

A Fifth-Order Korteweg–de Vries Equation for Shallow Water with Surface Tension: Multiple Soliton Solutions

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In this work we study a fifth-order Korteweg–de Vries equation for shallow water with surface tension derived by Dullin et al. The fifth-order Korteweg–de Vries equation, derived by using the nonlinear/non-local transformations introduced by Kodama, and the Camassa–Holm equation with linear dispersion, have very different behaviors despite being asymptotically equivalent. We use the simplified form of the Hirota direct method to derive multiple soliton solutions for this equation.

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1. Introduction

In the context of nonlinear fifth-order KdV type equations, studies are flourishing because these equations are able to describe the real features in a variety of scientific applications and engineering areas and would have much practical/physical meaning [1–17]. Fifth-order KdV (KdV5) type equations take the form

$$u_t = u_{xxxxx} + f(x, t, u_x, u_{xx}, u_{xxx}), \quad (1)$$

arising naturally in modeling many different wave phenomena such as gravity-capillary waves, the propagation of shallow water waves over a flat surface and magneto-sound propagation in plasmas [5]. The fifth-order KdV type equations appear in the literature in three main families, given as:

1. The first family reads [18]:

$$u_t + auu_x + bu_{3x} - ru_{5x} = 0, \quad (2)$$

where a, b, r are constants. This equation is called the Kawahara equation which models the dynamics of long waves in a viscous fluid.

2. The second family of nonlinear fifth-order KdV (fKdV) equations in its standard form reads [18–30]:

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma uu_{3x} + u_{5x} = 0, \quad (3)$$

where α, β , and γ are arbitrary nonzero and real parameters, and $u = u(x, t)$ is a sufficiently smooth function. The fifth-order KdV Eqs. (3) involve the linear dispersive term u_{5x} in addition to three nonlinear terms. Because the parameters α, β , and γ are arbitrary and take different values, this will drastically change the characteristics of the fKdV Eq. (3). A variety of the fKdV equations can be developed by changing the real values of the parameters α, β , and γ [18]. The most well-known fifth-order KdV equations which were examined thoroughly are the Sawada–Kotera (SK) equation, the Caudrey–Dodd–Gibbon equation, the Lax equation, the

Kaup–Kuperschmidt (KP) equation, and the Ito equation. These forms are:

(i) the Sawada–Kotera (SK) equation [18] is given by

$$u_t + 5u^2 u_x + 5u_x u_{xx} + 5uu_{3x} + u_{5x} = 0; \quad (4)$$

(ii) the Caudrey–Dodd–Gibbon equation (CDG) [18] is given by

$$u_t + 180u^2 u_x + 30u_x u_{xx} + 30uu_{xxx} + u_{xxxxx} = 0; \quad (5)$$

(iii) the Lax equation [18] reads

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10uu_{3x} + u_{5x} = 0; \quad (6)$$

(iv) the Kaup–Kuperschmidt (KK) equation [18] reads

$$u_t + 20u^2 u_x + 25u_x u_{xx} + 10uu_{3x} + u_{5x} = 0 \quad (7)$$

and is characterized by

$$\beta = \frac{5}{2}\gamma, \alpha = \frac{1}{5}\gamma^2; \quad (8)$$

(v) the Ito equation [18]

$$u_t + 2u^2 u_x + 6u_x u_{xx} + 3uu_{3x} + u_{5x} = 0. \quad (9)$$

It was found that the SK, CDG, Lax, and KK equations belong to the completely integrable hierarchy of higher-order KdV equations. These four equations have infinite sets of conservation laws, and therefore these equations have N -soliton solutions. However, the Ito equation is not completely integrable but has a limited number of special conservation laws.

3. The third family reads [19–30]:

$$u_{ttt} - u_{txxxx} - \alpha(u_t u_x)_{xx} - \beta(u_x u_{xt})_x = 0, \quad (10)$$

or equivalently

$$u_{ttt} - u_{txxxx} - \alpha(u_t u_x)_{xx} - \beta(u_x u_{xx})_t = 0. \quad (11)$$

The third family is integrable only if $\alpha = \beta$.

However, Dullin et al. [1, 2] used the Kodama transformation given in [3, 4] to transform the Camassa–Holm (CH) equation to a fifth-order KdV equation, referred to by KdV5, given by

$$u_t + cu_x + 3uu_x + 5\alpha^2(uu_{xx} + 2u_x u_{xx})$$

$$+ \frac{15}{2} \frac{\alpha^2}{\beta} u^2 u_x + \beta(\alpha^2 u_{xxxxx} + u_{xxx}) = 0, \quad (12)$$

which works as a model for the shallow water waves with

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surface tension, $u(x, t)$ is the fluid velocity, α^2 and β are the length scales, and c is the linear wave speed [1–6]. Unlike the second family of fifth-order KdV equations, the KdV5 includes two linear dispersive terms u_{xxx} and u_{xxxxx} . For $\alpha = 0$, the KdV5 (12) equation recovers the standard KdV equation.

Through the Kodama transformation [1–6]:

$$U = u + \xi \left(\frac{\alpha^2}{\beta} u^2 + 2 \frac{\alpha^2}{\beta} u_x \partial^{-1} u \right) + 2\delta^2 \alpha^2 u_{xx}, \quad (13)$$

the CH equation [9]

$$m_t + cm_x + Um_x + 2mU_x + \beta U_{xxx} = 0 \quad (14)$$

can be transformed to the KdV5 Eq. (12), where $m = U^2 - \alpha^2 U$ is a momentum variable, and $U(x, t)$ represents the height of the water free surface above a flat bottom. The CH equation can be derived as an asymptotic model for long gravity waves at the surface of shallow water. The CH equation retains non dominant terms with respect to the dominant balance between nonlinearity and dispersion modeled by the KdV equation. Besides being a model equation for water waves, the CH equation has its integrable bi-Hamiltonian structure, and has peaked solitary wave solutions [9].

In Refs. [1, 2], it was shown that the asymptotic equivalence of the CH equation to the KdV5 equation breaks down in the limit $\beta \rightarrow 0$, because the transformation as well as the resulting equation contain terms divided by β , and the CH equation provides peak on solutions whereas the KdV5 equation gives soliton solutions as will be discussed later. Moreover, the CH equation does not contain the term u_{xxxxx} that makes it easier to integrate numerically if compared with the KdV5 Eq. (12).

In Refs. [1, 2], it was shown that the CH equation is asymptotically equivalent to KdV5 equation by using the nonlinear/ nonlocal transformations introduced in the Kodama transformation [3, 4]. Moreover, Dullin et al. [1, 2] classified the travelling wave solutions of the KdV equation as a function of the Bond number by using phase plane analysis. In Ref. [5], the authors examined the binary Bell polynomials, Lax pair and the infinite conservation laws for the KdV5 equation.

It is interesting to note that the extended KdV equation, as first called by Marchant and Smyth [31], of second order in small parameters, was subjected to useful studies in [32–34] and some of the references therein. In Ref. [31], it is shown that the extended KdV equation can be transformed (to its order of approximation) to a higher-order member of the KdV hierarchy of integrable equations. However, in Ref. [32], the equations obtained in the higher orders of approximation, in general, also differ from the KdV equation with higher order corrections. Also, including surface tension in general does not alter the structure of the leading order and higher order equations [32]. Moreover, in [33], the derivation of a KdV type equation, second order in small parameters, containing terms from the bottom function was examined. The energy invariant for shallow-water waves and the Korteweg–de Vries equation were examined in detail

in [34]. Generally, KdV2 equation (second order with respect to small parameters) are examined thoroughly in [31–34] in addition to [1, 2].

Studies of finding soliton solutions of the nonlinear equations attracted huge number of works in a variety of fields in [19–30] and some of the references therein. Towards this goal, a variety of powerful methods to construct multiple soliton solutions has been established in the fields of mathematical physics and engineering. Examples of the methods that have been used are the Hirota bilinear method [7–30], the simplified Hirota method [8], the Bäcklund transformation method, the Darboux transformation, Pfaffian technique, the inverse scattering method, the Painlevé analysis, the generalized symmetry method, the subsidiary ordinary differential equation method, the coupled amplitude-phase formulation, the sine-cosine method, the sech-tanh method, the mapping and the deformation approach, and many other methods. The Hirota bilinear method [7], and the simplified Hirota method developed in [8] are rather heuristic and significant to handle equations with constant coefficients. These two methods possess powerful features that make it practical for the determination of single soliton and multiple soliton solutions for a wide class of nonlinear evolution equations. The simplified Hirota method [8] does not depend on the construction of the bilinear forms, instead it assumes that the multi-soliton solutions can be expressed as polynomials of exponential functions. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

In this work we plan to use the simplified Hirota method to formally derive multiple soliton solutions for the KdV5 Eq. (12). The multiple singular soliton solutions will be examined as well.

2. The KdV5 equation

In this section we will study the KdV5 equation

$$u_t + cu_x + 3uu_x + 5\alpha^2(uu_{xxx} + 2u_x u_{xx}) + \frac{15}{2} \frac{\alpha^2}{\beta} u^2 u_x + \beta(\alpha^2 u_{xxxxx} + u_{xxx}) = 0, \quad \beta \neq 0. \quad (15)$$

To determine the dispersion relation for (15) we substitute

$$u(x, t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t, \quad (16)$$

into the linear terms of (15) and solve the resulting equation for the dispersion relation c_i to find that

$$c_i = ck_i + \beta(k_i^3 + \alpha^2 k_i^5), \quad i = 1, 2, 3. \quad (17)$$

Consequently, the phase variables read

$$\theta_i = k_i x - [ck_i + \beta(k_i^3 + \alpha^2 k_i^5)] t, \quad i = 1, 2, 3. \quad (18)$$

To determine the single soliton solution, we use the transformation

$$u(x, t) = R(\ln f(x))_{xx}, \quad (19)$$

where the auxiliary function $f(x, t)$, for the single soliton solution is given by

$$f(x, t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - (ck_1 + \beta(k_1^3 + \alpha^2 k_1^5))t}. \quad (20)$$

Substituting (20) into (15) and solving for R we find

$$R = 4\beta. \quad (21)$$

This in turn gives the single soliton solution as

$$u(x, t) = \frac{4\beta k_1^2 e^{k_1 x - [ck_1 + \beta(k_1^3 + \alpha^2 k_1^5)]t}}{\left\{1 + e^{k_1 x - [ck_1 + \beta(k_1^3 + \alpha^2 k_1^5)]t}\right\}^2}. \quad (22)$$

Notice that using the auxiliary function as

$$f(x, t) = 1 - e^{\theta_1} = 1 - e^{k_1 x - [ck_1 + \beta(k_1^3 + \alpha^2 k_1^5)]t}, \quad (23)$$

gives the single singular soliton solution

$$u(x, t) = -\frac{4\beta k_1^2 e^{k_1 x - [ck_1 + \beta(k_1^3 + \alpha^2 k_1^5)]t}}{\left\{1 - e^{k_1 x - [ck_1 + \beta(k_1^3 + \alpha^2 k_1^5)]t}\right\}^2}, \quad (24)$$

which blows up when the denominator becomes zero for specific values of x, t, α^2, β and k_1 . For the two soliton solutions we set the auxiliary function as

$$f(x, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2}, \quad (25)$$

where the phase variables $\theta_i, i = 1, 2, 3$ are given earlier in (18), and a_{12} is the phase shift that will be determined. Substituting (25) and (19) into (15) and solving for the phase shift a_{12} , we find

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad (26)$$

which can be generalized to

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq 3. \quad (27)$$

Substituting (25), (26) into (19) gives the two soliton solutions.

However, using the auxiliary functions as

$$f(x, t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}\theta_1 + \theta_2, \quad (28)$$

gives the two singular soliton solutions.

For the three soliton solutions, we set the auxiliary function by

$$f(x, t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1 + \theta_2} + a_{13}e^{\theta_1 + \theta_3} + a_{23}e^{\theta_2 + \theta_3} + b_{123}e^{\theta_1 + \theta_2 + \theta_3}. \quad (29)$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \quad (30)$$

The three soliton solutions are obtained by substituting (29) into (19). This shows that the KdV5 equation (15) gives N -soliton solutions for finite N , where $N \geq 1$.

Moreover, using the auxiliary function in the form

$$f(x, t) = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12}e^{\theta_1 + \theta_2} + a_{13}e^{\theta_1 + \theta_3} + a_{23}e^{\theta_2 + \theta_3} - a_{12}a_{13}a_{23}e^{\theta_1 + \theta_2 + \theta_3}, \quad (31)$$

gives three singular soliton solutions.

3. Discussion

In this work we studied the KdV5 equation which works as a model for the shallow water waves with surface tension. The KdV5 equation and the CH equation with linear dispersion, have very different behaviors despite

being asymptotically equivalent. We should that the KdV5 equation gives multiple soliton solutions, whereas the Camassa–Holm equation provides peak on solutions. We used the simplified Hirota method to formally derive the multiple soliton solutions for the KdV5 equation.

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