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Null Quaternionic Bertrand Curves in Minkowski Space R_1^4

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In this study, we defined null quaternionic Bertrand curves in R_1^4. The only Bertrand null quaternionic curves in R_1^4 are null quaternionic helices with $(p-\tau)=0$.

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1. Introduction

The geometry of null curves in Minkowski spacetime has played an important role in the development of general relativity, as well as in mathematics and physics of gravitation. Many scientists have used Minkowski space to apply general relativity. There has been an increase in research on null curves in geometry and physics [1]. Bonnor [2] has introduced a Cartan frame for null curves in R_1^4 and proved the fundamental existence and congruence theorems. Bejancu [3] has given a method for the general study of the geometry of null curves in lightlike manifolds and in semi-Riemannian manifolds. A. Ferrandez, A. Gimenez and P. Lucas [4] have shown that a null Frenet curve, parametrized by the pseudo-arc parameter, is a null helix, if its lightlike curvature is constant. Çöken and Çiftçi [5] have reconstructed the Cartan frame of a null curve in Minkowski spacetime for an arbitrary parameter, and have characterized the Bertrand null curves. And then, Duggal and Jin [6] have studied major developments of null curves, hypersurfaces and their physical use, in their recent book with voluminous bibliography.

Quaternions were discovered by Hamilton as an extension to the complex numbers in 1843. Quaternions have found a broad application in many scientific areas: in mechanics of a solid body, for the description of rotation in space, in computer animation, etc. One of the most important tools used to analyze a quaternionic curve is the Frenet frame. Therefore, in [7], Bharathi and Nagaraj have defined Serret-Frenet formulas for a quaternionic curve in E^3 and E^4, and then Çöken and Tuna have studied Serret-Frenet formulas for quaternionic curves and quaternionic inclined curves in semi-euclidean spaces [8]. Moreover, we have studied the differential geometry of null quaternionic curves in semi-euclidean 3-spaces R_v^3 and gave the Frenet formula for null quaternionic curves by using spatial quaternions. We have constructed recently the Cartan frame for a null quaternionic curve in the 4-dimensional Minkowski space R_1^4 [9]. Then, we have established a relation of Bertrand pairs with null quaternionic Cartan helices in R_v^3 [10].

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The main goal of this paper is to define null quaternionic Bertrand curves in the four-dimensional Minkowski spaces. Here, by using the similar idea of Çöken and Çiftçi [5], we show that a null quaternionic curve is a Bertrand curve if and only if it has nonzero constant first Frenet curvature and it has a zero second Frenet curvature. Null quaternionic helices are the only null quaternionic Bertrand curves. We prove that the distance between null quaternionic Bertrand curves is a constant. Null quaternionic Bertrand curves are characterizated with Cartan curvatures.

2. Preliminaries

Let Q_H denotes a four dimensional vector space over the field H of characteristic grater than 2. Let e_i $(1 \leq i \leq 4)$ denote a basis for the vector space. Let the rule of multiplication on Q_H be defined on e_i $(1 \leq i \leq 4)$ and extended to the whole of the vector space by distributivity as follows.

The set of the semi-real quaternions is defined by

$$\begin{aligned} Q_H = \{ q | q = a e_1 + b e_2 + c e_3 + d; \\ a,b,c,d \in R, \quad e_1,e_2,e_3 \in R^3_{v(v=1,2)}, \\ h_v\left(e_i,e_i\right) = \varepsilon\left(e_i\right),\ 1 \leq i \leq 3 \}, \end{aligned}$$

where

$$\begin{aligned} &e_i \times e_i = -\varepsilon(e_i), \quad 1 \leq i \leq 3, \\ &e_i \times e_j = \varepsilon\left(e_i\right)\varepsilon\left(e_j\right) e_k \ \in\ R_1^3, \\ &e_i \times e_j = -\varepsilon\left(e_i\right)\varepsilon\left(e_j\right) e_k \ \in\ R_2^4 \end{aligned}$$

and (ijk) is an even permutation of (123). The multiplication of two semi real quaternions p and q is defined by

$$p \times q = S_p S_q + S_p V_q + S_q V_p + h(V_p, V_q) + V_p \wedge V_q$$

for every $p, q \in Q_H$, where we have used the inner and cross products in semi-euclidean space R_v^3. For a semi real quaternion $q = a e_1 + b e_2 + c e_3 + d \in Q_H$ the conjugate αq of q is defined by $\alpha q = -a e_1 - b e_2 - c e_3 + d$. Thus, we define symmetric, non-degenerate valued bilinear form h as follows:

$$h_v : Q_H \times Q_H \to R, \text{ by}$$

$$h_1(p,q) = \frac{1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) + \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]$$

for R_1^3 and

$$h_2(p,q)=\frac{1}{2}[-\varepsilon(p)\varepsilon(\alpha q)(p\times\alpha q)-\varepsilon(q)\varepsilon(\alpha p)(q\times\alpha p)]$$

for R_2^4, where

$$h_\upsilon(q,q)=a^2\varepsilon(e_1)+b^2\varepsilon(e_2)+c^2\varepsilon(e_3)+d^2$$

for $\upsilon=\{1,2\}$ and it is called the semi-real quaternion inner product. The vector product of two semi real quaternions $p=a_1e_1+b_1e_2+c_1e_3+d_1$ and $q=ae_1+be_2+ce_3+d$ is defined as

$$\begin{aligned}V_p\wedge V_q=&\varepsilon(e_2)\varepsilon(e_3)(b_1c-bc_1)e_1\\&-\varepsilon(e_1)\varepsilon(e_3)(a_1c-ac_1)e_2\\&+\varepsilon(e_1)\varepsilon(e_2)(a_1b-ab_1)e_3,\end{aligned}$$

for $\upsilon=1$ and

$$\begin{aligned}V_p\wedge V_q=&-\varepsilon(e_2)\varepsilon(e_3)(b_1c-bc_1)e_1\\&+\varepsilon(e_1)\varepsilon(e_3)(a_1c-ac_1)e_2\\&-\varepsilon(e_1)\varepsilon(e_2)(a_1b-ab_1)e_3,\end{aligned}$$

for $\upsilon=2$. And then, the norm of semi real quaternion q is denoted by

$$\parallel q\parallel^2=|h_\upsilon(q,q)|=|a^2\varepsilon(e_1)+b^2\varepsilon(e_2)+c^2\varepsilon(e_3)+d^2|,$$

for $\upsilon=\{1,2\}$.

The use of the concept of a spatial quaternion will be made throughout our work. q is called a spatial quaternion whenever $q+\alpha q=0$. It is a temporal quaternion whenever $q-\alpha q=0$ [7–10].

2.1. Null quaternionic curves in R_1^4

Let h, denote the semi quaternionic metric on R_1^4. A curve $C(s)$ in R_1^4 is a null quaternionic curve if $h(C'(s),C'(s))=0$ and $C'(s)\neq 0$ for all s. We note, that a null quaternionic curve $C(s)$ in R_1^4 satisfies $h(C''(s),C''(s))\neq 0$. We say that a null quaternionic curve $C(s)$ in R_1^4 is parametrized by the pseudo-arc if $h(C''(s),C''(s))=\pm 1$.

Let $C(s)=C_1(s)e_1+C_2(s)e_2+C_3(s)e_3+C_4(s)$ be a null quaternionic curve in R_1^4. The three-dimensional semi-euclidean space R_υ^3 is identified with the space of null spatial quaternions $C\in Q_{R_1^4}|C+\alpha C=0\}$ in an obvious manner $C:I\subset R\to Q_{R_1^4}$, $s\to C(s)=\sum\limits_{i=1}^{4}C_i(s)e_i$, $1\leq i\leq 4$. Let $\{L,N,U,W\}$ be the Frenet trihedron of the differentiable for null spatial quaternionic curve in Minkowski space R_1^4.

Null quaternionic Frenet curves parametrized by the pseudo-arc parameter are called null quaternionic Cartan curves. Let $C(s)$ is a null quaternionic Cartan curve in (R_1^4,h), where the quaternionic metric h represents the background gravitational field and $\{C'(s)=L,N,U,W\}$ is its Cartan frame with respect to a special paramerter s. Again, for physical reasons, the orientation is such that $h(L,N)=-1$, $h(W,W)=h(U,U)=1$ and all other products vanish. The Cartan equations are

$$\begin{aligned}&L'=W,\ N'=(p-\tau)U+pW,\\&U'=(p-\tau)L,\ W'=pL+N\end{aligned}$$

and

$$\begin{aligned}&L'=W,\ N'=(\tau+p)U+pW,\\&U'=(\tau+p)L,\ W'=pL+N,\end{aligned}$$

where

$$\begin{aligned}&h(L,L)=h(N,N)=h(L,U)=h(N,U)=\\&\qquad h(W,U)=h(N,W)=h(L,W)=0,\\&h(U,U)=h(W,W)=+1\wedge h(L,N)=-1.\end{aligned}$$

3. Characterization of null quaternionic Bertrand curves

Now we define Bertrand null quaternionic curve in Minkowski space R_1^4 and study Bertrand property for null quaternionic Cartan curves.

Definition 3.1. Let $(C,\bar{C})$ be a pair of framed null quaternionic curves in R_1^4, with distinguished parameters s and $\bar{s}$, respectively. This pair is said to be a null quaternionic Bertrand pair if their principal normal vector fields are linearly dependent. A Cartan null quaternionic curve that has a Bertrand null quaternionic mate is said to be a Bertrand null quaternionic curve (see [5, 6] for null curves).

The curve $\bar{C}$ is called a Bertrand mate of C and vice versa. A null quaternionic Frenet curve is said to be a null quaternionic Bertrand curve if it admits a quaternionic Bertrand mate. By the above Definition 3.1., there exists a functional relation $\bar{s}(s)=s$ for a null quaternionic Bertrand pair $(C,\bar{C})$, such that $\bar{W}(\bar{s})=\pm W$, the normal lines coincide at their respective points. To show that, the null quaternionic Bertrand curves have been used in characterizing null quaternionic helices.

We give the following characterization theorems for null quaternionic Bertrand pair in R_1^4

Lemma 3.2. The distance $|\lambda|$ between corresponding points of a null quaternionic Bertrand curve and of its Bertrand mate (or Bertrand conjugate) is a constant.

Proof. Let $C(s)$ and $\bar{C}(s)$ be null quaternionic Bertrand curves, with respect to a special paramerter s and suppose that $\{L,N,U,W\}$ and $\{\bar{L},\bar{N},\bar{U},\bar{W}\}$ are their quaternionic Cartan frames, respectively. Then we can write

$$\bar{C}(s)=C(s)+\lambda(s)W(s), \tag{3.1}$$

since the normal lines coincide. Suppose s and $\bar{s}$ are the pseudo-arc paramerters of C and $\bar{C}$, respectively, then by taking derivative of (3.1) with respect to s and using null quaternionic Cartan frame, we get

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s}\bar{L}=(1+\lambda p)L+\lambda N+\lambda' W.$$

On the other hand, the condition

$$h(\bar{L},W)=\frac{1}{2}(\bar{L}\times\alpha W+W\times\alpha\bar{L})=0$$

holds for null quaternionic Bertrand curves, hence $\lambda'=0$, we deduce that λ is a nonzero constant. This means that the norm $\parallel\bar{C}-C\parallel$ is a constant.

Thus, we can obtain the following theorem.

Theorem 3.3. Null quaternionic Cartan curve in R_1^4 is a null quaternionic Bertrand curve if and only if p is a non-zero constant and $(p-\tau)$ is a zero.

Proof. Suppose that $\bar{C}$ is a null quaternionic Bertrand mate for C. Then the previous theorem allows us to write

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s}\bar{L} = (1+\lambda p)L + \lambda N, \tag{3.2}$$

where λ is the distance between the curves. Since $\bar{L}$ is null, $h\left(\bar{L},\bar{L}\right) = \bar{L}\times\alpha\bar{L} = 0$, $h(L,W) = h(N,W) = 0$ and $h(L,N) = -1$ hold. We obtain

$$2(\lambda')^2 - 2\lambda(1+\lambda p) = 0.$$

So $\lambda = -\frac{1}{p}$ =constant. Using this value of λ in (3.2), we obtain

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s}\bar{L} = -\frac{1}{p}N. \tag{3.3}$$

Differentiating (3.3) with respect to s and using Cartan frame, we get

$$-W = \bar{W}(\frac{\mathrm{d}\bar{s}}{\mathrm{d}s})^2 + \bar{L}(\frac{\mathrm{d}^2\bar{s}}{\mathrm{d}s^2}) + \frac{(p-\tau)}{p}U. \tag{3.4}$$

Since $\bar{W} = \pm W$, we get $h(\bar{W},W) = \pm 1$ and $\left(\frac{\mathrm{d}\bar{s}}{\mathrm{d}s}\right)^2 = 1$, $\frac{\mathrm{d}^2\bar{s}}{\mathrm{d}s^2} = 0$. Since $h(\bar{W},U) = 0$, we get $\bar{W} = -W$. Thus, $\bar{C}$ has same constant curvatures $p \neq 0$ and $(p-\tau) = 0$.

Conversely, let C be a null quaternionic Cartan curve with p and $(p-\tau)$ as in the hypothesis. Then consider a null quaternionic curve $\bar{C}$ with coordinate function

$$\bar{C} = C - \frac{1}{p}W. \tag{3.5}$$

Differentiating (3.5) with respect to pseudo arc parameter s and using Cartan frame, then $(\bar{C})' = \frac{1}{p}N$, shows that $\bar{C}$ is a null quaternionic Cartan curve. Since differentiating the last equations gives $\left(\bar{C}\right)'' = -W$. Thus the pseudo arc parameters of C and $\bar{C}$ are the same, and the normal vector of C is equal to the normal vector of C at the same paramerter values. These two facts with (3.1) imply that the normal lines of C and $\bar{C}$ coincide at their corresponding points.

Consequently, it follows from above Theorem 3.3. that the only Bertrand null quaternionic curves in R_1^4 are null quaternionic helices with $(p-\tau) = 0$.

Proposition 3.4. A null quaternionic Cartan curve C in R_1^4 is a three-dimensional null helix if and only if there exist a fixed direction X such that

$$h\left(L,X\right) = a \wedge h\left(N,X\right) = b, \tag{3.6}$$

where a and b are nonzero constants and $\{L,N,U,W\}$ is the quaternionic Cartan frame of C.

Proof. Suppose that X is a fixed direction satisfying (3.6). Then, differentiating (3.6) with respect to the pseudo-arc parameter, we obtain

$$p = -\frac{b}{a} \wedge (p-\tau) = 0.$$

Conversely, assume that C is a null quaternionic helix with $(p-\tau) = 0$. Then, if we set

$$X = -pL + N,$$

it is easy to see that X is a fixed direction and Eq. (3.6) holds, which completes the proof.

4. Conclusions

Here, by using the similar idea of above theorems for other null quaternionic Cartan frames, we can obtain the following result. The only Bertrand null quaternionic curves in R_1^4 are null quaternionic helices with $(p+\tau) = 0$.

References

[1] B.O. Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York 1983.

[2] W.B. Bonnor, *Tensor N.S.* **20**, 229 (1969).

[3] A. Bejancu, *Publ. Math. Debrecen* **44** 145 (1994).

[4] A. Fernandez, A. Gimenez, P. Lucas, *Int. J. Mod. Phys. A.* **16** 4845 (2001).

[5] A.C. Çöken, Ü. Çiftçi, *Geometriae Dedicata* **114**, 71 (2005).

[6] K.L. Duggal, D.H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific Publishing, 2007.

[7] K. Bharathi, M. Nagaraj, *Indian J. Pure Appl. Math.* **18**, 507 (1987).

[8] A.C. Çöken, A. Tuna, *Appl. Math. Comp.* **155**, 373 (2004).

[9] A. Tuna Aksoy, A.C. Çöken, *Acta Phys. Pol. A* **128**, B-286 (2015).

[10] A. Tuna Aksoy, A.C. Çöken, *Null Quaternionic Cartan Helices in R3v*, (2015), submitted.