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On the Dual Quaternionic $\bar{N}_3$ Slant Helices in D^4

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In this paper, we define the harmonic curvature functions for dual quaternionic curves. Moreover, we also study some characterizations for dual quaternionic slant helices according to dual quaternionic frame.

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1. Introduction

The quaternions were first introduced by Hamilton in 1843. The theory of Frenet frames for a quaternionic curve and dual quaternionic curve has been studied and developed by several researchers in this field [1–3, 5]. In 1987, the Frenet formulae for quaternionic curves in E^3 and E^4 were given by Bharathi and Nagaraj [5], and then many studies have been published on the quaternionic curves, using this study. Some of these Frenet formulas, in D^3 and D^4 dual spaces, had been defined by Sivridağ [3]. Characterization of quaternionic Lorentz manifolds was given in 1999 by Karadağ [6], and Frenet formulas for quaternionic curves in semi-Euclidean space have been defined by Çöken and Tuna's study [2], in which they gave inclined curves, harmonic curvatures and some characterizations for a quaternionic curve in the semi-Euclidean spaces E^4_2. After that, in 2009, formulas for dual-split quaternionic curves were obtained by Çöken et al [1]. And then, in 2011, Gök et al. gave a new kind of slant helix in Euclidean 4-space E^4 and obtained some characterizations for the quaternionic slant helices in terms of the harmonic curvatures.

The main goal of this paper is to define slant helix for dual quaternionic curves in D^4. Here, by using the idea similar to that of Gök and et al. [5], we show that a dual quaternionic curve is a dual quaternionic slant helix if and only if $H_2' - \bar{K} H_1 = 0$, where $\{H_1, H_2\}$ denotes the harmonic curvature functions and $\bar{K}$ is the principal curvature function of the dual quaternionic curve.

2. Preliminaries

Let Q_H denotes a four dimensional vector space over the field H of characteristic grater than 2. Let e_i, $(1 \le i \le 4)$ denote a basis for the vector space. Let the rule of multiplication on Q_H be defined on e_i, $(1 \le i \le 4)$ and extended to the entire vector space by distributivity as follows.

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The set of the real quaternions is defined by $Q_H = \{q | q = ae_1 + be_2 + ce_3 + de_4;\ a, b, c, d \in \mathbb{R},\ e_1, e_2, e_3 \in \mathbb{R}^3,\ e_4 = 1,\ 1 \le i \le 3\}$, where $e_i \times e_i = -1$, $e_i \times e_j = e_k = -(e_j \times e_i)$, $1 \le i \le 3$ and (ijk) is an even permutation of (123).

The multiplication of two real quaternions p and q is defined by $p \times q = S_p S_q + S_p V_q + S_q V_p + h(V_p, V_q) + V_p \Lambda V_q$ for every $p, q \in Q_H$. Here we have used the inner and cross products in Euclidean space $\mathbb{R}^3$. For a real quaternion $q = ae_1 + be_2 + ce_3 + d \in Q_H$ the conjugate αq of q is defined by $\alpha q = -ae_1 - be_2 - ce_3 + d$. This defines the symmetric, real valued, non-degenerate, bilinear form h as follows:

$$h(p, q) = \frac{1}{2} \left[p \times \alpha q + q \times \alpha p \right].$$

And then, the norm of real quaternion q is denoted by $\|q\|^2 = |h(q, q)| = |a^2 + b^2 + c^2 + d^2|$.

The concept of a spatial quaternion will be used throughout our work. q is called a spatial quaternion, whenever $q + \alpha q = 0$. It is a temporal quaternion, whenever $q - \alpha q = 0$ [2, 4, 5].

A dual number has the form $a + \xi a^*$ where a and a^* are real numbers and $\xi = (0, 1)$ is the dual unit having property $\xi^2 = 0$. The set of all dual numbers forms a commutative ring over the real number field and is denoted by Veldkamp [7].

ID^3 dual vector space (ID-module) can be written as $ID^3 = \{(A1, A2, A3) : A1, A2, A3 \in ID\}$

Similarly, ID^4 dual vector space can be written as $ID4 = \{(A_1, A_2, A_3, A_4) : A_1, A_2, A_3, A_4 \in ID$. A dual quaternion Q is written as $Q = Ae_1 + Be_2 + Ce_3 + D$. As a consequence of this definition, a dual quaternion Q can also be written as $Q = q + \xi q^*$, $\xi^2 = 0$, where $q = ae_1 + be_2 + ce_3 + d$ and $q^* = a^* e_1 + b^* e_2 + c^* e_3 + d^*$ are, respectively, real and dual quaternion components. Let p and p^* be two real quaternions. We define the dual quaternion by $P = p + \xi p^*$, and denote the set of dual quaternions by Q_{ID}, such that

$$Q_{ID} = \{P | P = Ae_1 + Be_2 + Ce_3 + D;$$
$$A, B, C, D \in ID,\ e_1, e_2, e_3 \in \mathbb{R}^3,$$

where $e_i \times e_i = -1$, $e_i \times e_j = e_k = -(e_j \times e_i)$, $1 \le i \le 3$.

The multiplication of two dual quaternions P and Q is defined by:

$$P \times Q = p \times q + \xi(p \times q^* + p^* \times q),$$

where $P = p + \xi p^*$ and $Q = q + \xi q^*$ and $\times$ shows the dual quaternion multiplication. It is clear that

$$P \times Q = S_P S_Q + S_P V_Q$$
$$+ S_Q V_P - < V_P V_Q > + V_P \Lambda V_Q,$$

where $<,>$ is the inner-product and Λ is the cross-product on D^3. The conjugate of $P = S_P + V_P$ is denoted by $\alpha P = S_P - V_P$. For $\forall\ P, Q \in Q_{ID}$, we define the symmetric dual-valued bilinear form $H : Q_{ID} \times Q_{ID} \to ID$ by $H(P,Q) = \frac{1}{2}[P \times \alpha Q + Q \times \alpha P]$.

The following results may be obtained:

1) For $\forall\ P, Q$ of Q_{ID} we have $H(P,Q) = h(p,q) + \xi[h(p,q^*) + h(p^*,q)]$, where h is the symmetric real-valued bilinear form,

2) If $P = Ae_1 + Be_2 + Ce_3 + D$, then $H(P,P) = A^2 + B^2 + C^2 + D^2$,

3) $\forall\ P, Q \in Q_{ID}$ scalar part and vector part of P are $S_P = \frac{1}{2}(P + \alpha P)$, $V_P = \frac{1}{2}(P - \alpha P)$.

The concept of a dual spatial quaternion will be used throughout our work. P is called a dual spatial quaternion whenever $P + \alpha P = 0$. It is a dual temporal quaternion whenever $P - \alpha P = 0$. Let P and Q be two dual spatial quaternions. If $H(P,Q) = 0$, then P and Q are H-ortogonal [1, 3, 6].

Theorem 2.1. Let us consider the smooth curve $\beta \subset ID^3$ given by $\beta(s) = A(s)e_1 + B(s)e_2 + C(s)e_3$. Let s be the paramerter along the smooth curve β and dual tangent vector T of β has unit length. Then Frenet equations are

$$T' = KN_1,\ N_1' = -KT + RN_2,\ N_2' = -RN_1, \tag{1}$$

where $K = k + \xi k^*$ and $R = r + \xi r^*$ are the principal curvature and torsion of β, respectively. Moreover, k and r are the principal curvature and torsion of the curve in $\mathbb{R}^3$, which is determined by the real part of β, respectively [3].

Theorem 2.2. Let a curve $\bar{\beta}$ in ID^4 be given by

$$\bar{\beta}(s) = D(s) + A(s)e_1 + B(s)e_2 + C(s)e_3,$$

where $s \in [0,1]$ and $D,\ A,\ B,\ C \in ID$. Let s be the paramerter along the smooth curve $\bar{\beta}$ and dual tangent vector $\bar{T}$ of $\bar{\beta}$ has unit length. The Serret-Frenet formulae for a curve $\bar{\beta}$ in ID^4 may be derived with the help of the Serret-Frenet formulae of a certain curve β in ID^3 and given by

$$\bar{T}'(s) = \bar{K}\bar{N}_1,\ \bar{N}_1'(s) = K\bar{N}_2 - \bar{K}\bar{T},$$
$$\bar{N}_2'(s) = -K\bar{N}_1 + (R - \bar{K})\bar{N}_3,$$
$$\bar{N}_3'(s) = -(R - \bar{K})\bar{N}_2,$$

where $\{\bar{T}, \bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{K}, K, R - \bar{K}\}$ gives the Frenet apparatus for the curve $\bar{\beta}$, such that K and R are the principal curvature and torsion of the curve β in ID^3, respectively [3].

Definition 2.3. Let $\bar{\beta} : I \to Q_{ID^4}$ be an arc-lengthen dual quaternionic curve with nonzero curvatures, and $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ denotes the Frenet frame of the curve $\bar{\beta}$ in ID^4. $\{H_1, H_2\}$ denotes the harmonic curvature functions of the quaternionic curve $\bar{\beta}$. If $\bar{\beta} : I \to Q_{ID^4}$ is a quaternionic $\bar{N}_3$ slant helix (see [5] for real quaternionic curves)

$$H(N_3(s), u) = \cos\Phi,$$
$$\cos\Phi = \cos(\varphi + \varepsilon\varphi^*) = \text{const},\ \varphi \neq (\pi/2). \tag{2}$$

3. Dual quaternionic $\bar{N}_3$ slant helices and their harmonic curvatures functions

In this section, we give some characterizations for dual quaternionic N_2 and $\bar{N}_3$ slant helices in ID^3 and in ID^4 with respect to harmonic curvature functions of dual quaternionic curve.

Definition 3.1. $\beta : I \to Q_{ID^3}$ regular dual quaternionic curve is given by arc-length paramerter s and $\{T(s), N_1(s), N_2(s)\}$ denotes the Frenet frame of the dual quaternionic curve β. We denote β as a dual quaternionic N_2 slant helix in ID^3 if the last Frenet vectors field N_2 makes a constant angle Φ with a fixed direction u, namely,

$$H(N_2(s), u) = \cos\Phi,$$
$$\cos\Phi = \cos(\varphi + \varepsilon\varphi^*) = \text{const},\ \varphi \neq \left(\frac{\pi}{2}\right), \tag{3}$$

where u is a constant and unit vector in Q_{ID^3} for $\forall s \in I$.

Definition 3.2. Let $\gamma : I \to \mathbb{R}^3$ be a real quaternionic curve, such that

$$\gamma(s) = \gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3.$$

$\beta : I \to Q_{ID^3}$, $\beta(s) = \gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3 + \xi(\gamma_1^*(s)e_1 + \gamma_2^*(s)e_2 + \gamma_3^*(s)e_3)$ or $\beta(s) = A(s)e_1 + B(s)e_2 + C(s)e_3$; $A(s), B(s), C(s) \in ID$, obtained from γ, is such, that $\{T(s), N_1(s), N_2(s)\}$ is the Frenet trihedron in the point $\beta(s)$ of the curve β and let $u = u_0 + \xi u_0^*$ be a unit and constant vector. Then, $\beta'(s) = T_0(s) + \xi T_0^*(s) = T(s)$, $\alpha\beta'(s) = -T_0(s) - \xi T_0^*(s)$, $u = u_0 + \xi u_0^*$, $\alpha u = -u_0 - \xi u_0^*$ (u is a dual spatial quaternion). Thus, $uT(s) \in Q_{ID^3}$, $H(u,T) = h(u_0, T_0) + \xi[h(u_0, T_0^*(s)) + h(u_0^*, T_0)]$.

$H(u,T) = \cos\Phi = \cos(\varphi + \varepsilon\varphi^*)$. Since β is an inclined curve, we have $\cos\varphi = \text{const}$. If φ^* is a constant, we have $H(u,T) = \cos\Phi = \text{const}$.

Definition 3.3. Let $\beta : I \to Q_{ID^3}$ be a dual spatial quaternionic inclined curve, given by arc-length parameter s. Curvatures at the point $\beta(s)$ of curve β are $K(s) = k(s) + \xi k^*(s)$, $R(s) = r(s) + \xi r^*(s)$. In that case H is harmonic curvature, it is

$$H(s) = \frac{R(s)}{K(s)}. \tag{4}$$

Theorem 3.1. Let $\beta : I \to Q_{ID^3}$ be a regular dual quaternionic curve in ID^3 with arclength parameter s and let $\{T(s), N_1(s), N_2(s)\}$ denote the Frenet trihedron in the point $\beta(s)$ of the curve β. If the curve β is a dual spatial quaternionic N_2 slant helices with $u = u_0 + \xi u_0^*$, as its axis, then we have $H : I \to ID$

$$H(T(s), u) = H(s)\, H(N_2(s), u) = H\cos\Phi,$$
$$\cos\Phi = \cos(\varphi + \varepsilon\varphi^*),\ \varphi \neq (\pi/2), \tag{5}$$

where H is a dual harmonic curvature function of the curve β.

Proof. Let $\varphi \neq \pi/2$ be a constant angle between the quaternion u and the last Frenet vector of the curve β quaternionic N_2-slant helix in Q_{ID^3}. Thus, we have $H\left(N_2\left(s\right), u\right) = \cos \Phi = \text{const.}$

Here, by differentiating with respect to s, we get $H(N_2'(s), u) = 0$. With the help of Eq. (1), we obtain that $H(N_1(s), u) = 0$ and $H(N_1'(s), u) = 0$. Here, using Eq. (1),

$$H\left(T\left(s\right), u\right) = \frac{R}{K} H\left(N_2\left(s\right), u\right),$$

and then Eq. (4) gives us $H\left(T\left(s\right), u\right) = H\left(s\right) H\left(N_2\left(s\right), u\right) = H \cos \Phi$, $\varphi \neq \pi/2$.

Theorem 3.2. Let $\beta : I \to Q_{ID^3}$ be a regular dual quaternionic curve in ID^3 with arclength parameter s and let $\{T(s), N_1(s), N_2(s)\}$ denote the Frenet trihedron in the point $\beta(s)$ of the curve β. If the curve β is a dual spatial quaternionic N_2 slant helices in ID^3, the axis of β is

$$u = (H(s)T(s) + N_2(s))H(N_2(s), u), \tag{6}$$

where $H(s)$ is a dual harmonic curvature function of the curve β.

Proof. If the axis of dual spatial quaternionic N_2 slant helix β is u, then we can write

$$u = \lambda_1 T + \lambda_2 N_1 + \lambda_3 N_2.$$

Then, by using Theorem (3.1.) $\lambda_1 = H(s)H(N_2(s), u)$, $\lambda_2 = H(N_1(s), u) = 0$, $\lambda_3 = H(N_2(s), u)$. Thus it is easy to obtain $u = (H(s)T + N_2)H(N_2(s), u)$.

Definition 3.3. Let $\beta : I \to Q_{ID^3}$ be a regular dual quaternionic curve in ID^3 with arclength parameter s and $\{T(s), N_1(s), N_2(s)\}$ denotes the Frenet trihedron of the curve β and H denotes the dual harmonic curvature function at the point $\beta(s)$. The dual quaternion

$$D = H(s)T(s) + N_2(s) \tag{7}$$

is called a Darboux dual quaternion of the dual spatial quaternionic N_2 slant helix β in ID^3.

Theorem 3.3. Let $\beta : I \to Q_{ID^3}$ be a regular dual quaternionic curve in ID^3 with arclength parameter s and $\{T(s), N_1(s), N_2(s)\}$ denotes the Frenet trihedron of the curve β and H denotes the dual harmonic curvature function of the curve β. Then the curve β is a spatial dual quaternionic N_2 slant helix in ID^3 if and only if D is a constant dual spatial quaternion.

Proof. $(\Rightarrow)$ Let β be a dual spatial quaternionic N_2 slant helix in ID^3 and u be the axis of β. From Theorem (3.2.), we have $u = \left(H\left(s\right)T(s) + N_2(s)\right) H\left(N_2\left(s\right), u\right) = D \cos \Phi$, where Φ and u are constant. Thus, D is a constant dual spatial quaternion.

$(\Leftarrow)$ In contrast, suppose that D is a constant vector field, we have $\|D\|^2 = H(D, D)$ is constant. By using Theorem (3.2.), we can write $\|u\|^2 = \|D \cos \Phi\|^2 = \cos^2 \Phi h\left(D, D\right)$.

Since u is a unit dual quaternion and $\|D\|$ is constant, we have $\cos \Phi = \frac{1}{\|D\|} = \left(HN_2\left(s\right), u\right)$ is constant. Therefore, β is a dual spatial quaternionic N_2 slant helix.

Theorem 3.4. Let $\beta : I \to Q_{ID^3}$ be a dual spatial quaternionic curve, given by arc-length parameter s. Let $H(s)$ be harmonic curvature and $\{T(s), N_1(s), N_2(s)\}$ be the Frenet frame at the point $\beta(s)$. If the curve β is a spatial dual quaternionic N_2 slant helix in ID^3, then $H^2(s)$ is constant.

Proof. Let β be a spatial dual quaternionic N_2 slant helix. Since the axis of β is $u = \left(H\left(s\right)T(s) + N_2(s)\right) \cos \Phi$ unit dual spatial quaternion, $\|u\|^2 = 1$. Hence, we have $\|u\|^2 = |H(u, u)| = |(u \times \alpha u)|$. With Eq. (6), by taking $\|T\left(s\right)\| = \|N_1\left(s\right)\| = \|N_2\left(s\right)\| = 1$ and $\|u\| = 1$ into consideration, $H^2\left(s\right) = \tan^2 \Phi = \text{const}$ is obtained.

Corollary 3.5. Let $\beta : I \to Q_{ID^3}$ be a dual spatial quaternionic curve with arc length parameter s and nonzero curvatures $\{K, R\}$. Then, curve β is a dual spatial quaternionic N_2 slant helix in ID^3 if and only if $\left(\frac{R(s)}{K(s)}\right)' = 0$.

Proof. It is obvious from Theorem (3.3.)

Definition 3.4. Let $\bar{\beta} : I \to Q_{ID^4}$ be an arc-lengthen dual quaternionic curve with nonzero curvatures $\bar{K}$, K, $R - \bar{K}$ and $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ denotes the Frenet frame of $\bar{\beta}$. We denote $\bar{\beta}$ as a quaternionic $\bar{N}_3$-slant helix in Q_{ID^4} if the last unit vector field $\bar{N}_3$ makes a constant angle ϕ with a fixed direction u, that is, $H(\bar{N}_3, u) = \cos \phi$, $\phi = \text{const}$, where u is a unit and constant dual quaternion, which is the axis of β for all $s \in I$.

Theorem 3.6. Let $\beta : I \subset R \to Q_{ID^3}$ be a regular dual spatial quaternionic N_2 slant helix, such that $\beta(s) = A(s)e_1 + B(s)e_2 + C(s)e_3$, $\bar{\beta}\left(s\right) = D\left(s\right) + A\left(s\right)e_1 + B\left(s\right)e_2 + C\left(s\right)e_3$ is obtained from β. Then $\bar{\beta}$ is a dual quaternionic inclined curve in Q_{ID^4}.

Proof. Let $\bar{\beta} : I \to Q_{ID^4}$ be an arc-lengthen dual quaternionic curve and u be a unit and constant dual spatial quaternion, which is the axis of $\bar{\beta}$, such that $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ be Frenet apparatus at the point $\bar{\beta}(s)$ of $\bar{\beta}$. Then we have

$$\begin{aligned} H\left(\bar{T}\left(s\right), u\right) &= \frac{1}{2}\left(T_0 \times \alpha u_0 + u_0 \times \alpha T_0\right) \\ &+ \frac{1}{2}\varepsilon[(T_0 \times \alpha u_0^* + u_0^* \times \alpha T_0) \\ &+ (T_0^* \times \alpha u_0 + u_0 \times \alpha T_0^*)], \end{aligned}$$

$$\begin{aligned} H(\bar{T}, u) &= h(u_0, T_0) + \xi[h(u_0, T_0^*\left(s\right)) + h(u_0^*, T_0)] = \\ &\cos \varphi \pm \varphi^* \sin \varphi = \cos \phi = \text{const.} \end{aligned}$$

Thus, $\bar{\beta}$ is a quaternionic inclined curve in Q_{ID^4}.

Definition 3.5. Let $\bar{\beta} : I \to Q_{ID^4}$ be an arc-lengthen dual quaternionic curve with nonzero curvatures K, $\bar{K}$, $R - \bar{K}$. In that case harmonic curvature functions in terms of $\bar{N}_3$ of $\bar{\beta}$ are defined by $H_i : I \to ID$

$$H_0 = 0,\ H_1 = \frac{R - \bar{K}}{K},$$

$$H_2 = -\frac{1}{\bar{K}} H_1' = -\frac{1}{\bar{K}}\left(\frac{R - \bar{K}}{K}\right)', \tag{8}$$

where $\bar{K}$ is the principal curvature, K is the torsion of $\bar{\beta}$, $(R - \bar{K})$ is the bi-torsion of $\bar{\beta}$.

Theorem 3.7. Let $\bar{\beta} : I \to Q_{ID^4}$ be a dual quaternionic curve, given by arc-length parameter s and u be a unit and constant quaternion of Q_{ID^4}. $\{H_1, H_2\}$ denotes the harmonic curvature functions of the quaternionic curve $\bar{\beta}$. If $\bar{\beta} : I \to Q_{ID^4}$ is a quaternionic $\bar{N}_3$ slant helix, with u as its axis, then we get

$$H\left(\bar{T}, u\right) = H_2 H(\bar{N}_3, u),\ H(\bar{N}_1, u) = H_1 H(\bar{N}_3, u),$$
$$H(\bar{N}_2, u) = H_0 H(\bar{N}_3, u),\ H(\bar{N}_3, u) = \cos\phi. \tag{9}$$

Proof. Let $\varphi \neq \pi/2$ be a constant angle between the quaternion u and the last Frenet vector of the curve $\bar{\beta}$ quaternionic $\bar{N}_3$-slant helix in Q_{ID^4}. Thus, we have $H(\bar{N}_3(s), u) = \cos\phi$, for all $s \in I$. Then by differentiating the equation above, with respect to s, we obtain $H\left(\bar{N}_3'(s), X\right) = 0$ or from Serret-Frenet formulas of $\bar{\beta}$, we get $\left(R - \bar{K}\right) H\left(\bar{N}_2(s), u\right) = 0$, where $\left(R - \bar{K}\right) \neq 0$, then

$$H\left(\bar{N}_2(s), u\right) = 0 = H_0 H(\bar{N}_3, u). \tag{10}$$

By taking the derivative of Eq. (10) and using the Frenet formulas, we obtain $H\left(\bar{N}_2'(s), u\right) = 0$,

$$H(K\bar{N}_1 + (R - \bar{K})\bar{N}_3, u) = 0,$$

and from Eq. (8), we have

$$H(\bar{N}_1, u) = H_1 H(\bar{N}_3, u). \tag{11}$$

If derivative of the last equation with respect to s is taken,

$$H\left(\bar{N}_1'(s), u\right) = H_1' H(\bar{N}_3, u)$$

is found. Here, by using Eq. (10) and Definition (3.5.)

$$H(T(s), u) = H_2 H(\bar{N}_3, u). \tag{12}$$

Thus, with the Eqs. (10)–(12) the proof is completed.

Corollary 3.8. Let $\bar{\beta} : I \to Q_{ID^4}$ be a dual quaternionic curve given by arc-length parameter s and u be a unit and constant dual quaternion of Q_{ID^4}. $\{H_1, H_2\}$ denotes the harmonic curvature functions of the quaternionic curve. If $\bar{\beta} : I \to Q_{ID^4}$ is a dual quaternionic $\bar{N}_3$-slant helix, the axis of $\bar{\beta}$ is $u = (H_2\bar{T}(s) + H_1\bar{N}_1(s) + \bar{N}_3(s))H(\bar{N}_3, u)$ or $u = (H_2\bar{T}(s) + H_1\bar{N}_1(s) + \bar{N}_3(s))\cos\phi$.

Proof. If u is the axis of a quaternionic $\bar{N}_3$-slant helix in Q_{ID^4}, then we can write $u = \lambda_1\bar{T}(s) + \lambda_2\bar{N}_1(s) + \lambda_3\bar{N}_2(s) + \lambda_4\bar{N}_3(s)$ and then by using Theorem (3.7.), we get

$$\lambda_1 = H(\bar{T}(s), u) = H_2 h(\bar{N}_3(s), u),$$
$$\lambda_2 = H(\bar{N}_1(s), u) = H_1 h(\bar{N}_3(s), u),$$
$$\lambda_3 = H(\bar{N}_2(s), u) = H_0 h(\bar{N}_3(s), u),$$
$$\lambda_4 = H(\bar{N}_3(s), u).$$

Thus we easily obtain that $u = (H_2\bar{T}(s) + H_1\bar{N}_1(s) + \bar{N}_3(s))\cos\phi$.

Definition 3.6. Let $\bar{\beta} : I \to Q_{ID^4}$ be a dual quaternionic curve given by arc-length parameter s. Such that $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ is Frenet apparatus and harmonic curvatures $\{H_1, H_2\}$. The dual quaternion $D = H_2\bar{T}(s) + H_1\bar{N}_1(s) + \bar{N}_3(s)$ is called the Darboux dual quaternion of the dual quaternionic $\bar{N}_3$-slant helix $\bar{\beta}$.

Theorem 3.9. Let $\bar{\beta} : I \to Q_{ID^4}$ be a dual quaternionic curve, given by arc-length parameter s. Such that $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ is Frenet apparatus and harmonic curvatures $\{H_1, H_2\}$. Then $\bar{\beta}$ is a dual quaternionic $\bar{N}_3$-slant helix if and only if D is a constant dual quaternion.

Proof. ($\Rightarrow$) Suppose that $\bar{\beta}$ is a dual quaternionic $\bar{N}_3$-slant helix in Q_{ID^4} and u is the axis of $\bar{\beta}$. From Corollary (3.8.), we have that $u = \left(H_2\bar{T}(s) + H_1\bar{N}_1(s) + \bar{N}_3(s)\right)\cos\phi = D\cos\phi$, where ϕ and u are constant and hence D is a constant dual quaternion.

($\Leftarrow$) Let D be a constant dual spatial quaternion. From Definition (3.6.) and Corollary (3.8.), we can write

$$u = D\cos\phi. \tag{13}$$

By derivating Eq. (13) with respect to s, we get $u' = D'\cos\phi + D(\cos\phi)'$ from the hypothesis, D is a constant dual spatial quaternion and $D(\cos\phi)' = 0$, where $D \neq 0$ or we get $\cos\phi = $ const. We can define a unique axis of the quaternionic $\bar{N}_3$ slant helix, where $h(\bar{N}_3(s), u) = h(\bar{N}_3(s), D\cos\phi) = \cos\phi h(\bar{N}_3(s), D)$. From Definition (3.6.), we get $h\left(\bar{N}_3(s), u\right) = \cos\phi$. Thus, u is a constant dual quaternion and $\bar{\beta}$ is a quaternionic $\bar{N}_3$ slant helix in Q_{ID^4}.

Theorem 3.10. Let $\bar{\beta} : I \to Q_{ID^4}$ be a dual quaternionic curve given by arc-length parameter s. Such that $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ is Frenet apparatus and harmonic curvatures $\{H_1, H_2\}$. Then $\bar{\beta}$ is a quaternionic $\bar{N}_3$ slant helix if and only if

$$H_2' - \bar{K}H_1 = 0. \tag{14}$$

Proof. ($\Rightarrow$) If we differentiate D along the curve $\bar{\beta}$, we get $D' = H_2'\bar{T}(s) + H_2\bar{T}'(s) + H_1'\bar{N}_1(s) + H_1\bar{N}_1'(s) + \bar{N}_3'(s)$. The Serret-Frenet formulas and Eq. (8) give us

$$D' = (H_2' - \bar{K}H_1)\bar{T}. \tag{15}$$

Since $\bar{\beta}$ is a quaternionic $\bar{N}_3$ slant helix, D is a constant dual quaternion. Thus, we can write $D' = 0$ or $\left(H_2' - \bar{K}H_1\right) = 0$.

($\Leftarrow$) If $(H_2' - \bar{K}H_1) = 0$, we can easily see that $D' = 0$ or D is a constant dual quaternion, and then from Theorem (3.9.), we have that $\bar{\beta}$ is a dual quaternionic $\bar{N}_3$-slant helix in Q_{ID^4}.

Corollary 3.11. Let $\bar{\beta} : I \to Q_{ID^4}$ be an arc-lengthen dual quaternionic curve with nonzero curvatures K, $\bar{K}$, $R - \bar{K}$ Let $\{\bar{T}(s), \bar{N}_1(s), \bar{N}_2(s), \bar{N}_3(s)\}$ and $\{H_1, H_2\}$ be the Frenet frame and the harmonic curvature functions of the quaternionic curve, respectively. Then $\bar{\beta}$ is a quaternionic $\bar{N}_3$-slant helix if and only if

$$\left[\frac{1}{\bar{K}}\left(\frac{R - \bar{K}}{K}\right)'\right]' + \bar{K}\left(\frac{R - \bar{K}}{K}\right) = 0. \tag{16}$$

Proof. ($\Rightarrow$) Let $\bar{\beta}$ be a quaternionic $\bar{N}_3$-slant helix in Q_{ID}^4, then from Theorem (3.10.) we have $\left(H_2' - \bar{K}H_1\right) = 0$. By using Definition (3.5) we have $[\frac{1}{\bar{K}}(\frac{R-\bar{K}}{K})']' + \bar{K}(\frac{R-\bar{K}}{K}) = 0$.

($\Leftarrow$) We suppose that the equation $\left[\frac{1}{\bar{K}}\left(\frac{R-\bar{K}}{K}\right)'\right]' + \bar{K}(\frac{R-\bar{K}}{K}) = 0$ holds, then from Theorem (3.10.) and Definition (3.5.), it is obvious that $\bar{\beta}$ is a quaternionic $\bar{N}_3$-slant helix in Q_{ID^4}.

Theorem 3.12. Let $\bar{\beta} : I \to Q_{ID^4}$ be a dual quaternionic curve with nonzero curvatures K, $\bar{K}$, $R-\bar{K}$. If $\bar{\beta}$ is a quaternionic $\bar{N}_3$-slant helix, then the following condition is satisfied, $H_1^2 + H_2^2 = \tan^2\phi = \text{const}$, where ϕ is the constant angle between the last Frenet vector $\bar{N}_3$ and a constant unit dual quaternion u.

Proof. Let $\bar{\beta}$ be a real quaternionic $\bar{N}_3$-slant helix. Since the axis of $\bar{\beta}$ is $u = \left(H_2\bar{T}(s) + H_1\bar{N}_1(s) + \bar{N}_3(s)\right)\cos\phi$ unit dual quaternion, $\|u\|^2 = 1$. Hence, using Eq. (3) we have

$$
\begin{aligned}
\|u\|^2 = H(u,u) = u \times \alpha u = \\
& H_2^2\cos^2\phi\left(\bar{T}(s) \times \alpha\bar{T}(s)\right) \\
& + H_1^2\cos^2\phi\left(\bar{N}_1(s) \times \alpha\bar{N}_1(s)\right) \\
& + (\bar{N}_3(s) \times \alpha\bar{N}_3(s))\cos^2\phi \\
& + H_2H_1\left(\bar{T}(s) \times \alpha\bar{N}_1(s) + \bar{N}_1(s) \times \alpha\bar{T}(s)\right)\cos^2\phi \\
& + H_2(\bar{T}(s) \times \alpha\bar{N}_3 + \bar{N}_3 \times \alpha\bar{T}(s))\cos^2\phi \\
& + H_1(\bar{N}_1(s) \times \alpha\bar{N}_3(s) + \bar{N}_3(s) \times \alpha\bar{N}_1(s))\cos^2\phi,
\end{aligned}
$$

where by using the properties of dual quaternion product we can easily write that $\|u\|^2 = 1 = (1 + H_1^2 + H_2^2)\cos^2\phi$ and then we get $H_1^2 + H_2^2 = \tan^2\phi = \text{const}$.

4. Conclusions

We give some characterizations of dual quaternionic N_2 and $\bar{N}_3$ slant helices in ID^3 and in ID^4 in terms of their dual quaternionic harmonic curvatures. Besides, by using Theorem (3.10.) and Definition (3.5.) the derivatives of harmonic curvatures are as follows:

$$
\begin{bmatrix} H_1' \\ H_2' \end{bmatrix} = \begin{bmatrix} 0 & -K \\ -\bar{K} & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.
$$

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