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Jack Polynomials and Fractional Quantum Hall Effect at $\nu = 1/3$

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We investigate properties of strongly correlated, spinless electrons confined within given Landau level at filling factor $\nu = 1/3$. Our analysis is based on the formalism of the Jack polynomials. Selected Jack polynomial wave functions are compared with ground states of the Coulomb interaction Hamiltonians, in different materials and the Landau levels, obtained by exact diagonalization. We show that certain Jack wave functions can be used as a description of fractional quantum Hall states.

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1. Introduction

The Jack polynomials (Jacks) [1–6] have been related to the fractional quantum Hall effect (FQHE) by a number of authors [7–15]. The Jack polynomial J^{α}_{λ} is indexed by a real parameter α and a partition λ. Partition is a series of nonnegative integers in a decreasing order. In FQHE partitions are usually represented in a following way: starting from zero, a number in the i-th place indicates a number of occurrence of i in the partition e.g. partition $\lambda = [102]$ denotes a series where number 0 occurs ones, number 1 does not occur (occurs zero times) and number 2 occurs twice. The Jack polynomials are eigenstates of the following Laplace–Beltrami Hamiltonian defined in the space of symmetric polynomials:

$$H_{\mathrm{LB}}(\alpha) = \alpha \sum_i (x_i\partial_i)(x_i\partial_i) + \sum_{i<j}(x_i + x_j)(x_i - x_j)^{-1}(x_i\partial_i - x_j\partial_j). \quad (1)$$

Fermionic Jacks denoted $S^{\alpha}_{\lambda+\delta}$ have an additional antisymmetrizing Vandermonde factor for $\delta = [111\ldots1]$. An explicit recursion construction of both Jack and fermionic Jacks was derived [5, 10, 11, 16]. We compare selected Jack-based wave functions with ground states of the Coulomb interaction of electrons confined in given Landau level (LL) and two materials: GaAs and graphene.

2. Theory

Mathematical analysis of the angular momentum operators on the sphere provides tools used in determination of FQH wave functions. Such analysis have been performed for the symmetric Jack polynomials (bosonic) [7–9, 12], necessary conditions for the parameters of a Jack to be considered as FQH state have been given. That is $\alpha = \alpha_{k,r} = -(k+1)/(r-1)$ for $(k+1)$ and $(r-1)$ both are positive integers and coprime, a partition has to have a form $\lambda = \lambda^0_{k,r,s} = [n_0 0^{s(r-1)} k 0^{r-1} k 0^{r-1} k \ldots k]$, where 0^{r-1} denotes the sequence of $(r-1)$ zeros. We examine the case of $s = 1$ (this implies $n_0 = k$). Filling factor ν of the Jack state is related to its partition λ in a following way. $\nu \approx \frac{\#\lambda}{\lambda_{max}}$ where λ_{max} is maximal element of partition λ and $\#\lambda$ is its number of elements. Relation is only approximate due to the presence of a shift. Proper relation states $\nu\lambda_{max} = \#\lambda - S$ and S is a constant shift which is usually the same as shift on the Haldane sphere. For example partition $\mu = [1010101]$ corresponds to the $\nu = \frac{\#\mu - S}{\mu_{max}} = \frac{4-1}{6}$ (the Jack polynomial indexed by this is related to the bosonic Laughlin state $\nu = 1/2$). Relations between parameters of fermionic Jack can be obtained straightforwardly [10, 11].

We perform our calculations for the family of Jack-related wave functions at filling factor $\nu = 1/3$. The most prominent fractional quantum Hall state at this filling is the famous Laughlin wave function [17, 18] which is a fermionic Jack $S^{-2}_{[1010\ldots01]+\delta} = S^{-2}_{[10^2 10^2 \ldots 0^2 1]}$. Using a particle–hole symmetry of a wave function of electrons confined within one LL at filling ν one can obtain valid description of the electron system at filing $(1-\nu)$. Thus we decided to discuss anti-Jack wave functions (the Jack wave functions reflected using particle–hole symmetry), proposed originally for the state $\nu = 2/3$. Our attention has been pointed onto an anti-Jack $S^{-5}_{[1^4 0^2 \ldots 0^2 1^4]}$ which is a member of a "parafermion" $\nu = k/(k+2), k = 1, 2\ldots$ family, for $k = 4$ [19, 20]. This series of states can be viewed as the densest ground states generated by the short range $(k+1)$-body repulsion [21]. One could expect that a fermionic anti-Jack generated by the partition $[1^2 01^2 0 \ldots 1^2] = [222\ldots2] + \delta$ should be examined, however it does not satisfy coprime conditions (i.e. for $k = 2, r = 1$ numbers $k+1$ and $r-1$ are not coprime), moreover, real parameter would have to take value $\alpha = -\infty$. This would correspond to the function being product of a monomial and the Vandermonde determinant. Such object is not interesting for the applications in FQHE due to the trivial form of a function. Similarly state generated by $[1^2 0^4 \ldots]$ does not give a proper fermionic Jack, due to the same condition $(k+1 = 3$ equals $r-1 = 3)$. Nonetheless we used recursion formula

for fermionic Jack $[1^2 0^4 \ldots]$ and numerically confirmed that the Jack polynomial indexed by this partition has a pole at $\alpha_{2,4} = -\frac{3}{3} = -1$ and is not well defined.

3. Results

Our numerical overlaps of two examined states (Jack $S^{-2}_{[10^2 10^2 \ldots 0^2 1]}$ and anti-Jack $S^{-5}_{[1^4 0^2 \ldots 0^2 1^4]}$) and selected Coulomb interaction states are presented in Table I. Data confirm well known fact that the Laughlin wave function is a valid description of a FQHE at the LLL in GaAs, obtained overlaps reach almost 99%. Similar values have been obtained for the first excited LL in graphene. On the other hand, values of overlaps in the first excited LL in GaAs are not satisfactory, but improve upon effects of nonzero layer width. Results for the anti-Jack $[1^4 0^2 \ldots]$ suggest that it may be considered as a description of FQH state at first excited LL in GaAS and graphene, however values are not overwhelming. Neither of considered wave functions seem to be a proper description of second excited LL in graphene. The results show what Jack polynomials fit FQHE for one particular filling factor. Nonetheless in order to get broader picture of Jack-based wave functions in FQHE further study is required.

TABLE I

Overlaps of two indicated Jack states with different Coulomb ground states with additional requirement of zero angular momentum. Consecutive columns stand for: partition $[\ldots]$, electron number N, magnetic flux on the sphere $2Q$, and the overlaps with Coulomb ground states in the $n = 0$ and 1 Landau levels in GaAs (LL_n) and in the $n = 1$ and 2 Landau levels in graphene ($\mathrm{G\text{-}LL}_n$). Calculations have been performed for zero layer width for each Coulomb system, with an exception of $\mathrm{LL}_1^{\mathrm{wide}}$ corresponding to layer width of 3 magnetic lengths.

Jack	N	$2Q$	LL_0	LL_1	$\mathrm{LL}_1^{\mathrm{wide}}$	$\mathrm{G\text{-}LL}_1$	$\mathrm{G\text{-}LL}_2$
Jack	14	39	0.9887	0.5771	0.7411	0.9858	0.0018
$[10^2 \ldots 0^2 1]$	15	42	0.9876	0.5298	0.7275	0.9845	0.0005
anti-Jack	7	27	0.6186	0.8675	0.8563	0.6161	0.5082
$[1^4 0^2 \ldots 0^2 1^4]$	9	33	0.7349	0.7697	0.7832	0.7358	0.1139

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