1. Introduction

The mixed-spin Ising models traditionally attract a great deal of research interest, because they may exhibit much more complex magnetic and critical behavior than their single-spin counterparts [1]. Despite considerable efforts, there exist only a few fully exactly tractable mixed-spin Ising models defined mostly on three-coordinated lattices such as the honeycomb [2], bathroom-tile [3] and square-hexagon-dodecagon [1] lattice. In the present work, we will exactly solve the mixed-spin Ising model on a three-coordinated Bethe lattice with the aim to study an influence of the single-ion anisotropy up on the critical behavior and temperature dependences of the spontaneous magnetization.

2. Mixed-spin Ising model and its exact solution

Let us consider the mixed spin-1/2 and spin-S Ising model on the Bethe lattice with the coordination number $q=3$, which is schematically illustrated on the left-hand-side of Fig. 1. The investigated Bethe lattice consists of two inequivalent interpenetrating sublattices A and B, which are occupied by the spin-1/2 and spin-S atoms depicted in Fig. 1 as filled and empty circles, respectively. The total Hamiltonian of the mixed spin-1/2 and spin-S Ising model on the three-coordinated Bethe lattice reads

\[ \mathcal{H} = J \sum_{\langle ij \rangle} S_i \sigma_j - D \sum_{i=1}^N S_i^2, \]  

where $N$ is the total number of lattice sites at each sublattice, $S_i = -S, -S + 1, \ldots, S$ and $\sigma_j = \pm 1/2$ are the Ising spins located at the $i$-th and $j$-th lattice point of the sublattice A and B, respectively. The first summation is carried out over all nearest-neighbor spin pairs coupled via the antiferromagnetic interaction $J > 0$ and the second summation takes into account the uniaxial single-ion anisotropy $D$ acting on the spins from the sublattice B.

The total Hamiltonian (1) can be rewritten as a sum over the site Hamiltonians $\mathcal{H}_i = \sum_{j=1}^N \mathcal{H}_i$, where each site Hamiltonian $\mathcal{H}_i$ involves all the interaction terms connected to the $i$-th spin-S atom from the sublattice B

\[ \mathcal{H}_i = J S_i (\sigma_1 + \sigma_2 + \sigma_3) - D S_i^2. \]

The partition function of the mixed spin-1/2 and spin-S Ising model on the Bethe lattice can be subsequently partially factorized into the product

\[ Z = \prod_{\{\sigma\}} \sum_{S_i = -S}^{S} \exp(-\beta \mathcal{H}_i) = \prod_{\{\sigma\}} \sum_{i=1}^N Z_i, \]

where $\beta = 1/(k_B T)$, $k_B$ is the Boltzmann constant, $T$ is the absolute temperature, $\sum_{\{\sigma\}}$ denotes a summation over all possible configurations of the spin-1/2 atoms from the sublattice A and the second summation $\sum_{S_i}$ is carried out over spin states of the $i$th spin-S atom from the sublattice B. After performing the latter particular summation one obtains the explicit form of the site partition function $Z_i$, which can be subsequently replaced through the star-triangle transformation (see Ref. [1]):

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The mapping parameters \( A \) and \( R \) can be directly obtained from a self-consistency condition [1] of the star-triangle transformation (4), which requires

\[
A = \left( V_1 V_2^2 \right)^{1/2}, \quad \beta R = 2 \ln(V_1/V_2),
\]

whereas the expressions \( V_1 \) and \( V_2 \) are defined as follows:

\[
V_1 = \sum_{n=\pm S} \exp(\beta Dn^2) \cosh(3\beta Jn/2),
\]

\[
V_2 = \sum_{n=\pm S} \exp(\beta Dn^2) \cosh(\beta Jn/2).
\]

The substitution of the star-triangle transformation (4) into the partition function (3) establishes a rigorous mapping relationship between the partition function \( Z \) of the mixed-spin Ising model on the three-coordinated Bethe lattice and the partition function \( Z_{\text{Husimi}} \) of the corresponding spin-1/2 Ising model on the triangular Husimi lattice with the effective pair interaction \( R \) (see Fig. 1):

\[
Z(\beta, J, D) = A^N Z_{\text{Husimi}}(\beta, R).
\]

Following the approach elaborated in Ref. [4] one may extract exact results for the mixed-spin Ising model on the Bethe lattice from a rigorous mapping correspondence (7) with the spin-1/2 Ising model on the triangular Husimi lattice, which can be exactly treated by the method of exact recursion relations [5, 6]. From this point of view, our exact solution of the mixed-spin Ising model on the Bethe lattice is formally completed, since the exact results for the critical temperature and spontaneous magnetization of the spin-1/2 Ising model on the triangular Husimi lattice were already reported in Refs. [4, 6]. For instance, the phase diagrams of the mixed-spin Ising model on the three-coordinated Bethe lattice can be obtained from a comparison of the effective interaction given by Eqs. (5),(6) with the critical value \( \beta D = \ln(7/3) \) [4], which provides the following critical condition:

\[
3 \sum_{n=\pm S} e^{\beta Dn^2} \cosh \left( \frac{3}{2} \beta Jn \right) = \]

\[
7 \sum_{n=\pm S} e^{\beta Dn^2} \cosh \left( \frac{1}{2} \beta Jn \right).
\]

The similar procedure can be utilized for the exact calculation of both spontaneous sublattice magnetizations \( m_A = \langle \sigma_3 \rangle \) and \( m_B = \langle S \rangle \), which can be computed from the spontaneous magnetization of the spin-1/2 Ising model on the triangular Husimi lattice [6].

\[3. \text{ Results and discussion}\]

Let us proceed to a discussion of the most interesting results for the phase diagrams and spontaneous magnetization of the mixed spin-1/2 and spin-\( S \) Ising model on the three-coordinated Bethe lattice. Figure 2 illustrates typical dependences of the reduced critical temperature \( k_B T_c/J \) on a relative strength of the uniaxial single-ion anisotropy \( D/J \) for several values of the quantum spin number \( S \). As one can see, the critical temperature monotonically decreases with decrease of the parameter \( D/J \) for any quantum spin number \( S \). However, the critical temperature tends to zero as \( D/J \to -3/2 \) for all integer spins \( S \) (Fig. 2a), while it tends towards the finite value \( k_B T_c/J = 1/\ln(9) \) in the asymptotic limit \( D/J \to -\infty \) for all half-odd-integer spins \( S \) (Fig. 2b).

This qualitative difference appears owing to a sufficiently strong negative single-ion anisotropy \( D/J \), which forces all integer spins towards their non-magnetic spin state \( S_i = 0 \) and all half-odd-integer spins towards their lowest magnetic spin state \( S_i = 1/2 \).
and $-1.4$, respectively. Temperature variations of the spontaneous magnetization for another particular spin case $S = 3/2$ depicted in Fig. 3b serve in evidence that the total magnetization may additionally display the peculiar L-type dependence. Under this circumstance, the total magnetization rises from zero due to thermal excitations from the lowest-energy spin state $S_i = 1/2$ towards the higher-energy state $S_i = 3/2$, whereas this effect is especially visible when $D/J \lesssim 0.75$ (see in Fig. 3b the curve for $D/J = -0.8$). Note that the similar P-type dependences with thermally-induced increase of the total magnetization starting from the non-zero asymptotic value can be found for the spin magnitudes $S > 3/2$.

4. Conclusion

The present work provides exact results for the phase diagrams, total and sublattice magnetizations of the ferromagnetic version of the mixed spin-1/2 and spin-$S$ Ising model on the three-coordinated Bethe lattice. In particular, we have focused our attention to the effect of uniaxial single-ion anisotropy upon temperature dependences of the spontaneous magnetization and finite-temperature phase diagrams, which were established for several spin magnitudes $S$. It has been shown that the critical temperature exhibits a monotonous decline with decrease of the single-ion anisotropy $D/J$, whereas the overall critical behavior basically depends on whether the quantum spin number $S$ is integer or half-odd-integer. Besides, we have also proved an extraordinary diversity in thermal dependences of the total magnetization, which were classified as Q-, R-, P-, L- or S-type thermal dependences within the extended Néel nomenclature [3, 4]. By contrast, the N- and W-type thermal dependences of the total magnetization with one or two compensation temperatures cannot be found within the investigated ferrimagnetic model. The present results for the mixed spin-1/2 and spin-$S$ Ising model on the three-coordinated Bethe lattice are in a good qualitative accord with the exact results obtained previously for the same model on the honeycomb [2] and bathroom-tile [3] lattices.

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References