions. These functions are

(i) $\beta : I \to Q_{D_1^1}$, which is a semi-dual quaternionic inclined curve if and only if $H^2(s)$ is constant. Here $H$ is harmonic curvature function of the curve $\beta$.

(ii) $\tilde{\beta}$ is an inclined line if and only if $\sum_{i=1}^{3} \varepsilon N_i H^2(N_i + 1)(s, u)$ is a constant. Here $H_1$ and $H_2$ are harmonic curvature functions of the curve $\beta$.

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