# Solutions for Conservative Nonlinear Oscillators Using an Approximate Method Based on Chebyshev Series Expansion of the Restoring Force

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Approximate solutions for small and large amplitude oscillations of conservative systems with odd nonlinearity are obtained using a "cubication" method. In this procedure, the Chebyshev polynomial expansion is used to replace the nonlinear function by a third-order polynomial equation. The original second-order differential equation, which governs the dynamics of the system, is replaced by the Duffing equation, whose exact frequency and solution are expressed in terms of the complete elliptic integral of the first kind and the Jacobi elliptic function cn, respectively. Then, the exact solution for the Duffing equation is the approximate solution for the original nonlinear differential equation. The coefficients for the linear and cubic terms of the approximate Duffing equation — obtained by "cubication" of the original second-order differential equation — depend on the initial oscillation amplitude. Six examples of different types of common conservative nonlinear oscillators are analysed to illustrate this scheme. The results obtained using the cubication method are compared with those obtained using other approximate methods such as the harmonic, linearized and rational balance methods as well as the homotopy perturbation method. Comparison of the approximate frequencies and solutions with the exact ones shows good agreement.

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## 1. Introduction

Nonlinear oscillations in mathematics, physics, engineering, and related fields have been the subject of intensive research for many years and several methods have been used to find approximate solutions to these dynamical systems [1, 2]. Physical and mechanical oscillatory systems are often governed by nonlinear differential equations. Unfortunately, with the exception of a number of particular cases, the exact analytical solutions of such equations cannot be determined. In many cases, it is possible to replace the nonlinear differential equation by a corresponding linear differential equation that approximates the original nonlinear equation closely to give useful results [2]. Often such linearization is not feasible and for this situation the original differential equation itself must be directly dealt with [2]. However, in many cases it is possible to compute accurate approximate analytical solutions of the equations. There are a large variety of approximate methods commonly used for solving nonlinear oscillatory systems such as perturbation [1, 2], harmonic and rational harmonic balance [2], homotopy perturbation [3], homotopy analysis [4], energy [5], variational formulation [6], variational iteration [7], linearization [8], Fourier-least squares [9] methods and so on. The knowledge both exact as approximate solutions of such dynamical system offer valuable information regarding the evolution of the modelled phenomena.

One important class of nonlinear oscillator are conservative oscillators in which the restoring force is not dependent on time, the total energy is constant [1, 2] and any oscillation is stationary. An important feature of the solutions for conservative oscillators is that they are periodic and range over a continuous interval of initial values [1, 2]. It has been shown that it is possible to replace the second-order nonlinear differential equation, which governs the behaviour of a conservative nonlinear oscillator, with another equation that approximates the original equation closely enough to give accurate results. One of these schemes uses the Chebyshev series expansion of the nonlinear function, which appears in the second-order differential equation [10]. In this review we apply this technique to some of the most common conservative nonlinear oscillators. In this technique the original second-order nonlinear differential equation is replaced by the well-known Duffing equation, with linear and cubic terms, and this differential equation is solved exactly. To do this, the Chebyshev series expansion of the nonlinear function $f(x)$ is truncated and only the

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first two terms are retained. Replacement of the original nonlinear differential equation by an "approximate cubic Duffing equation" allows us to obtain an approximate frequency–amplitude relation as a function of the complete elliptic integral of the first kind and the periodic solution in terms of the Jacobi elliptic function [2].

The accuracy of the "cubication" procedure is illustrated by obtaining the approximate frequency and periodic solution for six conservative nonlinear oscillators: (1) anti-symmetric, constant force oscillator, (2) anti-symmetric quadratic nonlinear oscillator, (3) oscillator with fractional-power restoring force, (4) oscillator typified as a mass attached to two stretched elastic springs, (5) oscillator with a linear term and a fractional-power nonlinear term, and (6) the finite extensibility nonlinear oscillator (FENO). These oscillators describe the dynamical behaviour response of several dynamic systems. The analytical approximate frequencies obtained using the cubication method are compared with those calculated using other approximate methods such as the harmonic, linearized, and rational balance methods as well as the homotopy perturbation method. This comparison allows us to conclude that the cubication method is as accurate as the second-order approximation of all of these methods, however to obtain the second-order (or higher) approximate frequency using all these methods is more difficult than to obtain it using the cubication method. This allows us to conclude that the method considered in this paper is very simple and easy to apply.

## 2. Solution method

Conservative single-degree-of-freedom nonlinear oscillators are modelled by second-order autonomous ordinary differential equations of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + f(x) = 0 \tag{1}$$

with initial conditions

$$x(0) = A, \quad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0, \tag{2}$$

where $x$ and $t$ are the non-dimensional displacement and time, respectively. In Eqs. (1) and (2), $f(x)$ is the nonlinear function and $A$ is the initial oscillation amplitude. From Eq. (1), we conclude that the conservative nonlinear restoring force is given by $F(x) = -f(x)$. We assume that the nonlinear function $f(x)$ is odd, i.e. $f(-x) = -f(-x)$ and satisfies $xf(x) > 0$ for $x \in [-A, A]$, $x \neq 0$, where $A$ is the oscillation amplitude. The motion is assumed to be periodic and the problem is to determine the angular frequency of oscillation, $\omega$, and corresponding solution, $x$, as a function of the time, $t$, the system parameters and the oscillation amplitude $A$.

Now we assume that the nonlinear function $f(x)$ can be replaced by an equivalent representation form. In this paper, we adopt the Chebyshev polynomials of the first kind to carry out this replacement [10]. The technique is based on the expansion of the nonlinear function $f(x)$ in terms of the Chebyshev polynomial series around the equilibrium point $x = 0$. To do this, a new reduced variable, $y = x/A$, is introduced into Eq. (1) to give

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{1}{A}f(Ay) = 0. \tag{3}$$

Therefore, the initial conditions of Eq. (2) become

$$y(0) = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 0. \tag{4}$$

Following Denman [11] and Jonckheere [12], the nonlinear function $f(Ay)$ is expanded in terms of the Chebyshev polynomials of the first kind $T_n(y)$ as

$$f(Ay) = \sum_{n=0}^{\infty} b_{2n+1}(A)T_{2n+1}(y), \tag{5}$$

where the coefficients $b_{2n+1}$ are calculated as follows [6, 7]:

$$b_{2n+1}(A) = \frac{2}{\pi}\int_{-1}^{+1} \frac{1}{\sqrt{1-y^2}} f(Ay)T_{2n+1}(y)\,\mathrm{d}y. \tag{6}$$

As we can see, these coefficients depend on the amplitude $A$. In Eq. (5) we took into account that $f(x)$ is an odd function of $x$ and then only the polynomials $T_1$, $T_3$, $T_5$, ..., appear in the series expansion. The first polynomials are [13, 14]:

$$T_1(y) = y, \quad T_3(y) = 4y^3 - 3y,$$
$$T_5(y) = 16y^5 - 20y^3 + 5y. \tag{7}$$

As Eq. (5) is a series with an infinite number of terms, it is necessary to do an approximation to solve the problem analytically. In the simplest approximation ("linearization" procedure [15]), only the first term in Eq. (5) is retained and the restoring force is replaced by a linear equation

$$f(Ay) \approx b_1(A)T_1(y) = b_1(A)y = \frac{b_1(A)}{A}x. \tag{8}$$

The nonlinear differential equation in Eq. (1) is approximated by the equation corresponding to the linear oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{b_1(A)}{A}x = 0. \tag{9}$$

Therefore, the approximate frequency $\omega(A)$ and the corresponding periodic solution $x(t)$ to Eq. (1) are, respectively

$$\omega(A) = \sqrt{\frac{b_1(A)}{A}} \tag{10}$$

and

$$x(t) = A\cos\left(\sqrt{\frac{b_1(A)}{A}}t\right). \tag{11}$$

To obtain a better approximation, the first two terms in Eq. (5) are retained

$$f(Ay) \approx b_1(A)T_1(y) + b_3(A)T_3(y) = [b_1(A) - 3b_3(A)]y$$
$$+4b_3(A)y^3 = \frac{b_1(A) - 3b_3(A)}{A}x + \frac{4b_3(A)}{A^3}x^3. \tag{12}$$

Notice that the nonlinear function $f(Ay)$ is replaced by an equivalent form by using only two terms of the Chebyshev polynomial expansion to ensure a polynomial cubic equation. Therefore, we may write Eq. (1) as an equivalent cubic Duffing oscillator as follows:

$$\frac{d^2x}{dt^2} + \alpha(A)x + \beta(A)x^3 \approx 0, \tag{13}$$

in which, coefficients $\alpha$ and $\beta$ are defined in terms of coefficients $b_1$ and $b_3$ of the Chebyshev polynomial expansion of the nonlinear function as

$$\alpha(A) = \frac{b_1(A) - 3b_3(A)}{A}, \quad \beta(A) = \frac{4b_3(A)}{A^3}. \tag{14}$$

Taking this into account, the "cubication" procedure consists in approximating the original nonlinear differential equation in Eq. (1) by Eq. (13) — which is the nonlinear differential equation for the Duffing oscillator. This means that the approximate frequency and solution for the original differential equation will be the exact frequency and solution for the Duffing equation [2]:

$$\omega(A) = \frac{\pi\sqrt{\alpha + \beta A^2}}{2K(m)}, \quad m = \frac{\beta A^2}{2(\alpha + \beta A^2)}, \tag{15}$$

$$x(t) = A\mathrm{cn}\left(t\sqrt{\alpha + \beta A^2}; m\right), \tag{16}$$

where $K(m)$ is the complete elliptic integral of the first kind given by the following equation:

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} \tag{17}$$

and $\mathrm{cn}(z; m)$ is the Jacobi elliptic function.

Substituting Eq. (14) into Eqs. (15) and (16), we can write the approximate equations for the frequency and periodic solution of the initial nonlinear oscillator as a function of the first two coefficients of the Chebyshev series expansion of the nonlinear function $f(x)$ [10]:

$$\omega(A) = \frac{\pi\sqrt{b_1 + b_3}}{2\sqrt{A}K\left(2b_3/(b_1 + b_3)\right)}, \tag{18}$$

$$x(t) = A\mathrm{cn}\left(t\sqrt{\frac{b_1 + b_3}{A}}; \frac{2b_3}{b_1 + b_3}\right). \tag{19}$$

Thus, the approximate frequency and periodic solution of Eq. (1) are given by Eqs. (18) and (19) with $b_1$ and $b_3$ given by Eq. (6) in terms of the oscillation amplitude $A$.

As can be seen, Eqs. (18) and (19) are very simple and they can be easily computed with the help of symbolic computation software such as MATHEMATICA.

In the next section, we compare the approximate frequency and periodic solution given by Eqs. (18) and (19) with the exact ones.

## 3. Examples

In this section, we present six examples to illustrate the usefulness and effectiveness of the "cubication" technique based on the truncated Chebyshev series expansion of the nonlinear function, $f(x)$.

*Example 1. Anti-symmetric, constant force oscillator*

For this dynamical system, the nonlinear function is

$$f(x) = \mathrm{sgn}(x). \tag{20}$$

From Eq. (6), the first two coefficients of the Chebyshev series expansion of Eq. (20) are given as follows:

$$b_1 = \frac{4}{\pi}, \quad b_3 = -\frac{4}{3\pi}. \tag{21}$$

Substituting Eq. (21) into Eqs. (18) and (19), we obtain the approximate frequency and periodic solution as

$$\omega(A) = \frac{1}{K(-1)}\sqrt{\frac{2\pi}{3A}} = \frac{2}{K(1/2)}\sqrt{\frac{\pi}{3A}} \approx \frac{1.103878}{\sqrt{A}}, \tag{22}$$

$$x(t) = A\mathrm{cn}\left(\sqrt{\frac{8}{3\pi A}}t; -1\right) \approx A\mathrm{cn}\left(0.921318A^{-1/2}t; -1\right). \tag{23}$$

The approximate frequency given in Eq. (22) has already been obtained in Ref. [10]. For the purpose of comparison, the following exact frequency $\omega_e(A)$ is [2]:

$$\omega_e(A) = \frac{\pi}{2\sqrt{2A}} \approx \frac{1.110721}{\sqrt{A}}. \tag{24}$$

The corresponding exact solution to Eqs. (1) and (20) is

$$x_e(t) = \begin{cases} -\frac{t^2}{2} + A, & 0 \le t \le \frac{T_e}{4} \\ \frac{t^2}{2} - 2\sqrt{2A}t + 3A, & \frac{T_e}{4} \le t \le \frac{3T_e}{4} \\ -\frac{t^2}{2} + 4\sqrt{2A}t - 15A, & \frac{3T_e}{4} \le t \le T_e \end{cases} \tag{25}$$

where $T_e = 2\pi/\omega_e$ is the exact period of the oscillation. For comparison, the ratio of the approximate frequency $\omega$ given by Eq. (22) to the exact frequency $\omega_e$ in Eq. (24) is

$$\frac{\omega(A)}{\omega_e(A)} = \frac{4\sqrt{2}}{\sqrt{3\pi}K(1/2)} \approx 0.993830. \tag{26}$$

It may be seen that the relative error of the approximate frequency is 0.6%.

Wu et al. [16] approximately solved the anti-symmetric nonlinear oscillator applying the Newton-harmonic balancing approach (NHBM) and they obtained that the relative errors for the approximate frequency were 1.6%, 0.3%, and 0.10% for the first-, second- and third-order approximation, respectively. As we can see, the second- and third-order approximate frequencies are better than that obtained in this paper, although this last frequency is close to the second-order approximate frequency obtained using the NHBM.

This oscillator has been also solved using a novel rational harmonic balance method (RHBM) [17] and the relative error for the approximate frequency obtained was 0.24%, which is similar to that obtained using the second-order NHBM.

Beléndez et al. [18] approximately solved the anti-symmetric, constant force oscillator, using a modify homotopy perturbation method (MHPM). They achieved relative errors of 1.6%, 0.3%, and 0.06% for the first-, second-, and third-order analytical approximations, respectively. Once again, the approximate frequency obtained in this paper is less accurate (0.6% versus 0.3%) than the second-order frequency obtained using the MHPM.

In summary, for this oscillator the approximate frequency obtained in this paper is more accurate than the first-order frequency obtained using the NHBM and the MHPM (0.6% versus 1.6%) and less accurate than the second-order frequency obtained using the NHBM, RHBM, and MHPM. However, we can see that the method considered in this paper is very simple and easy to apply than the other methods mentioned.

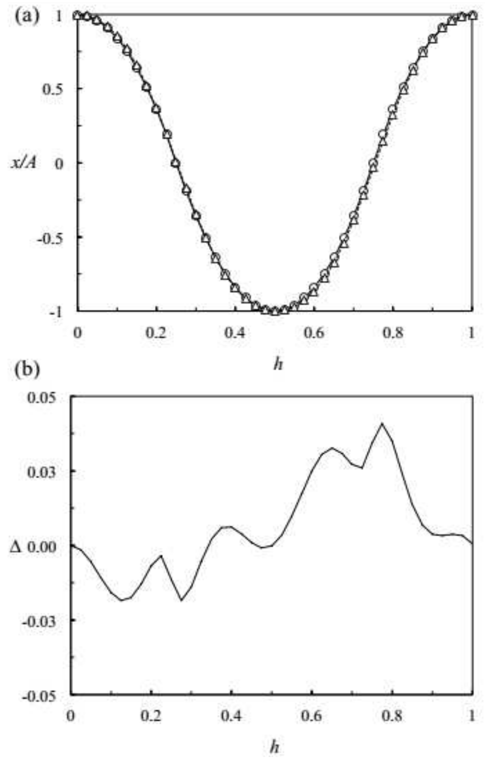


Fig. 1. (a) Comparison of the approximate solution (dashed line and triangles) with the numerical solution (continuous line and squares), and (b) difference between normalized exact and approximate solutions for the anti-symmetric, constant force oscillator.

The normalized periodic exact solution, $x_{\rm e}/A$ (circles), in Eq. (25), and the approximate solution, $x/A$ (triangles), calculated using Eq. (23), are shown in Fig. 1a, whereas in Fig. 1b the difference $\Delta = (x_{\rm e} - x)/A$ has been plotted. In these figures the non-dimensional parameter $h$ is defined as $h = t/T_{\rm e}$. These figures show that Eqs. (22) and (23) provide excellent approximations to the exact frequency and solution in Eqs. (24) and (25). The value for the "$L_2$–norm" is also analyzed in order to obtain a global estimation of the accuracy of the approximate solution, where $L_2$ is calculated as follows [19]:

$$L_2 = \sqrt{\int_0^1 \left| \frac{x_{\rm e}(h) - x(h)}{A} \right|^2 {\rm d}h}. \tag{27}$$

For the anti-symmetric, constant force oscillator, the value for $L_2$ is 0.0176.

*Example 2. Anti-symmetric quadratic nonlinear oscillator*

The nonlinear function for this oscillator is

$$f(x) = x\,|x|\,. \tag{28}$$

Then from Eq. (6) we obtain

$$b_1 = \frac{8A^2}{3\pi}, \quad b_3 = \frac{8A^2}{15\pi}. \tag{29}$$

Substituting Eqs. (28) and (29) into Eqs. (18) and (19) yields the following analytical approximations for the frequency and periodic solution of the anti-symmetric quadratic nonlinear oscillator:

$$\omega(A) = \frac{2}{K(1/3)}\sqrt{\frac{\pi A}{5}} \approx 0.914306\sqrt{A}, \tag{30}$$

$$x(t) = A{\rm cn}\left(\frac{4}{\sqrt{5\pi A}}t; \frac{1}{3}\right) \approx A{\rm cn}\left(1.009253A^{-1/2}t; \frac{1}{3}\right). \tag{31}$$

The exact frequency $\omega_{\rm e}(A)$ for this system is [20]:

$$\omega_{\rm e}(A) = \sqrt{\frac{3\pi A}{2}}\frac{\Gamma(5/6)}{\Gamma(1/3)} \approx 0.914681\sqrt{A} \tag{32}$$

and the ratio of the approximate frequency $\omega$ (Eq. (30)) to the exact frequency $\omega_{\rm e}$ (Eq. (32)) is

$$\frac{\omega(A)}{\omega_{\rm e}(A)} = \frac{\sqrt{15}K(1/3)\Gamma(5/6)}{2\sqrt{8}\Gamma(1/3)} \approx 1.00041. \tag{33}$$

Equation (33) allows us to conclude that the relative error of the approximate frequency is as low as 0.04% and that Eq. (30) provides an excellent approximation to the exact frequency in Eq. (32).

Beléndez et al. [21] approximately solved the anti-symmetric quadratic nonlinear oscillator applying a generalized RHBM and the relative error for the approximate frequency they obtained was 0.07%. In this case, the relative error obtained using the method considered in this paper is better than that obtained using the RHBM.

By applying the first approximation based on the harmonic balance method (HBM) and a second-order RHBM, Mickens [2] achieved analytical approximate frequencies which relative errors were 0.7% and 0.07% [22], respectively. Both of them are less accurate than that calculated in this paper.

Beléndez et al. [22] approximately solved this oscillator using the MHPM. They achieved relative errors of 0.7%, 0.05%, and 0.03% for the first-, second-, and third-order analytical approximations, respectively. As we can see, the approximate frequency obtained using the cubication method is more accurate than the first- and second-order frequencies obtained using the MHPM and slightly less accurate (0.04% versus 0.03%) than the third-order frequency obtained using the MHPM.

We can conclude than for this nonlinear oscillator, the cubication method gives an approximate frequency more accurate than those obtained using the second-order MHPM and RHMB and only slightly less accurate than the third-order approximation obtained using the MHPM.

Figure 2a shows the normalized exact periodic solution, $x_{\rm e}/A$ (circles), computed numerically, and the approximate solution, $x/A$ (triangles), given by Eq. (31), and Fig. 2b presents the difference $\Delta = (x_{\rm e} - x)/A$. These figures indicate that the approximation to the

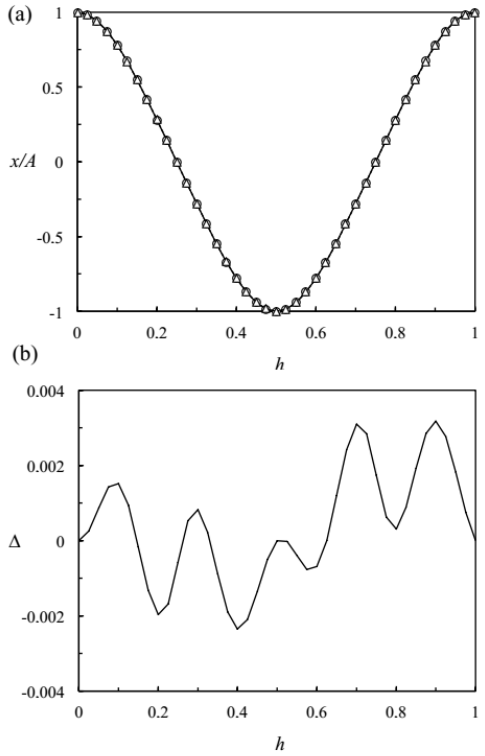


Fig. 2. (a) Comparison of the approximate solution (dashed line and triangles) with the numerical solution (continuous line and squares), and (b) difference between normalized numerical exact and approximate solutions for the anti-symmetric quadratic nonlinear oscillator.

(numerical) exact solution is excellent. For this oscillator the value for the $L_2$-norm is 0.00155.

*Example 3. Oscillator with fractional-power restoring force*

As an example of this type of oscillator, we consider the "cube root" oscillator, whose nonlinear function is [23]

$$f(x) = \operatorname{sgn}(x)\,|x|^{1/3}\,. \tag{34}$$

The Chebyshev polynomial expansion of $f(x)$ is given by Eq. (12), where coefficients $b_1$ and $b_3$ are

$$b_1 = \frac{2\Gamma(7/6)A^{1/3}}{\sqrt{\pi}\Gamma(5/3)}, \quad b_3 = -\frac{2\Gamma(7/6)A^{1/3}}{3\sqrt{\pi}\Gamma(8/3)}. \tag{35}$$

Using Eqs. (18) and (19), the approximate frequency is

$$\omega(A) = \frac{\pi^{3/4}\sqrt{2\Gamma(7/6)}}{K(-1/2)\sqrt{5\Gamma(5/3)}A^{1/3}} \approx \frac{1.068650}{A^{1/3}} \tag{36}$$

and the corresponding approximate periodic solution becomes

$$x(t) = A\mathrm{cn}\left(\sqrt{\frac{8\Gamma(7/6)}{5A^{2/3}\sqrt{\pi}\Gamma(5/3)}}t; -\frac{1}{2}\right) \approx$$

$$A\mathrm{cn}\left(0.963159A^{-1/3}t; -\frac{1}{2}\right). \tag{37}$$

For this nonlinear oscillator the exact frequency $\omega_{\mathrm{e}}(A)$ can be obtained as follows [20]:

$$\omega_{\mathrm{e}}(A) = \frac{2\sqrt{\pi}\Gamma(5/4)}{\sqrt{6}\Gamma(3/4)A^{1/3}} \approx \frac{1.070451}{A^{1/3}}. \tag{38}$$

The ratio of the approximate frequency to the exact frequency is

$$\frac{\omega(A)}{\omega_{\mathrm{e}}(A)} \approx 0.998317, \tag{39}$$

which means that the relative error of the approximate frequency is 0.17%.

Once again, we are going to compare this approximate frequency with other previously published. Lim and Wu [24] have approximately solved this oscillator by using an improved harmonic balance method in which linearization is carried out prior to harmonic balancing. They achieved relative error of 0.6% and 0.11% for the first and the second approximation orders and by applying the NHBM, Wu et al. [16] achieved frequencies with relative errors of 0.6%, 0.12%, and 0.03% for the first-, second- and third-order approximations, respectively. Now the approximate frequency obtained using the cubication method is less accurate than the third-order approximation and slightly less accurate (0.17% versus 0.12%) than the second-order frequency.

By applying the first and second approximations based on the HBM, Mickens [23] achieved for the frequency relative errors of 2.0% and 0.7%, respectively. As we can see, the analytical approximate frequency obtained

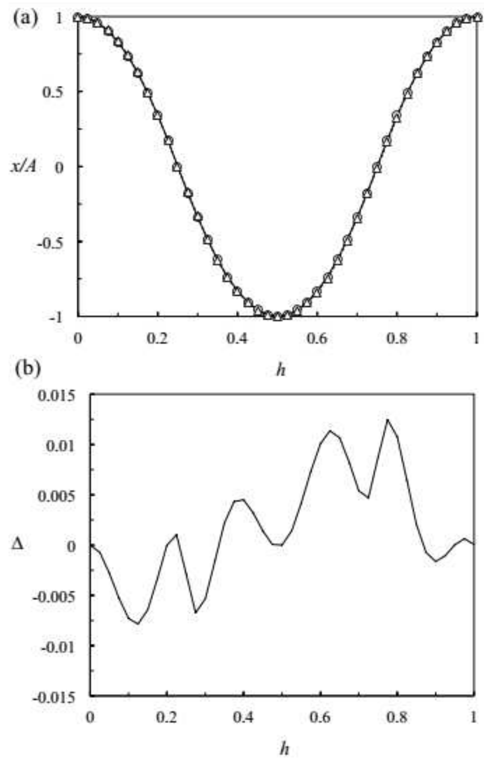


Fig. 3. (a) Comparison of the approximate solution (dashed line and triangles) with the numerical solution (continuous line and squares), and (b) difference between normalized numerical exact and approximate solutions for the oscillator with fractional-power restoring force.

in this paper is more accurate than the second order approximation obtained by Mickens (0.17% versus 0.7%).

Finally, this oscillator has been also solved using the MHPM [25] and the relative errors achieved for the first-, second-, and third-order approximations were 0.6%, 0.17%, and 0.024%, respectively.

As in previous examples, the normalized periodic (numerical) exact solution, $x_{\mathrm{e}}/A$ (circles), and the approximate solution, $x/A$ (triangles), calculated using Eq. (37), are plotted in Fig. 3a, and Fig. 3b shows the difference $\Delta = (x_{\mathrm{e}} - x)/A$. Once again, these figures show that the approximation to the exact solution is excellent. For this oscillator $L_2 = 0.00565$.

*Example 4. Mass attached to two stretched elastic springs*

The nonlinear function for this system is written as follows [2]:

$$f(x) = x - \frac{\lambda x}{\sqrt{1+x^2}}, \tag{40}$$

where $0 < \lambda \leq 1$. The approximate frequency for this oscillator has already been calculated in Ref. [10]. Now, the approximate solution is obtained. From Eq. (6) it is possible to obtain the following expressions for coefficients $b_1$ and $b_3$:

$$b_1 = A\left[1 - \frac{4\lambda}{\pi A^2\sqrt{1+A^2}}\left((1+A^2)E\left(\frac{A^2}{1+A^2}\right) - K\left(\frac{A^2}{1+A^2}\right)\right)\right], \tag{41a}$$

$$b_3 = \frac{4\lambda}{3\pi A^3\sqrt{1+A^2}}\left[(8+9A^2+A^4)E\left(\frac{A^2}{1+A^2}\right) - (8+5A^2)K\left(\frac{A^2}{1+A^2}\right)\right], \tag{41b}$$

where $E(m)$ is the complete elliptic integral of the second kind defined as follows:

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta. \tag{42}$$

Direct integration of Eq. (1) using Eq. (40) and taking into account the initial conditions in Eq. (2), gives [26]:

$$\omega_{\mathrm{e}}(A) = \frac{\pi}{2}\left[\int_0^1 \frac{A\,\mathrm{d}u}{\sqrt{A^2(1-u^2) - 2\lambda(\sqrt{1+A^2} - \sqrt{1+A^2u^2})}}\right]^{-1}, \tag{43}$$

which must be solved numerically for each value of the oscillation amplitude $A$.

For $\lambda = 1$ and small values of the oscillation amplitude $A$, it is possible to do the power-series expansions of the exact and approximate frequencies [26]:

$$\omega_{\mathrm{e}}(A) \approx \frac{\Gamma^2(3/4)}{\sqrt{2\pi}}A - \frac{\pi^{3/2}[3\Gamma^2(3/4) + 4\Gamma^2(5/4)]}{256\sqrt{2}\Gamma^2(5/4)}A^3 + \ldots \approx 0.59907A - 0.177537A^3 + \ldots \tag{44}$$

$$\omega(A) \approx \frac{\Gamma^2(3/4)}{\sqrt{2\pi}}A - \frac{15\Gamma^2(3/4)[12\Gamma^2(5/4) - \Gamma^2(3/4)]}{512\sqrt{2\pi}\Gamma^2(5/4)} \times A^3 + \ldots \approx 0.59907A - 0.178531A^3 + \ldots \tag{45}$$

On the other hand, for $0 < \lambda < 1$ and small values of $A$, the power series expansions of the frequencies are [26]

$$\omega_{\mathrm{e}}(A) \approx \sqrt{1-\lambda} + \frac{3\lambda}{16\sqrt{1-\lambda}}A^2 + \frac{3\lambda(33-40)}{1024(1-\lambda)^{3/2}}A^4 + \ldots \tag{46}$$

and the power series expansion for the approximate frequency is

$$\omega(A) \approx \sqrt{1-\lambda} + \frac{3\lambda}{16\sqrt{1-\lambda}}A^2 + \frac{3\lambda(33-40)}{1024(1-\lambda)^{3/2}}A^4 + \ldots \tag{47}$$

From Eqs. (44)–(47) we obtain

$$\lim_{A\to 0} \frac{\omega(A)}{\omega_{\mathrm{e}}(A)} = 1. \tag{48}$$

It is also easy to verify that

$$\lim_{A\to \infty} \frac{\omega(A)}{\omega_{\mathrm{e}}(A)} = 1, \tag{49}$$

because for large $A$, the nonlinear function ($f(x)$ in Eq. (40)) tends to $x$. From Eqs. (48) and (49), we conclude that the relative error of the angular frequency tends to zero when $A$ tends to zero and to infinity.

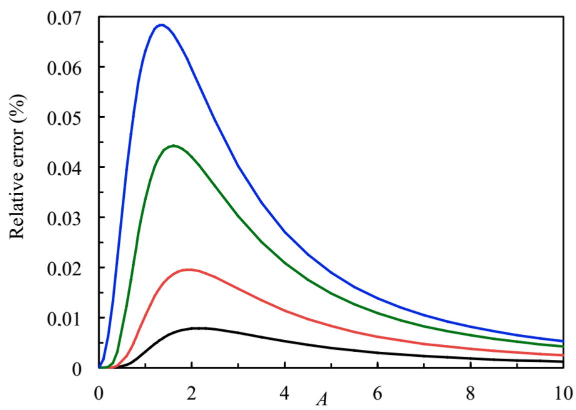


Fig. 4. Relative errors for the approximate and exact frequencies as a function of the oscillation amplitude for the oscillator typified as a mass attached to two stretched elastic springs for $\lambda = 0.5$ (black), 0.7 (red), 0.9 (green) and 1 (blue).

Figure 4 shows the relative errors of the approximate frequency as a function of $A$ for $\lambda = 0.5$, 0.7, 0.9, and 1. In this figure the relative errors are defined as $100\,|(\omega - \omega_{\mathrm{e}})/\omega_{\mathrm{e}}|$. The highest relative error (0.07%) is obtained for $\lambda = 1$ and $A = 1.36$. From Eqs. (48) and (49) and Fig. 4, we can conclude that $\omega(A)$ in Eqs. (18) and (43) gives excellent approximate frequencies for small as well as large values of oscillation amplitude $A$.

TABLE I

Comparison of the approximate frequencies with the exact one and relative errors in parentheses. Nonlinear oscillator typified as a mass attached to two stretched elastic springs for $\lambda = 1$.

| $A$ | $\omega_{\rm e}$ (Eq. (43)) | $\omega$ (cubication) | $\omega_{\rm NHBM}$ [27] | $\omega_{\rm RHBM}$ [28] |
|---|---|---|---|---|
| 0.1 | 0.05973045 | 0.59731442 (0.0017%) | 0.06026085 (0.9%) | 0.05973115 (0.0012%) |
| 0.4 | 0.22916187 | 0.22921186 (0.022%) | 0.23078447 (0.7%) | 0.22907153 (0.04%) |
| 1 | 0.48085077 | 0.48115453 (0.06%) | 0.48163170 (0.16%) | 0.48029191 (0.12%) |
| 4 | 0.83742053 | 0.83764734 (0.027%) | 0.83666882 (0.09%) | 0.83710095 (0.04%) |
| 7 | 0.90679079 | 0.90688561 (0.010%) | 0.90663702 (0.017%) | 0.90667190 (0.013%) |
| 10 | 0.93495724 | 0.93500683 (0.005%) | 0.93496420 (0.0007%) | 0.93489862 (0.006%) |

Now we compare the results obtained in this paper with those achieved using the NHBM [27] and the RHBM [28]. As can be seen in these two articles and from Fig. 4, the highest relative errors (for a value of $A$) are obtained for $\lambda = 1$. Due to this, we only compared the results obtained using the cubication method, the NHBM and the RHBM for this value of $\lambda$. Comparison of the exact frequency $\omega_{\rm e}(A)$ (Eq. (43)), with the frequency $\omega(A)$ obtained using the cubication method, the second-order approximate frequency $\omega_{\rm NHBM}(A)$ calculated using the NHBM and the frequency $\omega_{\rm RHBM}(A)$ obtained using he RHBM is shown in Table I for $\lambda = 1$. As we can see, the cubication method gives the more accurate results, only for $A = 10$ the linearized harmonic balance method (LHBM) is the best method but the relative errors are as low (0.0007%, 0.006%, and 0.005% for LHBM, RHBM, and cubication method, respectively) that the improvement is not significant.

For $\lambda = 1$ and $A = 1.36$, values that correspond to the highest relative error for the approximate frequency, the normalized periodic (numerical) exact solution, $x_{\rm e}/A$ (circles), and the approximate solution, $x/A$ (triangles), calculated using Eqs. (19) and (41), are plotted in Fig. 5a, whereas Fig. 5b shows the difference $\Delta = (x_{\rm e} - x)/A$. For $\lambda = 1$ and $A = 1.36$ we obtain $L_2 = 0.00245$.

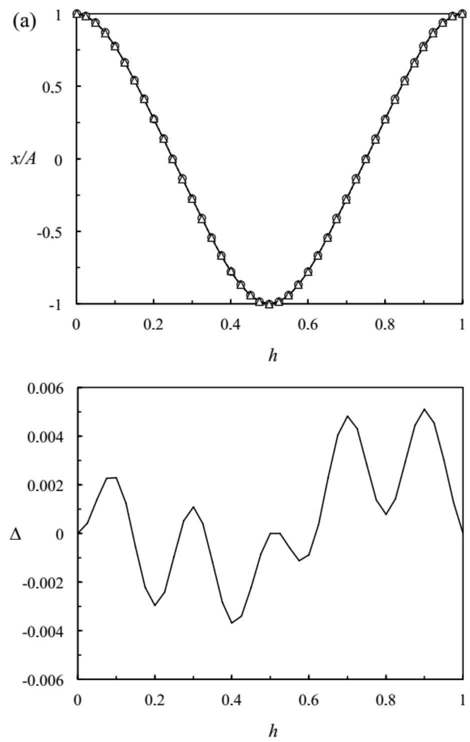


Fig. 5. (a) Comparison of the approximate solution (dashed line and triangles) with the numerical exact solution (continuous line and squares), and (b) difference between normalized numerical exact and approximate solutions for the oscillator typified as a mass attached to two stretched elastic springs ($\lambda = 1$ and $A = 1.36$).

*Example 5. Oscillator with a linear term and a fractional-power nonlinear term*

We consider an oscillator, which was analysed by Cveticanin and Pogány in [29]. The nonlinear function they studied is

$$f(x) = x + c_{5/3}^2 x\,|x|^{2/3}\,, \tag{50}$$

where $c_{5/3}^2$ is a positive constant. For this nonlinear oscillator we obtain the first two coefficients of the Chebyshev series expansion and their values are

$$b_1 = A + c_{5/3}^2 \frac{2\Gamma(11/6)A^{5/3}}{\sqrt{\pi}\Gamma(7/3)},$$
$$b_3 = c_{5/3}^2 \frac{2A^{5/3}\Gamma(11/6)}{7\sqrt{\pi}\Gamma(7/3)}. \tag{51}$$

For this oscillator, we calculated the exact frequency as follows [29]:

$$\omega_{\rm e}(A) = \frac{\pi}{2}\left[\int_0^1 \frac{{\rm d}u}{\sqrt{1 - u^2 + \frac{3}{4}c_{5/3}^2 A^{2/3}(1 - u^{8/3})}}\right]^{-1}, \tag{52}$$

which must be solved numerically for each value of $A$.

For this system, the following is satisfied:

$$\lim_{A\to 0} \frac{\omega(A)}{\omega_{\rm e}(A)} = 1. \tag{53}$$

For large values of $A$, it is possible to do the power-series expansions of the exact and the approximate frequencies, and we obtain

$$\omega_e(A) \approx 0.940814 c_{5/3} A^{1/3} + \dots, \quad (54)$$

$$\omega(A) \approx 0.940536 c_{5/3} A^{1/3} + \dots \quad (55)$$

From Eqs. (54) and (55), we have

$$\lim_{A\to\infty} \frac{\omega(A)}{\omega_e(A)} = 0.999704. \quad (56)$$

We conclude that the relative error of the angular frequency tends to zero when $A$ tends to zero and tends to 0.03% when $A$ tends to infinity.

We consider the same three examples as those analysed by Cveticanin and Pogány: $c_{5/3}^2 = 0.001$ and $c_{5/3}^2 = 1$ with $A = 0.1$, for which the procedure they developed in [29] is applicable, and $c_{5/3}^2 = 0.5$ and $A = 3$, for which the procedure proposed by these authors does not give accurate results.

The analytical approximations for the frequency and solution for $c_{5/3}^2 = 0.001$ and $A = 0.1$ are

$$\omega = 1.00096, \quad (57)$$

$$x(t) = 0.1\mathrm{cn}(1.00011t; 0.000547544) \quad (58)$$

and the exact frequency computed using Eq. (52) is also $\omega_e = 1.00096$.

For $c_{5/3}^2 = 1$ and $A = 0.1$ we obtain

$$\omega = 1.09171, \quad (59)$$

$$x(t) = 0.1cn(1.10431t; 0.0449976) \quad (60)$$

and the exact frequency is $\omega_e = 1.09172$, which implies that the relative error of the approximate frequency is 0.0009%.

Finally, for $c_{5/3}^2 = 0.5$ and $A = 3$ we obtain

$$\omega = 1.38699, \quad (61)$$

$$x(t) = 3\mathrm{cn}(1.43514t; 0.128618). \quad (62)$$

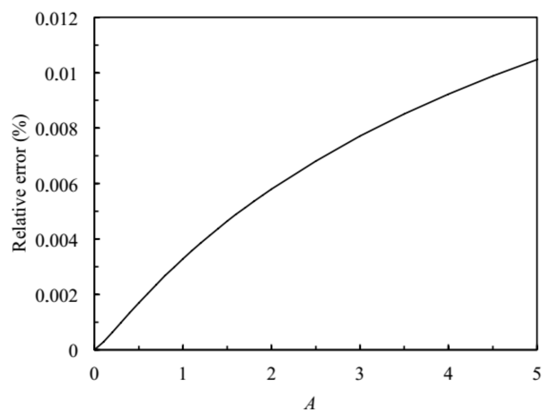


Fig. 6. Relative errors for the approximate and exact frequencies as a function of the oscillation amplitude for the oscillator with a linear term and a fractional-power nonlinear term ($c_{5/3}^2 = 0.5$).

In this nonlinear system the exact frequency is $\omega_e = 1.38710$, which implies that the relative error for the approximate frequency given in Eq. (61) is 0.008%. Figure 6 shows the relative errors of the approximate frequency as a function of $A$ for $c_{5/3}^2 = 0.5$. The graphs of the normalized periodic (numerical) exact solution, $x_e/A$ (circles), and the approximate solution, $x/A$ (triangles), calculated using Eq. (62), are plotted in Fig. 7a for $c_{5/3}^2 = 0.5$ and $A = 3$. Figure 7b shows the difference $\Delta = (x_e - x)/A$. In this case, $L_2 = 0.00058$.

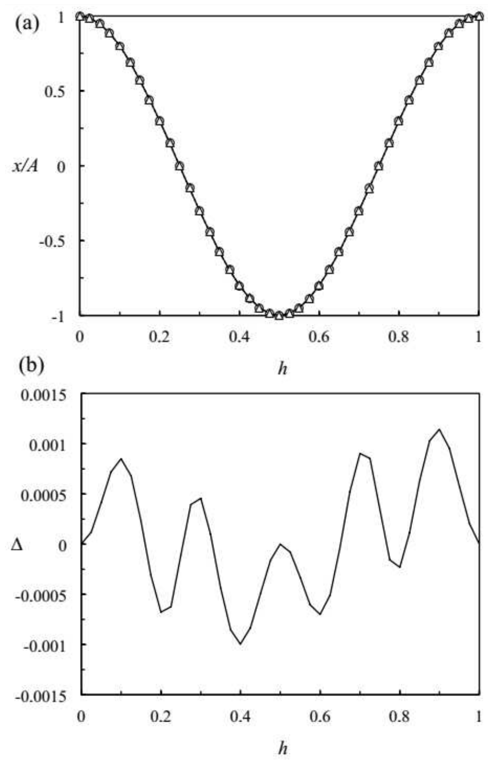


Fig. 7. (a) Comparison of the approximate solution (dashed line and triangles) with the numerical solution (continuous line and squares), and (b) difference between normalized numerical exact and approximate solutions for the oscillator with a linear term and a fractional-power nonlinear term ($c_{5/3}^2 = 0.5$ and $A = 3$).

*Example 6. Finite extensibility nonlinear oscillator, FENO*

The nonlinear function for this nonlinear system is [19]:

$$f(x) = \frac{x}{1 - x^2} \quad (63)$$

with $0 < A < 1$, where $A$ is the initial oscillation amplitude.

From Eq. (6) it is possible to obtain the following expressions for coefficients $b_1$ and $b_3$:

$$b_1 = \frac{2}{A}\left(-1 + \frac{1}{\sqrt{1 - A^2}}\right), \quad (64a)$$

$$b_3 = \frac{2}{A^3}\frac{4\lambda}{3\pi A^3\sqrt{1 + A^2}}\left[-4 + \frac{4}{\sqrt{1 - A^2}} + A^2\left(1 - \frac{3}{\sqrt{1 - A^2}}\right)\right]. \quad (64b)$$

The exact frequency can be computed as follows [13, 14]:

$$\omega_e(A) = \frac{\pi}{2}\left[\int_0^1 \frac{A\,du}{\sqrt{\log(1-A^2u^2)-\log(1-A^2)}}\right]^{-1}. \quad (65)$$

This equation must be solved numerically for each value of $A$. For small values of the oscillation amplitude $A$, it is possible to take into account the following power series expansions:

$$\omega_e(A) \approx 1 + \frac{3}{8}A^2 + \frac{59}{256}A^4 + \frac{337}{2048}A^6 + \ldots \quad (66)$$

$$\omega(A) \approx 1 + \frac{3}{8}A^2 + \frac{59}{256}A^4 + \frac{341}{2048}A^6 + \ldots \quad (67)$$

From Eqs. (66) and (67) we obtain

$$\lim_{A\to 0} \frac{\omega(A)}{\omega_e(A)} = 1. \quad (68)$$

Figure 8 shows the relative errors of the approximate frequency as a function of $A$. In this figure the relative errors are defined as $100(\omega - \omega_e)/\omega_e$. For $A < 0.9$, the relative error is less than 0.9%.

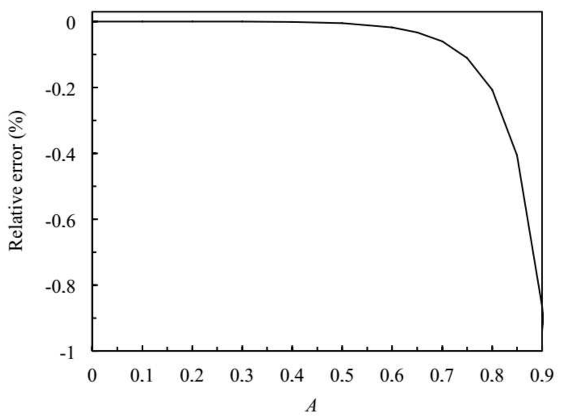


Fig. 8. Relative errors for the approximate and exact frequencies as a function of the oscillation amplitude for the finite extensibility nonlinear oscillator.

As in all previous examples considered, we compare the approximate frequency obtained using the cubication method with other methods of approximation. Febbo [19] analysed carefully this nonlinear oscillator and obtained the analytical approximate frequency using the second-order LHBM. He obtained a relative error less than 4% for oscillation amplitudes lower than 0.9. On the other hand, Beléndez et al. [30] solved this nonlinear oscillator using the second-order HBM and they reduced the relative error from 4% to 0.6%, which means that the approximate frequency they obtained is lightly more accurate than that calculated using the cubication method, but both of them have similar relative errors. For this nonlinear oscillator the best results were obtained by Elías-Zúñiga and Martínez-Romero [5] using an energy method.

For $A = 0.9$, the normalized periodic (numerical) exact solution, $x_e/A$ (circles), and the approximate

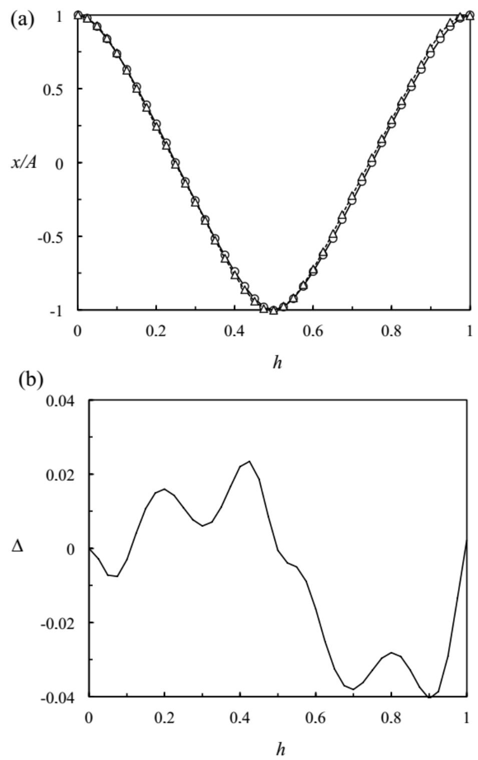


Fig. 9. (a) Comparison of the approximate solution (dashed line and triangles) with the numerical solution (continuous line and squares), and (b) difference between normalized numerical exact and approximate solutions for the finite extensibility nonlinear oscillator ($A = 0.9$).

solution, $x/A$ (triangles), calculated using Eqs. (19), (64a), and (64b) are plotted in Fig. 9a, whereas Fig. 9b shows the difference $\Delta = (x_e - x)/A$. For $A = 0.9$ we obtain $L_2 = 0.022$.

## 4. Generalization to other conservative nonlinear oscillators

It is possible to consider the following general expression for the nonlinear function [31]:

$$f(x) = \lambda_1 x + \lambda_2 \frac{\mathrm{sgn}(x)\,|x|^{\alpha_1}}{(1+\alpha_3 x^2)^{\alpha_2}}, \quad (69)$$

which corresponds to an extensive set of conservative nonlinear oscillators depending on the values of parameters $\lambda_1$, $\lambda_2$, $\alpha_1$, $\alpha_2$ and $\alpha_3$. Equation (69) includes the six oscillatory systems considered in this paper as well as a wide range of conservative nonlinear oscillators which have been analysed in the bibliography. Using Eq. (6) we obtain the following general expressions for $b_1(A)$ and $b_3(A)$:

$$b_1(A) = \lambda_1 A + \frac{2}{\sqrt{\pi}}\lambda_2 A^{\alpha_1}\Gamma\left(\frac{\alpha_1+2}{2}\right) \times {}_2F_1\left(\frac{\alpha_1+2}{2}, \alpha_2, \frac{\alpha_1+3}{2}; -\alpha_3 A^2\right), \quad (70)$$

$$b_3(A) = \frac{2}{\sqrt{\pi}}\lambda_2 A^{\alpha_1}\Gamma\left(\frac{\alpha_1+2}{2}\right)\Big[2(\alpha_1+2) \times {}_2F_1\left(\frac{\alpha_1+4}{2},\alpha_2,\frac{\alpha_1+5}{2};-\alpha_3 A^2\right) - 3\,{}_2F_1\left(\frac{\alpha_1+2}{2},\alpha_2,\frac{\alpha_1+3}{2};-\alpha_3 A^2\right)\Big], \quad (71)$$

where

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad (72)$$

is the Gauss hypergeometric function [32] and $(a)_n$ is the Pochhammer symbol.

## 5. Final remarks

In this section we briefly analyse how the usage of three terms in the Chebyshev polynomial expansion of the nonlinear function $f(x)$ in Eq. (5) instead of two, would change the maximum error attained in each oscillator. Retaining three terms in Eq. (5) and taking into account Eq. (7), it follows that:

$$f(Ay) \approx b_1(A)T_1(y) + b_3(A)T_3(y) + b_5(A)T_5(y) = \frac{b_1(A)-3b_3(A)+5b_5(A)}{A}x + \frac{4b_3(A)-20b_5(A)}{A^3}x^3 + \frac{16b_5(A)}{A^5}x^5. \quad (73)$$

Therefore, we may write Eq. (1) as an equivalent cubic-quintic Duffing oscillator as follows:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a_1(A)x + a_3(A)x^3 + a_5(A)x^5 \approx 0, \quad (74)$$

in which coefficients $a_1$, $a_3$ and $a_5$ are defined in terms of coefficients $b_1$, $b_3$ and $b_5$ of the Chebyshev polynomial expansion of the nonlinear function as

$$a_1(A) = \frac{b_1(A)-3b_3(A)+5b_5(A)}{A},$$

$$a_3(A) = \frac{4b_3(A)-20b_5(A)}{A^3}, \quad a_5 = \frac{16b_5(A)}{A^5}. \quad (75)$$

For the cubic-quintic oscillator given in Eq. (74), Beléndez et al. [33] have recently obtained its exact frequency — that we now call $\omega_3$ — which can be written as follows (Eq. (23) in Ref. [33]):

$$\omega_3(A) = \left(\frac{q_1q_2}{6}\right)^{1/4}\frac{\pi}{2K(m_3)}, \quad (76)$$

where

$$m_3(A) = \frac{1}{2} - \frac{q_3}{4}\sqrt{\frac{3}{2q_1q_2}}, \quad (77)$$

$$q_1(A) = a_1(A) + a_3(A)A^2 + a_5(A)A^4, \quad (78)$$

$$q_2(A) = 6a_1(A) + 3a_3(A)A^2 + 2a_5(A)A^4, \quad (79)$$

$$q_3(A) = 4a_1(A) + 3a_3(A)A^2 + 2a_5(A)A^4, \quad (80)$$

and $a_1(A)$, $a_3(A)$ and $a_5(A)$ are given by Eq. (75).

For the anti-symmetric, constant force oscillator (Example 1, Eq. (20)) and taking into account Eq. (6), the third coefficient of the Chebyshev series expansion of the nonlinear function is

$$b_5 = \frac{4}{5\pi}. \quad (81)$$

From Eqs. (75)–(80) we obtain

$$\omega_3(A) = \sqrt{\frac{\pi}{5A}}\left(\frac{91}{4}\right)^{1/4}K^{-1}\left(\frac{1}{2} - \frac{3}{2}\sqrt{\frac{3}{91}}\right) \approx \frac{1.111365}{\sqrt{A}}. \quad (82)$$

The relative error for the approximate frequency obtained using three terms in the Chebyshev series expansion in Eq. (5) is 0.06%, whereas the relative error for the approximate frequency obtained using only two terms is 0.6%. As we can see, the usage of three terms in Eq. (5) diminishes the error ten times in relation to the cubication method.

For the anti-symmetric quadratic nonlinear oscillator (Example 2, Eq. (28)) we obtain

$$b_5 = -\frac{8A^2}{105\pi} \quad (83)$$

and from Eqs. (75)–(80) we obtain

$$\omega_3(A) \approx 0.914745\sqrt{A}. \quad (84)$$

The relative errors for the approximate frequencies that we obtained were 0.04% and 0.007% for two and three terms in Eq. (5), respectively. For this example the usage of three terms diminishes the relative error 5.7 times, however the usage of two terms also provides an excellent approximation to the exact frequency in Eq. (32).

For the oscillator with fractional-power restoring force (Example 3, Eq. (34)) it follows that:

$$b_5 = \frac{\Gamma(7/6)A^{1/3}}{5\sqrt{\pi}\Gamma(5/3)} \quad (85)$$

and from Eqs. (75)–(80) we obtain

$$\omega_3(A) \approx \frac{1.07072}{A^{1/3}}. \quad (86)$$

The relative errors for the approximate frequencies that we obtained were 0.17% and 0.03% for two and three terms in Eq. (5), respectively.

For the mass attached to two stretched elastic springs (Example 4, Eq. (40)) the highest relative error for the approximate frequency when two terms are used in Eq. (5) is obtained for $\lambda = 1$ and $A = 1.36$ and its value is 0.07%. For these values of $\lambda$ and $A$ the relative error obtained when three terms are considered in Eq. (5) is 0.008%. Now the usage of three terms provides a relative error which is nine times lower than the relative error obtained using two terms in Eq. (5). However, as happened for example 2, the usage of only two terms provides a very acceptable maximum relative error.

For the oscillator with a linear term and a fractional-power nonlinear term (Example 5, Eq. (50)) with $c_{5/3}^2 = 0.5$ and $A = 3$, the relative error for the approximate frequency obtained using Eq. (76) and three terms in Eq. (5) is 0.0012%. Finally, for the finite extensibility nonlinear oscillator (Example 6, Eq. (63)) the relative error for the approximate frequency when two terms are used in Eq. (5) is less than 0.9% for $A < 0.9$, whereas this relative error is less than 0.21% when three terms are used in the Chebyshev series expansion of the nonlinear function.

## 6. Conclusions

A "cubication" method for conservative nonlinear oscillators with odd nonlinearity based on the Chebyshev series expansion of the nonlinear function $f(x)$ was analyzed and discussed and an approximate frequency–amplitude relationship and periodic solution were obtained. In this procedure, instead of approximately solving the original nonlinear differential equation, an accurate analytical approximate solution can be obtained by exactly solving an approximated nonlinear differential equation: the Duffing equation. Then, the exact solution for the Duffing equation is the approximate solution for the original differential equation. To do this, the Chebyshev polynomial expansion was used to replace the original nonlinear function by an approximate equivalent cubic polynomial equation using the first two terms of its Chebyshev polynomial expansion. Six examples of conservative nonlinear oscillators that describe the dynamical behaviour response of several physical and engineering systems were presented. These examples — together with the two systems previously analysed (Duffing-harmonic [34] and cubic-quintic [35, 36] nonlinear oscillators) — illustrate the accuracy of the analytical approximate frequencies and the corresponding periodic solutions obtained following this procedure. The analytical approximate frequencies for these six examples have been compared with those obtained using other approximate method. This comparison allows us to conclude that the results obtained using the cubication method are similar to those obtained using the second-order approximation of the harmonic balance method, the linearized harmonic balance method, the rational harmonic balance method and the homotopy perturbation method. Finally, we briefly analysed how the usage of three terms in the Chebyshev polynomial expansion of the nonlinear function $f(x)$ in Eq. (5) instead of two, changes the maximum error attained in each oscillator. The approximate method considered in this paper has been combined with the nonlinearization method [37, 38] to develop a new procedure to approximately solve conservative nonlinear oscillators. Finally, the "cubication" method has been recently extended by Elías-Zúñiga [39] considering three terms in the Chebyshev polynomial expansion of the nonlinear function $f(x)$ in Eq. (5).

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