Embedded Solitons and Conservation Law with $\chi^{(2)}$ and $\chi^{(3)}$ Nonlinear Susceptibilities

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This paper studies embedded solitons that are confined to continuous spectrum, with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinear susceptibilities. Bright and singular soliton solutions are obtained by the method of undetermined coefficients. Subsequently, the Lie symmetry analysis and mapping method retrieves additional solutions to the model such as shock waves, singular solitons, cnoidal waves, and several others. Finally, a conservation law for this model is secured through the Lie symmetry analysis.

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1. Introduction

Optical solitons is one of the most active areas of research in the field of nonlinear optics [1–14]. It is evident, from a variety of reported results, that the main focus is on integrability issue in fiber optics, photonic crystal, metamaterials and metasurfaces. This paper, however, changes gear to focus on embedded solitons that come with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities.

Embedded solitons are nonlinear waves that become confined to the continuous spectrum of a nonlinear system. Such solitons arise in presence of opposing dispersion and competing nonlinearities at fundamental harmonics (FH) and second harmonics (SH). This is an area of nonlinear optics where very minimal results are visible [11]. In the past, it is only solitons with quadratic nonlinearity that were analyzed in detail and quite a few results were disseminated [1, 11].

The focus of this paper will be the integrability aspect and conservation law of these embedded solitons that are modeled by coupled nonlinear Schrödinger's equation (NLSE) with such nonlinearities. The method of undetermined coefficients retrieves bright and singular soliton solutions to the model. The mapping principle will extract additional solutions that include cnoidal waves and snoidal waves. In the limiting case, when the modulus of ellipticity approaches zero or unity, bright, dark and singular soliton solutions or periodic waves fall out. Subsequently, the Lie symmetry analysis retrieves bright solitons and rational solutions. Finally, using the multipliers approach and double reduction technique, conservation law is derived. These are detailed in the next upcoming sections.

1.1. Governing equation

The governing equation for solitons in quadratic nonlinear media is given by [1, 10, 11]:

$$\mathrm{i}\,q_t + a_1 q_{xx} + b_1 q_{xt} + c_1 q^* r + d_1 |q|^2 q = 0, \tag{1}$$

$$\mathrm{i}\,r_t + a_2 r_{xx} + b_2 r_{xt} + c_2 r + d_2 q^2 + \delta |q|^2 r = 0. \tag{2}$$

In Eqs. (1) and (2), the dependent variables are $q(x,t)$ and $r(x,t)$ which are complex valued functions representing FH and SH, respectively. The independent variables x and t are spatial and temporal variables, respectively. The coefficients a_j and b_j are from group velocity dispersion and spatio-temporal dispersion, respectively. Also c_j is the group-velocity mismatch because of frequency difference between FH and SH fields.

2. Soliton solutions

The method of undetermined coefficients is one of the most popular modern approaches to obtain the soliton solutions to such governing equations since its first appearance a couple of years ago. In fact, this approach has been successfully applied to extract the soliton solutions to water wave model, plasma physics model as well as in nonlinear optics and nuclear physics. In order to study the details of this model, the starting hypothesis is [1, 6, 10, 11]:

$$q(x,t) = P_1(x,t)\,\mathrm{e}^{\mathrm{i}\,\phi(x,t)}, \tag{3}$$

$$r(x,t) = P_2(x,t)\,\mathrm{e}^{2\,\mathrm{i}\,\phi(x,t)}, \tag{4}$$

where $P_l(x,t)$ is the amplitude component of the soliton and $\phi(x,t)$ represents the phase component with

$$\phi(x,t) = -\kappa x + \omega t + \theta. \tag{5}$$

Here, κ is the soliton frequency, ω is the soliton wave number and θ is the phase constant. Substituting (3), (4) and (5) into (1) and (2) and decomposing into real and imaginary parts gives

$$\left(\omega + a_1\kappa^2 - b_1\kappa\omega\right) P_1 - a_1\frac{\partial^2 P_1}{\partial x^2} - b_1\frac{\partial^2 P_1}{\partial x \partial t} - c_1 P_1 P_2 - d_1 P_1^3 = 0 \tag{6}$$

and

$$v = \frac{b_1\omega - 2a_1\kappa}{1 - b_1\kappa}, \tag{7}$$

respectively, from the first component. Here v is the speed of the soliton. Then, the second component respectively gives

$$\left(2\omega + 4a_2\kappa^2 - 4b_2\kappa\omega - c_2\right) P_2 - a_2\frac{\partial^2 P_2}{\partial x^2} - b_2\frac{\partial^2 P_2}{\partial x \partial t} - d_2 P_1^2 - \delta P_1^2 P_2 = 0 \tag{8}$$

and

$$v = \frac{2b_2\omega - 4a_2\kappa}{1 - 2b_2\kappa}. \tag{9}$$

From (7) and (9) equating the two values of the soliton speed v leads to

$$a_1 = 2a_2 \tag{10}$$

and

$$a_1 = 2a_2. \tag{11}$$

Thus, the governing equations modify to

$$\mathrm{i}q_t + 2aq_{xx} + 2bq_{xt} + c_1 q^* r + d_1|q|^2 q = 0, \tag{12}$$

$$\mathrm{i}r_t + ar_{xx} + br_{xt} + c_2 r + d_2 q^2 + \delta|q|^2 r = 0. \tag{13}$$

Hence, the real part equations from the two components can be written as

$$\left(\omega + 2a\kappa^2 - 2b\kappa\omega\right) P_1 - 2a\frac{\partial^2 P_1}{\partial x^2} - 2b\frac{\partial^2 P_1}{\partial x \partial t} - c_1 P_1 P_2 - d_1 P_1^3 = 0 \tag{14}$$

and

$$\left(2\omega + 4a\kappa^2 - 4b\kappa\omega - c_2\right) P_2 - a\frac{\partial^2 P_2}{\partial x^2} - b\frac{\partial^2 P_2}{\partial x \partial t} - d_2 P_1^2 - \delta P_1^2 P_2 = 0. \tag{15}$$

These two equations will now be studied further along in the next two subsections for bright and singular solitons.

2.1. Bright solitons

For bright solitons, the starting hypothesis is given by [1, 10, 11]:

$$P_l = A_l \operatorname{sech}^{p_l}\tau \tag{16}$$

for $l = 1, 2$ where

$$\tau = B(x - vt). \tag{17}$$

Here A_l represents the amplitudes of the solitons in the two components and B is the inverse width of these solitons and v, as defined before is the speed of the solitons. The unknown exponents p_l will be determined. Substituting (16) into the two components (14) and (15) simplifies them to

$$\left\{\omega\left(2b\kappa - 1\right) - 2a\kappa^2 + 2(a - bv)p_1^2 B^2\right\} \operatorname{sech}^{p_1}\tau + c_1 A_2 \operatorname{sech}^{p_1+p_2}\tau - 2p_1(p_1+1)(a - bv)B^2 \times \operatorname{sech}^{p_1+2}\tau + d_1 A_1^2 \operatorname{sech}^{3p_1}\tau = 0 \tag{18}$$

and

$$\left[p_2^2(a - bv)A_2 B^2 + A_2\left\{2\omega(2b\kappa - 1) - 4a\kappa^2 - c_2\right\}\right] \operatorname{sech}^{p_2}\tau - p_2(p_2+1)(a - bv)A_2 B^2 \operatorname{sech}^{p_2+2}\tau + d_2 A_1^2 \operatorname{sech}^{2p_1}\tau + \delta A_1^2 A_2 \operatorname{sech}^{2p_1+p_2}\tau = 0, \tag{19}$$

respectively. The balancing principle applied to (18) and (19) leads to

$$p_1 = 1 \tag{20}$$

and

$$p_2 = 2. \tag{21}$$

From (18), setting the coefficients of linearly independent functions to zero leads to

$$\omega = \frac{4a\kappa^2 - c_1 A_2 - d_1 A_1^2}{2\left(2b\kappa - 1\right)} \tag{22}$$

and

$$v = \frac{4aB^2 - c_1 A_2 - d_1 A_1^2}{4bB^2}. \tag{23}$$

Similarly, from (19), the coefficients of linearly independent functions give

$$\omega = \frac{3A_2\left(4a\kappa^2 - c_2\right) - 3d_2 A_1^2 - 2\delta A_1^2 A_2}{6A_2\left(2b\kappa - 1\right)} \tag{24}$$

and

$$v = \frac{6aB^2 - \delta A_1^2}{6bB^2}. \tag{25}$$

Now equating the two expressions for the soliton speed (v) from (23) and (25), gives the soliton amplitude ratios

$$\frac{A_1^2}{A_2} = \frac{3c_1}{2\delta - 3d_1}. \tag{26}$$

Again equating the wave numbers of the two components from (22) and (24) yields another relation between the amplitudes A_1 and A_2:

$$3d_2 A_1^2 - A_1^2 A_2(3d_1 - 2\delta) + 3c_2 A_2 - 3c_1 A_2^2 = 0. \tag{27}$$

From (26) and (27), one retrieves another constraint condition for bright solitons to exist that is given by

$$3d_2 c_1 + c_2\left(2\delta - 3d_1\right) = 0. \tag{28}$$

Thus, bright 1-soliton solution to the model is given by

$$q(x,t) = A_1 \operatorname{sech}\left[B(x - vt)\right] \mathrm{e}^{\mathrm{i}\,(-\kappa x + \omega t + \theta)}, \tag{29}$$

$$r(x,t) = A_2 \operatorname{sech}^2\left[B(x - vt)\right] \mathrm{e}^{2\,\mathrm{i}\,(-\kappa x + \omega t + \theta)}, \tag{30}$$

with all necessary constraints and parameter definitions in place as discussed.

2.1.1. Conservation law

To obtain conservation laws for the coupled system (12)–(13), it can be shown that this system is conserved for $c_1 = d_2$. That is the system is a total divergence $D_t T^t + D_x T^x$, where T^t is the density and T^x is the flux. For this combination, we get

$$T^t = \frac{1}{2}\Big[|q|^2 + |r|^2 + \mathrm{i}b\,(qq_x^* - q^*q_x) + \frac{\mathrm{i}b}{2}\,(rr_x^* - r^*r_x)\Big],$$

$$T^x = \frac{1}{2}\Big[\frac{\mathrm{i}b}{2}\,(qq_t^* - q^*q_t) + \frac{\mathrm{i}b}{4}\,(rr_t^* - r^*r_t) + \mathrm{i}a\,(qq_x^* - q^*q_x) + \frac{\mathrm{i}a}{2}\,(rr_x^* - r^*r_x)\Big].$$

Therefore the conserved quantity is given by

$$I = \int_{-\infty}^{\infty}\Big\{2\left(|q|^2 + |r|^2\right) + 2\mathrm{i}b\,(qq_x^* - q^*q_x) + \mathrm{i}b\,(rr_x^* - r^*r_x)\Big\}\mathrm{d}x = \frac{4}{B}\left\{A_1^2\,(1 - 2\kappa b) + A_2^2\,(1 - 4\kappa b)\right\},$$

where the integral is evaluated using the soliton solutions from (29) and (30). This represents the sum of total power and linear momentum of the two pulses.

2.2. Singular solitons

In this case the starting hypothesis is [1, 10, 11]:

$$P_l = A_l \mathrm{csch}^{p_l}\tau \tag{31}$$

for $l = 1, 2$. The real part, from the two component, gives

$$\left\{\omega\,(2b\kappa - 1) - 2a\kappa^2 + 2(a - bv)p_1^2B^2\right\}\mathrm{csch}^{p_1}\tau + c_1A_2\mathrm{csch}^{p_1+p_2}\tau - 2p_1(p_1+1)(a - bv)B^2\mathrm{csch}^{p_1+2}\tau + d_1A_1^2\mathrm{csch}^{3p_1}\tau = 0 \tag{32}$$

and

$$\left[p_2^2(a - bv)A_2B^2 + A_2\left\{2\omega(2b\kappa - 1) - 4a\kappa^2 - c_2\right\}\right]\times\mathrm{csch}^{p_2}\tau - p_2(p_2+1)(a - bv)A_2B^2\mathrm{csch}^{p_2+2}\tau + d_2A_1^2\mathrm{csch}^{2p_1}\tau + \delta A_1^2A_2\mathrm{csch}^{2p_1+p_2}\tau = 0. \tag{33}$$

Balancing principle, applied to (32) and (33) leads to the same values of p_j for $j = 1, 2$ as given by (20) and (21), respectively. The coefficients of linearly independent functions from (32), upon setting to zero, leads to the same value of ω as in (22). However, the speed of the soliton is given by

$$v = \frac{4aB^2 + c_1A_2 + d_1A_1^2}{4bB^2}. \tag{34}$$

Similarly, from the coefficients of linearly independent functions in (33) gives

$$\omega = \frac{3A_2\left(4a\kappa^2 - c_2\right) - 3d_2A_1^2 + 2\delta A_1^2A_2}{6A_2\,(2b\kappa - 1)}, \tag{35}$$

and

$$v = \frac{6aB^2 + \delta A_1^2}{6bB^2}. \tag{36}$$

Equating the two expressions for the soliton speed from (34) and (36) also gives (26). Next, equating the two expressions of the soliton wave number (ω) from (22) and (35) gives

$$3d_2A_1^2 - A_1^2A_2(3d_1 + 2\delta) + 3c_2A_2 - 3c_1A_2^2 = 0. \tag{37}$$

Finally from (26) and (37), one can recover

$$A_1 = \frac{1}{2}\sqrt{\frac{3\left\{3\,(c_1d_2 - c_2d_1) + 2\delta c_2\right\}}{\delta\,(2\delta - 3d_1)}}, \tag{38}$$

and

$$A_2 = \frac{3\,(c_1d_2 - c_2d_1) + 2\delta c_2}{4\delta c_1}. \tag{39}$$

The free parameter relation A_1 given by (38) introduces the constraint condition

$$\delta\,(2\delta - 3d_1)\left\{3\,(c_1d_2 - c_2d_1) + 2\delta c_2\right\} > 0, \tag{40}$$

for singular solitons to exist.

Finally, the singular 1-soliton for quadratic nonlinear media is given by

$$q(x,t) = A_1\mathrm{csch}\,[B(x - vt)]\,\mathrm{e}^{\mathrm{i}(-\kappa x + \omega t + \theta)} \tag{41}$$

$$r(x,t) = A_2\mathrm{csch}^2\,[B(x - vt)]\,\mathrm{e}^{2\mathrm{i}(-\kappa x + \omega t + \theta)}. \tag{42}$$

3. Mapping methods

This section will focus on the integrability aspect of the model (1) and (2) by the aid of mapping method. This method leads to solutions in terms of the Jacobi elliptic functions (JEFs) which leads to solutions in terms of solitons and shock waves in limiting case of the modulus of ellipticity. These are detailed in the following subsections.

3.1. Mathematical analysis

A nonlinear evolution equation (NLEE) in two variables x and t can be written in the form

$$F(u, u_t, u_x, \ldots) = 0. \tag{43}$$

We look for its travelling wave solution (TWS) of the form

$$u(x,t) = u(\xi), \xi = k(x - ct), \tag{44}$$

where k, c are constants to be determined. Substituting Eq. (44) into Eq. (43), we obtain an ordinary differential equation (ODE) and we search for its solution in the form

$$u(\xi) = \sum_{i=0}^{n} A_i f^i, \tag{45}$$

where n is a positive integer which may be determined by balancing the linear term of the highest order with the nonlinear term in the reduced ODE, A_i are constants to be determined and f satisfies the elliptic equation of the first kind

$$f'' = pf + qf^3, f'^2 = pf^2 + \frac{1}{2}qf^4 + r. \tag{46}$$

The prime denotes derivative with respect to ξ. We substitute Eq. (45) into the ODE and use Eq. (46) to get the constants A_i, k, c and the parameters p, q, r. Thus a mapping relation is established between Eq. (46) and Eq. (43) through Eq. (45).

The advantage of this method is that it gives a variety of solutions in terms of JEFs because the squares of first derivatives of all twelve (12) JEFs can be expressed in powers of themselves as given by Eq. (45). The JEFs $\mathrm{sn}(\xi, \mathrm{m})$, $\mathrm{cn}(\xi, \mathrm{m})$ and $\mathrm{dn}(\xi, \mathrm{m})$ where m is known as the modulus of the elliptic function with $0 < m < 1$ has the following properties:
when $m \to 0$,

$$\mathrm{sn}\xi \to \sin\xi, \mathrm{cn}\xi \to \cos\xi, \mathrm{dn}\xi \to 1, \tag{47}$$

and when$m \to 1$,

$$\mathrm{sn}\xi \to \tanh\xi, \mathrm{cn}\xi \to \mathrm{sech}\xi, \mathrm{dn}\xi \to \mathrm{sech}\xi. \tag{48}$$

Three other JEFs $\mathrm{ns}\xi, \mathrm{nc}\xi$ and $\mathrm{nd}\xi$ are reciprocals of $\mathrm{sn}\xi, \mathrm{cn}\xi$ and $\mathrm{dn}\xi$, respectively. The different ratios of these six (6) JEFs give rise to six (6) other JEFs $\mathrm{sc}\xi, \mathrm{cd}\xi, \mathrm{ds}\xi, \mathrm{cs}\xi, \mathrm{dc}\xi$ and $\mathrm{sd}\xi$.

3.2. JEF solutions

We follow the analysis in Sect. 2 and use Eq. (26) to reduce Eq. (14) into the form

$$(\omega + 2a\kappa^2 - 2b\kappa\omega)P_1 - 2a\frac{\partial^2 P_1}{\partial x^2} - 2b\frac{\partial^2 P_1}{\partial x \partial t} - \frac{2\delta}{3}P_1^3 = 0. \tag{49}$$

Considering the TWS given by Eq. (17), Eq. (49) can be reduced to

$$L_1 P_1'' + L_2 P_1 + L_3 P_1^3 = 0 \tag{50}$$

where

$$L_1 = 2B^2(bv - a), \quad L_2 = \omega + 2a\kappa^2 - 2b\kappa\omega,$$

$$L_3 = -2\delta \tag{51}$$

and prime denotes differentiation with respect to τ.

Following the analysis in Sect. 4.1, we can write the solution of Eq. (50) as

$$P_1(\tau) = A_0 + A_1 f(\tau). \tag{52}$$

We substitute Eq. (52) into Eq. (50) and use Eq. (46) to generate a system of algebraic equations which are obtained by equating the coefficients of powers of f to zero as follows:

$$f^3 : qL_1A_1 + L_3A_1^3 = 0, \tag{53}$$

$$f^2 : 3L_3A_0A_1^2 = 0, \tag{54}$$

$$f^1 : (pL_1 + L_2)A_1 + 3L_3A_0^2A_1 = 0, \tag{55}$$

$$f^0 : L_2A_0 + L_3A_0^3 = 0. \tag{56}$$

Equations (53)–(56) give

$$A_0 = 0, A_1 = \pm\sqrt{\frac{qL_2}{pL_3}}, pL_1 + L_2 = 0. \tag{57}$$

Thus our solutions given by Eqs. (3) and (4) can be written as

$$q(x,t) = \pm\sqrt{\frac{qL_2}{pL_3}} f(B(x - vt))\mathrm{e}^{\mathrm{i}(-\kappa x + \omega t + \theta)}, \tag{58}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{qL_2}{pL_3} f^2(B(x - vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x + \omega t + \theta)}. \tag{59}$$

Case 1. $f(\tau) = \mathrm{sn}\tau$ and $f(\tau) = \mathrm{cd}\tau$

In this case, we can see from Eq. (46) that $p = -(1 + m^2), q = 2m^2, r = 1$.

Thus the PWSs of Eqs. (1) and (2) can be written as

$$q(x,t) = \pm\sqrt{\frac{m^2(\omega + 2a\kappa^2 - 2b\kappa\omega)}{1 + m^2}}\mathrm{sn}(B(x - vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x + \omega t + \theta)}, \tag{60}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{m^2(\omega + 2a\kappa^2 - 2b\kappa\omega)}{1 + m^2} \times \mathrm{sn}^2(B(x - vt))\mathrm{e}^{2\mathrm{i}(-\kappa x + \omega t + \theta)}. \tag{61}$$

and

$$q(x,t) = \pm\sqrt{\frac{m^2(\omega + 2a\kappa^2 - 2b\kappa\omega)}{1 + m^2}}\mathrm{cd}(B(x - vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x + \omega t + \theta)}, \tag{62}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{m^2(\omega + 2a\kappa^2 - 2b\kappa\omega)}{1 + m^2}\mathrm{cd}^2(B(x - vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x + \omega t + \theta)}. \tag{63}$$

As $m \to 1$, Eq. (60) gives rise to the shock wave solution

$$q(x,t) = \pm\sqrt{\frac{(\omega + 2a\kappa^2 - 2b\kappa\omega)}{2}}\tanh(B(x - vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x + \omega t + \theta)}, \tag{64}$$

and Eq. (61) gives rise to the SWS

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{(\omega + 2a\kappa^2 - 2b\kappa\omega)}{2}\tanh^2(B(x - vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x + \omega t + \theta)}. \tag{65}$$

Case 2. $f(\tau) = \mathrm{cn}\tau$

Here, we can obtain from Eq. (46) that $p = 2m^2 - 1, q = -2m^2, r = 1 - m^2$.

Now, the PWSs of Eqs. (1) and (2) can be written as

$$q(x,t) = \pm\sqrt{\frac{m^2(\omega + 2a\kappa^2 - 2b\kappa\omega)}{\delta(2m^2 - 1)}}\mathrm{cn}(B(x - vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x + \omega t + \theta)}, \tag{66}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{m^2(\omega + 2a\kappa^2 - 2b\kappa\omega)}{\delta(2m^2 - 1)} \times \mathrm{cn}^2(B(x - vt))\mathrm{e}^{2\mathrm{i}(-\kappa x + \omega t + \theta)}. \tag{67}$$

As $m \to 1$, Eq. (66) leads us to the SWSs

$$q(x,t) = \pm\sqrt{\frac{(\omega + 2a\kappa^2 - 2b\kappa\omega)}{\delta}}\mathrm{sech}(B(x - vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x + \omega t + \theta)}, \tag{68}$$

and Eq. (67) gives rise to the SWS

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{(\omega + 2a\kappa^2 - 2b\kappa\omega)}{\delta} \times \mathrm{sech}^2(B(x - vt))\mathrm{e}^{2\mathrm{i}(-\kappa x + \omega t + \theta)}. \tag{69}$$

Case 3. $f(\tau) = \mathrm{cs}\tau$

In this case, we deduce from eq. (46) that $p = 2 - m^2, q = 2, r = 1 - m^2$.

Therefore, the PWSs of Eqs. (1) and (2) can be written as

$$q(x,t) = \pm\sqrt{-\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta(2-m^2)}}\mathrm{cs}(B(x-vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{70}$$

$$r(x,t) = -\frac{2\delta - 3d_1}{3c_1}\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta(2-m^2)}\mathrm{cs}^2(B(x-vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x+\omega t+\theta)}. \tag{71}$$

As $m \to 0$, Eqs. (70) and (71) give rise to the trigonometric function solutions

$$q(x,t) = \pm\sqrt{-\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{2\delta}}\cot(B(x-vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{72}$$

$$r(x,t) = -\frac{2\delta - 3d_1}{3c_1}\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{2\delta}\cot^2(B(x-vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x+\omega t+\theta)}. \tag{73}$$

As $m \to 1$, Eqs. (70) and (71) lead to the singular wave solutions

$$q(x,t) = \pm\sqrt{-\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta}}\mathrm{csch}(B(x-vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{74}$$

$$r(x,t) = -\frac{2\delta - 3d_1}{3c_1}\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta} \times \mathrm{csch}^2(B(x-vt))\,\mathrm{e}^{2\mathrm{i}(-\kappa x+\omega t+\theta)}. \tag{75}$$

Case 4. $f(\tau) = \mathrm{ns}\tau$ and $f(\tau) = \mathrm{dc}\tau$

Here, we get from Eq. (46) that $p = -(1+m^2), q = 2, r = m^2$.

Thus the PWSs of Eqs. (1) and (2) can be written as

$$q(x,t) = \pm\sqrt{\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta(1+m^2)}}\mathrm{ns}(B(x-vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{76}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta(1+m^2)}\mathrm{ns}^2(B(x-vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{77}$$

and

$$q(x,t) = \pm\sqrt{\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta(1+m^2)}}\mathrm{dc}(B(x-vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{78}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta(1+m^2)}\mathrm{dc}^2(B(x-vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x+\omega t+\theta)}. \tag{79}$$

As $m \to 0$, Eqs. (76)–(79) give rise to the trigonometric function solutions

$$q(x,t) = \pm\sqrt{\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta}}\csc(B(x-vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{80}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta}\csc^2(B(x-vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{81}$$

and

$$q(x,t) = \pm\sqrt{\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta}}\sec(B(x-vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{82}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{\delta}\sec^2(B(x-vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x+\omega t+\theta)}. \tag{83}$$

As $m \to 1$, Eqs. (76) and (77) generate the singular wave solutions

$$q(x,t) = \pm\sqrt{\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{2\delta}}\coth(B(x-vt)) \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{84}$$

$$r(x,t) = \frac{2\delta - 3d_1}{3c_1}\frac{\omega + 2a\kappa^2 - 2b\kappa\omega}{2\delta}\coth^2(B(x-vt)) \times \mathrm{e}^{2\mathrm{i}(-\kappa x+\omega t+\theta)}. \tag{85}$$

4. Lie symmetry analysis

In this section, we will perform the Lie classical method [2, 7–9] on system of Eqs. (12) and (13). For complex-valued functions $q(x,t)$ and $r(x,t)$, the split into real and imaginary functions is carried out as

$$q(x,t) = P_1(x,t)\mathrm{e}^{\mathrm{i}\phi},\ r(x,t) = P_2(x,t)\mathrm{e}^{2\mathrm{i}\phi}, \tag{86}$$

where phase component is given by

$$\phi = -kx + \omega t + \theta. \tag{87}$$

Here θ is phase constant.

Using (86), Eqs. (12) and (13) decompose into the following system of equations:

$$(\omega + 2ak^2 - 2bk\omega)P_1 - 2aP_{1xx} - 2bP_{1xt} - c_1P_1P_2 - d_1P_1^3 = 0, (2\omega + 4ak^2 - 4bk\omega - c_2)P_2 - aP_{2xx} - bP_{2xt} - d_2P_1^2 - \delta P_1^2P_2 = 0, (2bk-1)P_{1t} + 2(2ak - b\omega)P_{1x} = 0,$$

$$(2bk-1)P_{2t} + 2(2ak - b\omega)P_{2x} = 0. \tag{88}$$

Let us consider the Lie group of point transformations

$$t^* = t + \epsilon\tau(x,t,P_1,P_2) + O(\epsilon^2)x^* = x + \epsilon\xi(x,t,P_1,P_2) + O(\epsilon^2)P_1^* = P_1 + \epsilon\eta_1(x,t,P_1,P_2) + O(\epsilon^2)P_2^* = P_2 + \epsilon\eta_2(x,t,P_1,P_2) + O(\epsilon^2) \tag{89}$$

with small parameter $\epsilon \ll 1$.

We find that the infinitesimal functions ξ, τ, η_1 and η_2 must satisfy the invariance conditions

$$(\omega + 2ak^2 - 2bk\omega)\eta_1 - 2a\eta_1^{xx} - 2b\eta_1^{xt} - c_1(\eta_1P_2 + \eta_2P_1) - 3d_1P_1^2\eta_1 = 0,$$

$$(2\omega + 4ak^2 - 4bk\omega - c_2)\eta_2 - a\eta_2^{xx} - b\eta_2^{xt} - 2d_2P_1\eta_1 - \delta(P_1^2\eta_2 + 2P_1P_2\eta_1) = 0,$$

$$(2bk-1)\eta_1^t + 2(2ak-b\omega)\eta_1^x = 0,$$

$$(2bk-1)\eta_2^t + 2(2ak-b\omega)\eta_2^x = 0, \tag{90}$$

where $\eta_1^t, \eta_2^t, \eta_1^{xx}, \eta_2^{xx}, \eta_1^{xt}$ and η_2^{xt} are extended infinitesimals. For reduction of system of Eqs. (88), we will consider the following two cases:

Case (i). When $k = \frac{1}{2b}$ and $\omega = \frac{a}{b^2}$. In this case we obtain the following symmetries:

$$\xi = C_1 + C_3(-x + \frac{2a}{b}t), \quad \tau = C_2 + tC_3, \eta_1 = 0,$$

$$\eta_2 = 0, \tag{91}$$

where C_1, C_2 and C_3 are arbitrary constants.

Corresponding infinitesimal generators are given by

$$V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial t}, V_3 = \left(-x + \frac{2a}{b}t\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}. \tag{92}$$

To obtain the symmetry reductions of Eqs. (88), we have to solve the characteristic equation

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}P_1}{\eta_1} = \frac{\mathrm{d}P_2}{\eta_2} \tag{93}$$

where ξ, τ, η_1 and η_2 are given by (91). To solve (93), we consider the following cases: (i) $V_2 + \mu V_1$, (ii) V_3.

Sub case (i). Vector field $V_2 + \mu V_1$

Solving characteristic Eq. (93), we have the following similarity variables:

$$\sigma = x - \mu t, P_1 = F(\sigma), P_2 = G(\sigma), \tag{94}$$

where σ is new independent variable and F, G are new dependent variables.

Using (94) in Eqs. (88), we obtain the following system of ordinary differential equations (ODEs):

$$(\omega + 2ak^2 - 2bk\omega)F - 2aF'' + 2b\mu F'' - c_1FG$$

$$-d_1F^3 = 0,$$

$$(2\omega + 4ak^2 - 4bk\omega - c_2)G - aG'' + b\mu G'' - d_2F^2$$

$$-\delta F^2G = 0, \tag{95}$$

where $'$ denotes derivative with respect to σ.

For this system, let us consider a special solution of the form

$$F(\sigma) = A\ \mathrm{sech}(\sigma), G(\sigma) = B\ \mathrm{sech}^2(\sigma), \tag{96}$$

where A and B are arbitrary real constants.

Substituting (96) into system of Eqs. (95), we obtain the following solution:

$$F(\sigma) = \pm\frac{1}{2b}\sqrt{\frac{6a}{\delta}}\,\mathrm{sech}(\sigma), G(\sigma) =$$

$$-\frac{3ad_2}{2\delta c_2 b^2}\mathrm{sech}^2(\sigma) \tag{97}$$

with

$$\mu = \frac{a\left(-1+4b^2\right)}{4b^3}, c_1 = \frac{c_2\left(-2\delta + 3d_1\right)}{3d_2}. \tag{98}$$

Corresponding solution of main system (12) and (13) is as follows:

$$q(x,t) = \pm\frac{1}{2b}\sqrt{\frac{6a}{\delta}}\,\mathrm{sech}\left(x - \left[\frac{a\left(4b^2-1\right)}{4b^3}\right]t\right)$$

$$\times\,\mathrm{e}^{\mathrm{i}\left(-\frac{1}{2b}x + \frac{a}{b^2}t + \theta\right)},$$

$$r(x,t) = -\frac{3ad_2}{2\delta c_2 b^2}\mathrm{sech}^2\left(x - \left[\frac{a\left(4b^2-1q\right)}{4b^3}\right]t\right)$$

$$\times\,\mathrm{e}^{2\mathrm{i}\left(-\frac{1}{2b}x + \frac{a}{b^2}t + \theta\right)} \tag{99}$$

with constraint (98).

Sub case (ii). Vector field V_3

In this case, we obtain the following similarity variables:

$$\sigma = xt - \frac{at^2}{b}, P_1 = F(\sigma), P_2 = G(\sigma), \tag{100}$$

where σ is new independent variable and F, G are new dependent variables. Using similarity variables (100) into the system of Eqs. (88), we have

$$-aF + 4b^3(\sigma F')' + 2c_1b^2FG + 2d_1b^2F^3 = 0,$$

$$(c_2b^2 - a)G + b^3(\sigma G')' + d_2b^2F^2$$

$$+\delta b^2F^2G = 0, \tag{101}$$

where $'$ denotes derivative with respect to σ.

We obtain the following solution of the system (101):

$$F(\xi) = \frac{1}{2}\sqrt{-2\frac{b}{d_1\sigma}}, G(\xi) = -\frac{d_2}{\delta}. \tag{102}$$

Corresponding solution of main system of Eqs. (12) and (13) is as follows:

$$q(x,t) = \frac{1}{2}\sqrt{\frac{2b}{d_1(\frac{at^2}{b} - xt)}}\,\mathrm{e}^{\mathrm{i}\left(-\frac{1}{2b}x + \frac{a}{b^2}t + \theta\right)}, r(x,t) =$$

$$-\frac{d_2}{\delta}\mathrm{e}^{2\mathrm{i}\left(-\frac{1}{2b}x + \frac{a}{b^2}t + \theta\right)}. \tag{103}$$

Case (ii). When k and ω are arbitrary

In this case we obtain the following symmetries:

$$\xi = F\left(t - \frac{bx}{2a}\right), \tag{104}$$

$$\tau = \frac{(2bk-1)}{(-2b\omega + 4ak)}F\left(t - \frac{bx}{2a}\right) + 1, \tag{105}$$

$$\eta_1 = 0, \tag{106}$$

$$\eta_2 = 0, \tag{107}$$

where $F\left(t - \frac{bx}{2a}\right)$ is arbitrary function.

We consider the special case for $F\left(t - \frac{bx}{2a}\right) = \sin\left(t - \frac{bx}{2a}\right)$ with constraint $\omega = \frac{a}{b^2k}$. In this case, we have the following similarity variables:

$$\zeta = \frac{b}{2a}\sin\left(t - \frac{bx}{2a}\right) + 1, P_1(x,t) = F(\zeta), P_2(x,t) =$$

$$G(\zeta), \tag{108}$$

where ζ is new independent variable and F, G are new dependent variables.

Corresponding reduction of Eqs. (88) to system of ODEs is as follows:

$$F'(k-1) = 0, G'(k-1) = 0,$$

$$8a^4(2bk - 2b^2k^3 - 1)F - b^6k(1 - 4\frac{a^2}{b^2}(\zeta-1)^2)F''$$

$$+4ka^2b^4(\zeta-1)F' + 8c_1kb^2a^3FG + 8d_1kb^2a^3F^3 = 0,$$

$$(64kba^4 + 16c_2kb^2a^3 - 32a^4 - 64a^4k^3b^2)G$$
$$-b^6k(1 - 4\frac{a^2}{b^2}(\zeta-1)^2)G'' + 4ka^2b^4(\zeta-1)G'$$
$$+16d_2kb^2a^3F^2 + 16\delta kb^2a^3F^2G = 0, \quad (109)$$

where $'$ denotes derivative with respect to ζ.

We obtain constant solution of the system of Eqs. (109). Corresponding solutions of main system of Eqs. (12)–(13) is as follows:

$$p(x,t) = l\,\mathrm{e}^{\mathrm{i}\left(-kx+\frac{a}{b^2k}t+\theta\right)}, q(x,t) =$$
$$m\,\mathrm{e}^{2\mathrm{i}\left(-kx+\frac{a}{b^2k}t+\theta\right)}, \quad (110)$$

where constants l, m satisfies the following conditions:

$$a + 2ab^2k^3 - 2abk - c_1b^2km - d_1b^2kl^2 = 0,$$
$$(2a + 4ab^2k^3 - 4abk - c_2b^2k)m - d_2b^2kl^2$$
$$-\delta b^2kl^2m = 0. \quad (111)$$

5. Conclusions

This paper obtained soliton solutions such as cnoidal waves and snoidal waves with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinear susceptibilities. Bright, dark, and singular soliton solutions are retrieved using several integration algorithms. These are method of undetermined coefficients, mapping method as well as the Lie symmetry analysis. In the limiting case, when the modulus of ellipticity approaches zero or unity, soliton solutions or periodic wave solutions emerge as the case may be. The conserved quantity appears with a limitation on parameter coefficients. These results pave the way to further research in this direction. Later, additional integration tool will be applied to this model to secure soliton and other possible solutions. These results will be disseminated elsewhere.

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