Effects of Material Non-Homogeneity and Two Parameter Elastic Foundation on Fundamental Frequency Parameters of Timoshenko Beams

M. Avcar*
Suleyman Demirel University, Engineering Faculty, Civil Engineering Department, Isparta, Turkey

In the present study, effects of material non-homogeneity and two-parameter elastic foundation on the fundamental frequency parameters of the simply supported beams are examined. Material non-homogeneity is characterized taking into account the parabolic variations of Young’s modulus and density along the thickness direction of the beam while the value of Poisson’s ratio is assumed to remain constant. The foundation medium is assumed to be linear, homogeneous and isotropic, and it is modeled by the Pasternak model with two parameters for describing the reaction of the elastic foundation on the beam. At first, the equation of the motion including the effects of the material non-homogeneity and two-parameter elastic foundation is provided. Then, the solutions including fundamental frequency parameters versus various non-homogeneity, density and foundation parameters, and length to depth ratio adopting the Timoshenko beam theory as well as the Euler–Bernoulli beam theory are presented. To show the accuracy of the present results, a comparison is carried out and a good agreement is found.

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1. Introduction

The concept of beams on elastic foundations has been widely used in different fields of engineering, e.g., railroad tracks, highway pavements and pipelines. Since soil exhibits a very complex behavior, various foundation models have been developed for modeling it. Among these models, the Pasternak two-parameter elastic foundation (TPEF), which accounts for foundation shearing-stiffness neglected in Winkler one-parameter foundation model, is the commonly used one [1].

On the other hand, in the studies relating to the vibration analysis of beams resting on elastic foundation generally slender beams have been considered so that the Euler–Bernoulli beam theory (EBBT) is usually adopted [2, 3]. However, EBBT slightly overestimates the frequency parameters and the error increases with the increase of modes and thickness of the beam. Therefore, if the beam is moderately short and thick, the Timoshenko beam theory (TBT) gives more accurate results due to that it takes into account both the shear deformation and rotary inertia of the beam. Several researches have been conducted on the vibration of beams resting on TPEF adopting TBT [4–7].

All of the above mentioned studies are carried out for homogeneous (H) beams in sense that mechanical properties of the beam are taken to be constant throughout. However, plenty of materials exist in the nature, which are non-homogeneous (NH), either by working under high temperature environment or due to the physical composition and imperfections in the underlying materials.

Because the use of these advanced materials in various fields of engineering has been increasing day by day, the vibration problems of NH beams resting on elastic foundation have also received the attention of numerous researchers [8–10].

From the review of available literature it is observed that the effects of material non-homogeneity (MNH) and TPEF on the values of fundamental frequency parameters (FFPs) of simply supported beam in which the MNH is characterized with the parabolic variation of the Young modulus and density along the thickness direction have not been dealt yet. In the present study an attempt is made to address this problem.

2. Formulation and solution of the problem

Consider an elastic beam of length $L$, height $h$, resting on TPEF as shown in Fig. 1a.

Fig. 1. (a) Coordinate and geometry of beam on TPEF, (b) beam element.

Pressure–displacement relation of the TPEF models are assumed to be [1]:

$$p(x, t) = k_1 w(x, t) - k_2 w''(x, t), \quad (1)$$

where $p(x, t)$ is the vertical foundation reaction, $w(x, t)$ is the function of transverse displacements of beam, $k_1$ is the modulus of sub-grade reaction, $k_2$ — shear...
foundation modulus. Note that, as \( k_2 = 0 \), the Pasternak foundation reduces to the Winklter foundation.

The equations including translational and rotational equilibrium conditions of the beam element are (Fig. 1b):

\[
\frac{dQ}{dx} = \rho A \left( \frac{\partial^2 w}{\partial t^2} \right) + k_1 w - k_2 \left( \frac{\partial^2 w}{\partial x^2} \right), \tag{2}
\]

\[
Q \Delta x - (\partial M/\partial x) \Delta x = \rho I \left( \frac{\partial^2 \Phi}{\partial t^2} \right), \tag{3}
\]

where \( Q, M, \rho, A, I, \) and \( \Phi \) are shear force, bending moment, density, area, moment of inertia, and angle of rotation of beam element due to bending, respectively.

The constitutive equations of flexural and shear stiffnesses are as follows, respectively:

\[
M = -EI \left( \frac{\partial^2 \Phi}{\partial x^2} \right) \quad \text{and} \quad Q = kGA \gamma, \tag{4}
\]

where \( E, \kappa, G, \) and \( \gamma \) are the Young modulus, shear correction factor, shear modulus and increase in the slope due to shear deformation.

The total slope of the beam is

\[
\gamma = \frac{\partial w}{\partial x} - \Phi. \tag{5}
\]

Substitution of Eq. (5) into Eqs. (2) and (3) yield the following equations:

\[
(\kappa GA + k_2) \left( \frac{\partial^2 w}{\partial x^2} \right) - \kappa GA \left( \frac{\partial \Phi}{\partial x} \right)
\]

\[-\rho A \left( \frac{\partial^2 w}{\partial t^2} \right) - k_1 w = 0, \tag{6}
\]

\[
EI \left( \frac{\partial^2 \Phi}{\partial x^2} \right) + \kappa GA \left( \frac{\partial \Phi}{\partial x} \right) - \kappa GA \Phi
\]

\[-\rho I \left( \frac{\partial^2 \Phi}{\partial t^2} \right) = 0. \tag{7}
\]

Using Eqs. (6) and (7) after some mathematical operations the equation of motion for the free vibration of a \( H \) beam resting on TPEF is obtained as follows:

\[
\left( EI + \frac{EIk_2}{\kappa GA} \right) \frac{\partial^4 w}{\partial x^4} - \left( \frac{EIk_1}{\kappa GA} + k_2 \right) \frac{\partial^2 w}{\partial x^2}
\]

\[-\left( \frac{EIP}{\kappa G} + I + \frac{\rho Ik_2}{\kappa GA} \right) \frac{\partial^4 w}{\partial x^2 \partial t^2}
\]

\[+ \left( \rho A + \frac{\rho Ik_1}{\kappa GA} \right) \frac{\partial^2 w}{\partial t^2} + \left( \frac{\rho I}{\kappa G} \right) \frac{\partial^4 w}{\partial t^4} + k_1 w = 0 \]  \tag{8}

It is assumed that the MNH of the beam stems from the variation of Young’s modulus and mass density along the thickness direction [9]:

\[
E_3 = E[1 + \alpha_1 \lambda(z)], \quad \rho_1 = \rho[1 + \alpha_2 \lambda(z)], \tag{9}
\]

where \( \bar{z} = z/h \), \( \lambda(z) \) is the continuous function of MNH defining the variation of Young’s modulus and density, \( \alpha_1 \) and \( \alpha_2 \) are the non-homogeneity and density parameters (\( -0.5 \leq \alpha_i \leq 1, i = 1, 2 \)). Note that the value of Poisson’s ratio is assumed to be constant.

The MNH functions of the beam are taken to be parabolic functions

\[
\lambda(z) = \bar{z}^2. \tag{10}
\]

Considering Eqs. (9) and (10) in Eq. (8), the governing equation for the free vibration of a NH Timoshenko beam resting on TPEF is obtained as follows:

\[
\left( D_1 + \frac{D_1k_2}{D_2\kappa A} \right) \frac{\partial^4 w}{\partial x^4} - \left( \frac{D_1k_1}{D_2\kappa A} + k_2 \right) \frac{\partial^2 w}{\partial x^2}
\]

\[-\left( \frac{D_1D_3}{D_2\kappa} + ID_3 + \frac{D_1Ik_2}{D_2\kappa A} \right) \frac{\partial^4 w}{\partial x^2 \partial t^2}
\]

\[+ \left( D_3A + \frac{D_3Ik_1}{D_2\kappa A} \right) \frac{\partial^4 w}{\partial t^4} + k_1 w = 0, \tag{11}
\]

where \( D_1, D_2 \) and \( D_3 \) are flexural rigidity, shear modulus, and density of the NH beam, and the following definitions apply:

\[
D_1 = Ebh^3 \int_{-1/2}^{1/2} \bar{z}^2 (1 + \alpha_1 \lambda(z)) d\bar{z}, \tag{12}
\]

\[
D_2 = \frac{E}{2(1 + \nu)} \int_{-1/2}^{1/2} (1 + \alpha_1 \lambda(z)) d\bar{z}, \tag{13}
\]

The solution of Eq. (11) is sought by separation of variables. Assume that the displacement can be separated into spatial and temporal variables [11]:

\[
w(x,t) = \xi(x)\eta(t), \tag{14}
\]

where \( \xi \) and \( \eta \) are dependent on position and time, respectively.

Besides, the following expressions are satisfied for the boundary conditions of the simply supported beam:

\[
\xi(0) = \xi(L) = \xi''(L) = 0. \tag{15}
\]

Substituting Eqs. (13) and (14) in Eq. (11), respectively, and after some mathematical operations lead to the following equation:

\[
\left( D_1 + \frac{D_1k_2}{D_2\kappa A} \right) \left( \frac{n^2}{L^2} \right)^4 + \left( \frac{D_1k_1}{D_2\kappa A} + k_2 \right) \left( \frac{n^2}{L^2} \right)^2
\]

\[-\omega^2 \left[ \frac{D_1D_3}{D_2\kappa} + ID_3 + \frac{D_1Ik_2}{D_2\kappa A} \right] \left( \frac{n^2}{L^2} \right)^2
\]

\[+ \left( D_3A + \frac{D_3Ik_1}{D_2\kappa A} \right) \left( \frac{n^2}{L^2} \right) \]

\[+ \omega^4 \left( \frac{D_3I}{\kappa D_2} \right) + k_1 = 0. \tag{16}
\]

Consequently, Eq. (15) is a quadratic equation in \( \omega^2 \) and yields two values of \( \omega^2 \), in which the smaller one corresponds to the free vibration of the NH Timoshenko beam resting on TPEF. Note that the equation for the free vibration of the NH Euler–Bernoulli beam resting on TPEF can be obtained setting the terms including \( D_3I \) equal to zero and letting \( D_2 \to \infty \) in Eq. (11) and reapplying the above given solution procedure.
Effects of Material Non-Homogeneity...

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TABLE II

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TABLE III

3. Numerical results and discussion

In this section illustrative studies are given to examine the present problem. All numerical results are expressed according to the following non-dimensional parameters:

\[ \Omega = \omega L^2 \sqrt{\frac{\rho A}{EI}}, \quad K_W = k_1 L^4 / EI, \]

\[ K_P = k_2 L^2 / \left( \pi^2 EI \right). \] (16)

3.1. Comparative study

The FFPs of homogeneous beam resting on the Pasternak type TPEF versus elastic foundation parameters are compared with those of Refs. [4, 5] in Table I for $n = 1$, $\kappa = 5/6$, $L/h = 5$, $\alpha_1 = \alpha_2 = 0$, $\nu = 0.3$. The obtained results validate the accuracy of present formulations.

3.2. Numerical examples

Example 1. Table II shows the FFPs of beams with and without elastic foundations versus non-homogeneity and density parameters $\alpha_1$ and $\alpha_2$ for $L/h = 5$, $\kappa = 5/6$, $\nu = 0.25$. It is seen that values of FFPs increase with the increase of $\alpha_1$, while they decrease with the increase of $\alpha_2$ in all cases. Besides, the effect of MNH on the values of FFPs varies not only according to $\alpha_1$ and $\alpha_2$, but also presence and type of elastic foundations and adopted beam theory.

Example 2. Table III shows FFPs of beams with and without elastic foundations versus foundation parameters, $K_W$ and $K_P$ for $\alpha_1 = 1$, $\alpha_2 = -0.5$, $L/h = 5$, $\kappa = 5/6$, $\nu = 0.25$. It is found that values of FFPs increase with consideration of the foundation parameters. Moreover, the effect of MNH on the values FFPs decreases with the increase of $K_W$ and $K_P$, and becomes least efficient for the Pasternak foundation.

Example 3. Table IV shows FFPs of beams with and without elastic foundations for $\alpha_1 = -0.5$, $\alpha_2 = 1$, $\kappa = 5/6$, $\nu = 0.25$ versus the length to depth ratio, $L/h$. It is observed that values of FFPs for TBT increase with the increase in the ratio $L/h$. Besides, the effect of variation of ratio, $L/h$, on the values of FFPs is more pronounced in foundationless case. Moreover, the effect of MNH on the values of FFPs increases with the increase in the ratio, $L/h$, in foundationless case, while it changes irregularly with consideration of elastic foundations.
FFPs of beams versus the length to depth ratio.

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</table>

4. Conclusions

In this study, effects of MNH and TPEF on the FFPs of the simply supported beams are examined using TBT as well as EBBT. Solutions including FFPs for various combinations of non-homogeneity, density and foundation parameters, and length to depth ratios are reported. Briefly, the obtained results can be summarized as follows:

1. The values of FFPs increase with consideration of the elastic foundations;
2. The variation of non-homogeneity and density parameters have adverse effects on the values of FFPs;
3. The presence of elastic foundation decreases the effects of MNH and length to depth ratio on the values of FFPs;
4. MNH is more influential on the values of FFPs in the Winkler foundation and EBBT cases in comparison with the Pasternak foundation and TBT cases.

The presented results prove the importance of the effects of MNH and TPEF on FFPs of beams.

References