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Pseudo-Spherical Null Quaternionic Curves in Minkowski Space $\boldsymbol{R}_1^4$

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In this study, we define the osculating pseudo-sphere of a null quaternionic Cartan curve in Minkowski space R_1^4. We give a characterization for pseudo-spherical null quaternionic Cartan curves.

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1. Introduction

In the first half of the twentieth century, Einstein has formulated general relativity as a theory of space, time and gravitation in semi-Riemannian space. Einstein's theory has opened a door for use of new geometries. One of them is simultaneously the geometry of special relativity. Since the second half of the twentieth century, the semi-Riemannian geometry has been among active areas of research in differential geometry and its applications in a variety of subjects in mathematics and physics. In a semi-Riemannian manifold, there are three different families of curves, that is, spacelike, timelike and null (lightlike) curves according to their causal characters. In the geometry of null curves the natural parameter is the pseudo-arc (see [1, 2]). In [1], Bonnor has introduces the Cartan frame as the most useful one and has used this frame to study the behavior of a null curve in Minkowski space R_1^4. Thus, one can use these fundamental results as the basic tools in researching the geometry of null curves. The theory of Frenet frames for a null curve has been studied and developed by several researchers in this field [2–4]. In [4] Çöken and Çiftci have studied null curves in the 4-dimensional Minkowski space R_1^4, and have characterized pseudo-spherical null curves. Recently Duggal and Jin [2] have studied the geometry of null curves and their physical use. And then, Sakaki has characterized pseudo-spherical null curves in the n-dimensional Minkowski space in terms of the curvature functions [5].

In mathematics, as an extension of the complex number, the quaternions are the number system in four dimensional vector space. The concept of number was further extended by W.R. Hamilton, whose theory of quaternions (1843) provided the first example of a noncommutative algebra (i.e., one in which $pq \neq qp$). There are different types of quaternions, namely real, complex and dual quaternions. A real quaternion, defined as $q = ae_1 + be_2 + ce_3 + d$, is composed of four units $(1,e_1,e_2,e_3)$, where e_1, e_2, e_3 are orthogonal unit spatial vectors and a, b, c, d are real numbers. This quaternion may be written as a linear combination of a real part and vectorial part. The space of quaternions Q is isomorphic to E^4, the four-dimensional vector space over the real numbers. Quaternions have played a significant role recently in several areas of the physical science, namely in differential geometry, in analysis and quaternionic formulation of equation of motion in theory of relativity.

The theory of Frenet frames for a quaternionic curve has been studied and developed by several researchers in this field [6, 7]. After them, Tuna Aksoy and Çöken have studied differential geometry of null quaternionic curves in semi-euclidean 3-spaces R_v^3 and gave the Frenet formula for null quaternionic curves by using spatial quaternions. And then, they have recently constructed Cartan frame for a null quaternionic curve in the 4-dimensional Minkowski space R_1^4 [8]. Bektaş et al. have defined the osculating spheres of a semi real quaternionic curve in semi-euclidean spaces E_1^3 and E_2^4 [9].

In this work, we introduce the geometric properties of null quaternionic curves in 4-dimensional Minkowski space. We use the null quaternionic Cartan frame (and the null quaternionic Cartan curvatures), use them to generate pseudo sphere null quaternionic curves in Minkowski space and investigate its properties. Here, by using the similar idea of Çöken and Çiftçi [4], we show that a null quaternionic curve is a a pseudo-spherical curve if and only if $(p-\tau)$ is a nonzero constant.

2. Preliminaries

Let us give the basic concepts about the semi-real quaternions. The set of the semi real quaternions

$$\begin{aligned} Q_v = \{q | q = ae_1 + be_2 + ce_3 + d;\\ a,b,c,d \in R, \quad e_1,e_2,e_3 \in R^3_{v(v=1,2)},\\ h_v\left(e_i,e_i\right) = \varepsilon\left(e_i\right), \quad 1 \leq i \leq 3\}, \end{aligned}$$

where

$$\begin{aligned} e_i \times e_i &= -\varepsilon(e_i), \quad 1 \leq i \leq 3,\\ e_i \times e_j &= \varepsilon\left(e_i\right)\varepsilon\left(e_j\right)e_k \ \in R_1^3,\\ e_i \times e_j &= -\varepsilon\left(e_i\right)\varepsilon\left(e_j\right)e_k \ \in R_2^4 \end{aligned}$$

and (ijk) is an even permutation of (123). The multiplication of two semi real quaternions p and q is defined by

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$$p \times q = S_p S_q + S_p V_q + S_q V_p + h(V_p V_q) + V_p \wedge V_q$$

for every $p, q \in Q_v$, where we have used, the inner and cross products in semi-euclidean space R_v^3. For a semi real quaternion $q = ae_1 + be_2 + ce_3 + d \in Q_v$ the conjugate αq of q is defined by $\alpha q = -ae_1 - be_2 - ce_3 + d$. Thus, we define symmetric, non-degenerate valued bilinear form h as follows:

$$h_v : Q_v \times Q_v \to R$$

by

$$h_1(p,q) = \frac{1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) + \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]$$

for R_1^3 and

$$h_2(p,q) = \frac{1}{2}[-\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) - \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]$$

for R_2^4, where

$$h_v(q,q) = a^2\varepsilon(e_1) + b^2\varepsilon(e_2) + c^2\varepsilon(e_3) + d^2$$

for $v = \{1,2\}$ and it is called the semi-real quaternion inner product. The vector product of two semi real quaternions $p = a_1e_1 + b_1e_2 + c_1e_3 + d_1$ and $q = ae_1 + be_2 + ce_3 + d$ is defined as

$$\begin{aligned} V_p \wedge V_q = {} & \varepsilon(e_2)\varepsilon(e_3)(b_1c - bc_1)e_1 \\ & -\varepsilon(e_1)\varepsilon(e_3)(a_1c - ac_1)e_2 \\ & +\varepsilon(e_1)\varepsilon(e_2)(a_1b - ab_1)e_3, \end{aligned}$$

for $v = 1$ and

$$\begin{aligned} V_p \wedge V_q = {} & -\varepsilon(e_2)\varepsilon(e_3)(b_1c - bc_1)e_1 \\ & +\varepsilon(e_1)\varepsilon(e_3)(a_1c - ac_1)e_2 \\ & -\varepsilon(e_1)\varepsilon(e_2)(a_1b - ab_1)e_3, \end{aligned}$$

for $v = 2$. And then, the norm of semi real quaternion q is denoted by

$$\| q \|^2 = |h_v(q,q)| = |a^2\varepsilon(e_1) + b^2\varepsilon(e_2) + c^2\varepsilon(e_3) + d^2|$$

for $v = \{1,2\}$.

The use of the concept of a spatial quaternion will be made throughout our work. q is called a spatial quaternion whenever $q + \alpha q = 0$. It is a temporal quaternion whenever $q - \alpha q = 0$ [6–9].

2.1. Null quaternionic curves in R_1^4

Let h, denote the semi quaternionic metric on R_1^4. A curve $\beta(s)$ in R_1^4 is a null quaternionic curve if $h(\beta'(s), \beta'(s)) = 0$ and $\beta'(s) \neq 0$ for all s. We note that a null quaternionic curve $\beta(s)$ in R_1^4 satisfies $h(\beta''(s), \beta''(s)) \neq 0$. We say that a null quaternionic curve $\beta(s)$ in R_1^4 is parametrized by the pseudo-arc if $h(\beta''(s), \beta''(s)) = \pm 1$. Let $\beta(s) = \gamma^1(s)e^1 + \gamma^2(s)e^2 + \gamma^3(s)e^3 + \gamma^4(s)$ be a null quaternionic curve in R_1^4. The three-dimensional semi-euclidean space R_v^3 is identified with the space of null spatial quaternions $\left\{\beta \in Q_{R_1^4} | \beta + \alpha\beta = 0\right\}$ in an obvious manner $\beta : I \subset R \to Q_{R_1^4}$, $s \to \beta(s) = \sum\limits_{i=1}^{4} \gamma_i(s)e_i$, $1 \le i \le 4$. Let $\{L, N, U, W\}$ be the Frenet trihedron of the differentiable for null spatial quaternionic curve in Minkowski space R_1^4. We consider a Cartan null quaternionic curve β in the 4-dimensional Minkowski space (R_1^4, h) with a Cartan frame $\{L, N, U, W\}$ with respect to a pseudo-arc parameter s so that its Cartan equations are

$$\begin{aligned} & L' = W,\ N' = (p - \tau)U + pW, \\ & U' = (p - \tau)L,\ W' = pL + N \end{aligned}$$

and

$$\begin{aligned} & L' = W,\ N' = (\tau + p)U + pW, \\ & U' = (\tau + p)L,\ W' = pL + N, \end{aligned}$$

where

$$\begin{aligned} h(L,L) &= h(N,N) = h(L,U) = h(N,U) = h(W,U) \\ &= h(N,W) = h(L,W) = 0, \\ h(U,U) &= h(W,W) = +1 \wedge h(L,N) = -1. \end{aligned}$$

3. Pseudo-sphere null quaternionic curves in R_1^4

In this section, we define pseudo-sphere null quaternionic curves. Null quaternionic curves that completely lie on a pseudo-sphere of radius $r > 0$ and of center A are given by $S_1^3(r) = \{X \in R_1^4 : h(X - AX - A) = r^2\}$ (see [10] for null curve definition). We define the osculating pseudo-sphere as follows.

Definition 3.1. Let β be a null quaternionic Cartan curve in R_1^4. Then the pseudo-sphere having 5 point contact with β is called the osculating pseudo-sphere of β (see [4] for null curve).

We assume $\beta(s)$ be a null quaternionic Cartan curve in R_1^4 with respect to a special paramerter s having Cartan curvatures p, $(p - \tau)$.

Lemma 3.2. Let $\beta(s)$ be a null quaternionic Cartan curve in R_1^4. The center point of the osculating pseudo-sphere at a point $\beta(s)$ is $A(s_0) = \beta(s_0) - \frac{1}{p-\tau}U(s_0)$.

Proof. For any point $\beta(s_0)$ the position vector $A(s_0) - \beta(s_0)$ can be written as a linear combination of the Cartan frame $\{L, N, U, W\}$ as follows:

$$A(s_0) - \beta(s_0) = m_1L + m_2N + m_3U + m_4W, \qquad (1)$$

where m_i for $1 \le i \le 4$ are differentiable functions. Consider the function

$$f(s) = h(A(s_0) - \beta(s_0),\ A(s_0) - \beta(s_0)) - r^2,$$

where r is the radius of the osculating pseudo-sphere, thus the equations

$$f(s_0) = f'(s_0) = f''(s_0) = f'''(s_0) = f^{(4)}(s_0) = 0,$$

are satisfied due to the definition of the osculating pseudo-sphere at s_0, then a straightforward computation leads to

$$\begin{aligned} & h(L(s_0),\ A(s_0) - \beta(s_0)) = 0 \wedge m_2 = 0, \\ & h(W(s_0),\ A(s_0) - \beta(s_0)) = 0 \wedge m_4 = 0, \\ & h(N(s_0),\ A(s_0) - \beta(s_0)) = 0 \wedge m_1 = 0, \\ & h(U(s_0,\ A(s_0) - \beta(s_0)) = 0 \wedge m_3(s_0) = -\tfrac{1}{p-\tau}. \end{aligned}$$

Thus we obtain

$$m_1 = m_2 = m_4 = 0 \wedge m_3(s_0) = -\frac{1}{p-\tau}.$$

Therefore, Eq. (1) implies that $A(s_0) - \beta(s_0) = -\frac{1}{p-\tau}U$ and $r = |-\frac{1}{p-\tau}|$ in the notations above.

Theorem 3.3. Let β be a null quaternionic Cartan curve. Then β is a pseudo-spherical curve if and only if $(p-\tau)$ is a nonzero constant.

Proof. Suppose that β lies on $S_1^3(r)$. Then the osculating pseudo-spheres at all points of the curve are exactly $S_1^3(r)$, and so r and $\frac{1}{p-\tau}$ are constant.

Conversely, assume that $\frac{1}{p-\tau}$ is a nonzero constant, then all of the osculating pseudo-spheres have the same radius. Moreover, if we consider the function $A(s_0) = \beta(s_0) - \frac{1}{p-\tau}U(s_0)$, giving the central point of the osculating pseudo-sphere, its derivative is zero everywhere, so it is constant. Consequently, the curve β lies on $S_1^3(r)$, since the equation $h(X-A,\ X-A) = r^2$ is valid for all s, which completes the proof.

Since $h(X-A,\ X-A) = |-\frac{1}{p-\tau}|^2$, null quaternionic curves with $\frac{1}{p-\tau}$ =const. lie on pseudo-sphere $S_1^3(r)$.

Corollary 3.4. A quaternionic Cartan curve $\beta \subset R_1^4$ fully lies on a pseudo-sphere if and only if there exists a fixed point A such that f or each $s \in I$

$$h(X(s) - \beta(s), -\beta'(s)) = 0.$$

4. Conclusions

We can characterize pseudo-sphere null quaternionic curve by means of the quaternionic Cartan curvatures in R_1^4. Null quaternionic curves that have $(p-\tau)$ as a nonzero constant, lie on a pseudo-sphere. More generally, there is no three dimensional null quaternionic curve that lies on a pseudo-sphere in Minkowski space-time (see [11] for null curves).

Similar results were also obtained for other null quaternionic Cartan frame. Let e_2 be a time-like vector in R_1^4 with other Cartan frame by

$$L' = W,\ N' = (\tau + p)U + pW,\ U' = (\tau + p)L,$$
$$W' = pL + N.$$

Here, by using the similar idea of above theorems, we can obtain the following results

$$A(s_0) = \beta(s_0) - \frac{1}{p+\tau}U(s_0).$$

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