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Existence of p-Adic Quasi Gibbs Measures for Mixed Type p-Adic Ising λ-Model

M. Dogan[a] H. Akin[a,*] and F. Mukhamedov[b]

[a]Zirve University, Faculty of Education, Department of Mathematics, Gaziantep, 27260, Turkey

[b]Department of Computational and Theoretical Sciences, Faculty of Science, IIUM, 25200 Kuantan, Malaysia

We consider nearest-neighbors and next nearest-neighbors p-adic Ising λ-model with spin values $\{\mp 1\}$ on a Cayley tree of order two. First we prove that the model satisfies the Kolmogorov consistency condition and then we prove that the nonlinear equation corresponding to the model has at least two solutions in Q_p, where p is a prime number $p \geq 3$. One of the roots is in ε_p and the others are in $\mathrm{Q}_p \backslash \varepsilon_p$. If the nonlinear equation has more than one non-trivial solutions for the model then we conclude that p-adic quasi Gibbs measure exists for the model.

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1. Introduction

The p-adic numbers were first defined by the German mathematician K. Hensel as a branch of pure mathematics [1–5]. However, a lot of applications of these numbers in theoretical physics have been proposed (see [1–5]). A number of p-adic models in physics cannot be described using ordinary probability theory based on the Kolmogorov axioms [2, 5]. New probability model is called "p-adic probability model" since it takes values in Q_p. The non-Archimedean analogy of the Kolmogorov probability theory gives us the possibility to construct wide classes of stochastic processes by using finite dimensional probability distributions instead of the infinite one, as was proved before.

This gives a possibility to develop the theory of statistical mechanics in the context of the p-adic theory, since it lies on the base of the theory of probability and stochastic processes. One of the central problems of the theory of statistical mechanics is the study of infinite-volume Gibbs measures corresponding to a given Hamiltonian. However, a complete analysis of the set of Gibbs measures for a specific Hamiltonian is often a difficult problem. This problem includes the study of phase transition problems. Recall that for a given Hamiltonian there is a phase transition if there exist at least two distinct p-adic Gibbs measures, of which one is bounded and the other is unbounded.

In the present work, we first establish the Hamiltonian for the mixed type p-adic Ising λ-model. After that we prove that the model satisfies the Kolmogorov consistency condition. Finally we prove the existence of the p-adic quasi Gibbs measures.

*corresponding author; e-mail: hasan.akin@zirve.edu.tr

2. Definitions and preliminaries and the model

2.1. p-adic numbers and measures

Let Q be the field of rational numbers. Every rational number $x \neq 0$ can be represented in the form of $x = p^r m/n$, where $r, m \in \mathrm{Z}$, n is a positive integer, $(p, m) = 1$, $(p, n) = 1$ and p is a fixed prime number. The p-adic norm of x is given by $|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$

It satisfies the following properties: 1) $|xy|_p = |x|_p |y|_p$; 2) The strong triangle inequality $|x+y|_p \leq \max\left\{|x|_p, |y|_p\right\}$; hence this is a non-Archimedean norm. Existence of two types of the completions of the rational numbers with respect to p-adic norm $|x|_p$ and usual norm $|x|_\infty$ was proved by Ostrowsky. p-adic norms are non-equivalent norms on Q. Any p-adic number can be uniquely represented as in the following canonical form $x = p^{\gamma(x)}\left(x_0 + x_1 p + x_2 p^2 + ...\right)$, where $\gamma = \gamma(x) \in \mathrm{Z}$ and x_j are integers $0 \leq x_j \leq p-1$, $x_0 > 0$ $(j = 0, 1, 2, \ldots)$. In this case $|x|_p = p^{-\gamma(x)}$. Let $B(a, r) = \left\{x \in \mathrm{Q}_p : \ |x-a|_p \leq r\right\}$, where $a \in \mathrm{Q}_p$, $r > 0$ be a disc. The p-adic logarithm is defined by $\log_p(x) = \log_p(1 + (x-1)) = \sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$, which converges for $x \in B(1, 1)$. The p-adic exponential is defined by $\exp_p(x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$, which converges for $x \in B\left(0, p^{-(p-1)^{-1}}\right)$.

Lemma 2.1 For the equation $x^2 = a$, where $0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + ...)$ [3], $0 \leq a_j \leq p-1$, $a_0 > 0$, to have a solution $x \in \mathrm{Q}_p$, it is necessary and sufficient that the following conditions are fulfilled: 1) $\gamma(a)$ is even; 2) a_0 is a quadratic residue modulo p if $p \neq 2$, and moreover $a_1 = a_2 = 0$ if $p = 2$.

Lemma 2.2 Let $x \in B\left(0, p^{-(p-1)^{-1}}\right)$ then $|\exp_p(x)|_p = 1$, $|\exp_p(x) - 1|_p = |x|_p < 1$, $|\log_p(1+x)|_p =$

$|x|_p < p^{-(p-1)^{-1}}$ and $\log_p(\exp_p(x)) = x$, $\exp_p(\log_p(1+x)) = 1+x$. Put

$$\varepsilon_p = \{x \in Q_p : |x|_p = 1, \quad |x-1|_p < p^{-1/(p-1)}\}. \quad (1)$$

Thus, from this lemma we may conclude that if $x \in \varepsilon_p$ then there is an element $h \in B\left(0, p^{-1/(p-1)}\right)$ such that $x = \exp_p(h)$.

2.3. p-adic measures

Let $(X, \boldsymbol{B})$ be a measurable space, where $\boldsymbol{B}$ is the algebra of subsets of X. A function $\mu : \boldsymbol{B} \to Q_p$ is called a p-adic measure if the equality $\mu(\bigcup\limits_{j=1}^{n} A_j) = \sum\limits_{j=1}^{n} \mu(A_j)$ holds for any $A_1, ..., A_n \subset \boldsymbol{B}$ such that $A_i \cap A_j = \emptyset$. If $\mu(X) = 1$ then the p-adic measure is called a probability measure.

2.4. Cayley tree

A Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles with exactly $k+1$ edges issuing from each vertex. Cayley tree is denoted by $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. Two vertices $x, y \in V$ are called *nearest-neighbors* if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l = \langle x, y\rangle$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree Γ^k, is the number of edges in the shortest path from x to y. For a fixed vertex $x^{(0)} \in V$ we set $W_n = \{x \in V | d(x, x^{(0)}) = n\}$, $V_n = \{x \in V | d(x, x^{(0)}) \leq n\}$ and L_n denotes the set of edges in V_n. The fixed vertex $x^{(0)}$ is called the zero-th level and the vertices in W_n are called the n-th level. For the sake of simplicity we put $|x| = d(x, x^{(0)})$, $x \in V$. Two vertices $x, y \in V$ are called *the next nearest neighbors* if $d(x, y) = 2$. The next nearest neighbor vertices x and y are called *prolonged next-nearest-neighbors* if $|x| \neq |y|$, $\rangle x, y\langle$.

2.5. Mixed type p-adic Ising λ-model

We consider the mixed type p-adic Ising λ-model, where spin takes values in $\Phi = \{-1, 1\}$ and these values are assigned to the vertices of the Cayley tree. A configuration σ on V is defined $\sigma : x \to \sigma(x) \in \{\mp 1\}$ as a function. Therefore, by using the same sense we may define a configuration on V_n and W_n respectively. We denote the set of all configurations as $\Omega_n = \Phi^V$. The Hamiltonian $H_n : \Omega_n \to Q_p$ of the p-adic Ising λ-model is

$$H_n(\sigma) = \sum_{\langle x,y\rangle \in V_n} \lambda_{x,y}(\sigma(x), \sigma(y)) + J \sum_{\substack{\rangle x,y\langle \\ x \in W_{n-2}, y \in W_n}} \sigma(x)\sigma(y), \quad (2)$$

where the sum is taken over all nearest $\langle x, y\rangle$ and prolonged next nearest $\rangle x, y\langle$ neighbors for the model. Here $\lambda : \Phi \times \Phi \to Q_p$ is a function such that $|\lambda_{x,y}(a, b)|_p < p^{-(p-1)^{-1}}$ for all $a, b \in \{\mp 1\}$, $x, y \in V$ and p is a fixed prime.

3. Construction of Gibbs measures

In this section we will give a construction of Gibbs measures for the mixed type p-adic Ising λ-model on the Cayley tree in p-adic setting. For the definition of Gibbs measure we need the following lemma:

Lemma 3.1 Let h_x, $x \in V$ be p-adic valued function that $h_x \in B\left(0, p^{-1/(p-1)}\right)$ for all $x \in V$ and $|\lambda_{x,y}(a, b)|_p < p^{-1/(p-1)}$ for all $a, b \in \Phi = \{\mp 1\}$. Then the relation $H_n(\sigma) + \sum\limits_{x \in W_n} h_x\sigma(x) \in B\left(0, p^{-1/(p-1)}\right)$ is valid for any $n \in N$. It can be easily proved by strong triangle inequality for p-adic norm $|x|_p$. Let $\mathbf{h} : x \in V \to h_x \in Q_p$ be a function of any vertex $x \in V$, where $|h_x|_p < p^{-1/(p-1)}$ for all $x \in V$. According to the given p-adic probability measure $\mu^{(n)}$ on Φ^{V_n} can be defined as following

$$\mu_h^{(n)}(\sigma_n) = Z_n^{-1} \exp_p\{H_n(\sigma_n) + \sum_{x \in W_n} h_x\sigma(x)\}. \quad (3)$$

Here, as before $\sigma_n : x \in V_n \to \sigma_n(x)$ and the partition function Z_n is defined by

$$Z_n = \sum_{\widetilde{\sigma}_n \in \Omega_{V_n}} \exp_p\{H(\widetilde{\sigma}_n) + \sum_{x \in W_n} h_x\widetilde{\sigma}(x)\}.$$

The measure $\mu^{(n)}$ exists since Lemma 3.1. The compatibility conditions for $\mu_h^{(n)}(\sigma_n)$, $n \geq 1$ are

$$\sum_{\sigma^{(n)} \in \Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \sigma^{(n)}) = \mu_h^{(n-1)}(\sigma_{n-1}), \quad (4)$$

where $\sigma_{n-1} \in \Omega_{V_{n-1}}$. We know that the analogy of the Kolmogorov extension theorem and (3) was proven in [2]. Then according to the Kolmogorov theorem there exists a unique p-adic measure μ_h on $\Omega = \Phi^V$ such that for $n = 1, 2, ...$ and $\sigma_n \in \Phi^{V_n}$ the following equality holds

$$\mu_h(\{\sigma|_{V_n} = \sigma_n\}) = \mu_h^{(n)}(\sigma_n),$$

and it is called "p-adic Gibbs measure" for the considered mixed type p-adic Ising λ-model on the Cayley tree. The following theorem guaranties that the measures $\mu_h^{(n)}(\sigma_n)$, $n \geq 1$ for $h = (h_x, x \in V)$ satisfies the compatibility condition.

Theorem 3.2 The measures $\mu_{h,\rho}^{(n)}(\sigma_n)$, $n = 1, 2, ...$ based on the model in Eq. (2) satisfy the compatibility condition (4) if and only if the following equation holds for any $x \in V \backslash \{x^{(0)}\}$

$$h_x^2 = \frac{a^2\theta h_y^2 h_z^2 + ac(h_y^2 + h_z^2) + c^2\theta}{b^2\theta h_y^2 h_z^2 + bd(h_y^2 + h_z^2) + d^2\theta}, \quad (5)$$

where $a = \rho^{\lambda(1,1)}$, $b = \rho^{\lambda(-1,1)}$, $c = \rho^{\lambda(1,-1)}$, $d = \rho^{\lambda(-1,-1)}$, $\theta = \rho^{2J}$. We can prove the theorem by using the same arguments as in [1]. From the theorem 3.2, the existence of p-adic quasi Gibbs measures for the mixed type p-adic Ising λ-model on the Cayley tree is reduced to the solutions of Eq. (5).

4. The Existence of p-adic quasi Gibbs measures

Here we consider the mixed type p-adic Ising λ-model (2) on the Cayley tree of order two. The main goal in this section is to establish the existence of the generalized p-adic quasi Gibbs measures by analysing Eq. (5). Here, the existence of the generalized p-adic quasi Gibbs measures is reduced to the solutions of Eq. (5). If (5) has solutions then the generalized p-adic quasi Gibbs measures exists. Now let us recall that a function $\boldsymbol{h} = \{h_x\}_{x\in V-\{0\}}$ is called a translation-invariant function if $h_x = h_y$ for all $x, y \in V$. A p-adic measure μ_h, corresponding to a translation-invariant function $\boldsymbol{h}$, is called a translation-invariant generalized p-adic quasi Gibbs measure. In general, the solution of Eq. (5) is very complicated. Therefore we are going to rewrite Eq. (5) in a simpler form by taking $u_x = h_x^2$, $u_y = h_y^2$, $u_z = h_z^2$, and translation invariant condition $u_x = u_y = u_z = u$ in the function for all $x, y, z \in V$. Then from Eq. (5) we obtain

$$u = \frac{(a^2\theta u^2 + 2acu + c^2\theta)}{(b^2\theta u^2 + 2bdu + d^2\theta)}, \tag{6}$$

where $a = \rho^{\lambda(1,1)}$, $b = \rho^{\lambda(-1,1)}$, $c = \rho^{\lambda(1,-1)}$, $d = \rho^{\lambda(-1,-1)}$, $\theta = \rho^{2J}$. Now, to find the fixed points let us take $h_x^2 = f(u) = u$. Then Eq. (6) will be transformed to the following equation:

$$u = f(u) = \frac{a^2\theta u^2 + 2acu + c^2\theta}{b^2\theta u^2 + 2bdu + d^2\theta}, \tag{7}$$

where $a, b, c, d, \theta \in \varepsilon_p$. To find the fixed points of the function $f(u)$ in (7) we need the following lemmas:

Lemma 4.1 Let $p \geq 3$, then for any $a, b \in \varepsilon_p$ one has $| a + 1 |_p = 1$, $| a + b |_p = 1$.

Lemma 4.2 Let $x, y \in Q_p$ such that $|xy|_p = 1$ and $|x + y|_p < 1$, then $|x|_p = |y|_p = 1$.

Now let us express (7) such that

$$f(u) = (a^2\theta u^2 + 2acu + c^2\theta)(b^2\theta u^2 + 2bdu + d^2\theta)^{-1}. \tag{8}$$

Lemma 4.3 Let $p \geq 3$, $a, b, c, d, \theta \in \varepsilon_p$ and f is given by (8). Then $f(\varepsilon_p) \subset \varepsilon_p$ and $|f(u) - f(v)|_p \leq p^{-1} | u - v |_p$, for all $u, v \in \varepsilon_p$.

Proof: Assume that $u \in \varepsilon_p$. Then by the Lemma 4.1 we have $|f(u)|_p = 1$ and from Lemma 2.2 one can see that

$$| f(u)-1 |_p = \left|\frac{\theta(a^2-b^2)u^2+2(ac-bd)u+\theta(c^2-d^2)}{b^2\theta u^2+2bdu+d^2\theta}\right|_p =$$

$$| \theta(a^2 - b^2)u^2 + 2(ac - bd)u + \theta(c^2 - d^2) |_p \leq p^{-1},$$

and then $f(u) \in \varepsilon_p$. It is clear that $\left|b^2\theta u_0^2 + 2bdu_0 + d^2\theta\right|_p = 1$. To prove the second condition, let $u, v \in \varepsilon_p$, then by using Lemma 4.1 and Lemma 2.2 we find

$$|f(u) - f(v)|_p =$$

$$\left|\frac{a^2\theta u^2 + 2acu + c^2\theta}{b^2\theta u^2 + 2bdu + d^2\theta} - \frac{a^2\theta v^2 + 2acv + c^2\theta}{b^2\theta v^2 + 2bdv + d^2\theta}\right|_p =$$

$$\left|\frac{(ad-bc)(u-v)\theta\,(2cd+2abuv+(bc+ad)u\theta+(bc+ad)v\theta)}{(2bdu+d^2\theta+b^2u^2\theta)\,(2bdv+d^2\theta+b^2v^2\theta)}\right|_p =$$

$$|u - v|_p \left|2a^2\theta bduv + (u + v)a^2d^2\theta^2 - 2ab^2c\theta u + 2acd^2\theta\right.$$

$$\left.-(u + v)b^2c^2\theta^2 - 2bc^2d\theta\right|_p = |u - v|_p \left|2(k_1 - k_2)\right.$$

$$\left.+2(k_4 - k_3) + (k_2 - 1) - (k_5 - 1)\right|_p \leq p^{-1}|u - v|_p,$$

where $k_1 = a^2\theta bduv$, $k_2 = a^2\theta^2d^2$, $k_3 = ab^2c\theta u$, $k_4 = acd^2\theta$, $k_5 = b^2c^2\theta^2$, $k_6 = bc^2d\theta$ and u, v, k_1, k_2, k_3, k_4, k_5, $k_6 \in \varepsilon_p$. Hence the function f satisfies the Banach contraction principle.

Proposition 4.4 Let a, b, c, d, $\theta \in \varepsilon_p$ with $|a - 1|_p + |b - 1|_p + |c - 1|_p + |d - 1|_p + |\theta - 1|_p \neq 0$ and f be given by (8). Then the following statements hold:

1. The function f has a unique fixed point $u_0 \in \varepsilon_p$
2. The function f has at most two fixed points u_1, u_2 different from x_0 if $\sqrt{-1}$ exists.

Proof: 1) There exists at least one fixed point u_0 since Lemma 4.3. It is clear that again from Lemma 4.3 if u_0 is a fixed point then $u_0 \in \varepsilon_p$. 2) From the Eq. (8) and $f(u) = u$ we obtain

$$b^2\theta u^3 + (2bd - a^2\theta)u^2 + (d^2\theta - 2ac)u - c^2\theta = 0, \tag{9}$$

and then we rewrite (9) as

$$b^2\theta u^3 + (2bd - a^2\theta)u^2 + (d^2\theta - 2ac)u - c^2\theta =$$

$$(u - u_0)\big[(b^2\theta u^2 + (2bd - a^2\theta + b^2\theta u_0)u + d^2\theta$$

$$-2ac + 2bdu_0 - a^2\theta u_0 + b^2\theta u_0^2) + (d^2\theta - 2ac$$

$$+u_0(2bd - a^2\theta + b^2\theta u_0))u_0 - c^2\theta\big] = 0,$$

since $u - u_0$ is a factor of (9). We can express the last equation in the form of:

$$(u - u_0)\big[\,(b^2\theta u^2 + (2bd - a^2\theta + b^2\theta u_0)u + A)$$

$$+Au_0 - c^2\theta\big] = 0, \tag{10}$$

where $A = d^2\theta - 2ac + 2bdu_0 - a^2\theta u_0 + b^2\theta u_0^2$. Now we need to check the solutions of quadratic equation

$$(b^2\theta u^2 + (2bd - a^2\theta + b^2\theta u_0)u + A) + Au_0 - c^2\theta = 0. \tag{11}$$

If u_0 is a solution of (9) then $Au_0 - c^2\theta = 0$. So we have $A = c^2\theta(u_0)^{-1}$. When we substitute A in the Eq. (11) then the quadratic equation is obtained as

$$b^2\theta u^2 + (2bd - a^2\theta + b^2\theta u_0)u + c^2\theta(u_0)^{-1} = 0. \tag{12}$$

Now we are going to investigate the solutions of (12) in Q_p by the quadratic formula as in the real case.

For the existence of $\sqrt{\Delta}$, the discriminant $\Delta = p^{\gamma_L}(a_0 + a_1p^{\gamma_1} + a_2p^{\gamma_2} + a_3p^{\gamma_3} + ...)$ must satisfy the conditions in Lemma 2.1. Therefore if we perform some operations on (12), then we can find the discriminant such that: $\Delta = 0 + p^{\gamma}\varepsilon$. For the existence of $\sqrt{\Delta}$ we need a more detailed analysis of Δ. For that in (12) let us assign $a = 1 + \varepsilon_a p^{\gamma_a}$, $b = 1 + \varepsilon_b p^{\gamma_b}$, $c = 1 + \varepsilon_c p^{\gamma_c}$, $d = 1 + \varepsilon_d p^{\gamma_d}$, $\theta = 1 + \varepsilon_0 p^{\gamma_0}$, $u_0 = 1 + \varepsilon_1 p^{\gamma_1}$. For Eq. (12) the discriminant is

$$\Delta = \big(2(bd - 1) - (a^2\theta^2 - 1) + (b^2\theta^2u_0 - 1) + 2\big)^2$$

$$-\frac{4}{u_0}(b^2c^2\theta^4 - 1) - 4. \tag{13}$$

When we substitute a, b, c, d, θ, u_0 given above in (13) and after long calculations we obtain:

$$\Delta = 16\varepsilon_b^2 p^{2\gamma_b} + 4\varepsilon_d^2 p^{2\gamma_d} + 4\varepsilon_a^2 p^{2\gamma_a} + \varepsilon_1^2 p^{2\gamma_1} + 8\varepsilon_b p^{\gamma_b}$$
$$-8\varepsilon_a p^{\gamma_a} + 4\varepsilon_1 p^{\gamma_1} - 8\varepsilon_0 p^{\gamma_0}. \quad (14)$$

Assume that $\min\{\gamma_a, \gamma_b, \gamma_0, \gamma_1\} = \gamma_a < \gamma_b, \gamma_0, \gamma_1$ then $\Delta = 4p^{\gamma_a}(-2\varepsilon_a + p^{\bar{\gamma}}\eta)$ and here let γ_a be even. Now we must check whether condition $|a-1|_p > |u_0 - 1|_p$ can be satisfied. For this we state the following proposition.

Proposition 4.6 Assume that $|a-1|_p = |b-1|_p > \max\left\{|c-1|_p, |d-1|_p\right\}$ and $|4\varepsilon_1 + \varepsilon_2|_p \le p^{-1}$, where $a-1 = p^{\gamma_a}\varepsilon_1$; $b-1 = p^{\gamma_a}\varepsilon_2$, then $|u_0 - 1|_p < |a-1|_p$.

Proof: Let us take θ instead of θ^2 in (7) then $|u_0 - 1|_p = \left|\frac{a^2\theta u_0^2 + 2acu_0 + c^2\theta}{b^2\theta u_0^2 + 2bdu_0 + d^2\theta} - 1\right|_p$, since $u_0 = f(u_0)$ and then we obtain

$$|u_0 - 1|_p = \quad (15)$$
$$\left|\frac{\theta(a^2 - b^2)u_0^2 + 2(ac - bd)u_0 + (c^2 - d^2)\theta}{b^2\theta u_0^2 + 2bdu_0 + d^2\theta}\right|_p.$$

After a restriction $|a-1|_p = |b-1|_p$, where $a - 1 = p^{\gamma_a}\varepsilon_1$, $b - 1 = p^{\gamma_a}\varepsilon_2$, from (15) we get

$$|u_0 - 1|_p = \left|u_0^2(a^2-1) - u_0^2(b^2-1) + 2u_0(a-1)c\right.$$
$$\left.-2u_0(b-1)d\right|_p = \left|u_0^2 p^{\gamma_a}\varepsilon_1(a+1) + 2u_0 p^{\gamma_a}\varepsilon_1 c -\right.$$
$$\left.u_0^2 p^{\gamma_a}\varepsilon_2 - 2u_0 p^{\gamma_a}\varepsilon_2 d\right|_p = |u_0 p^{\gamma_a}\left[u_0 p^{\gamma_a}\varepsilon_1(a+1)\right.$$
$$\left.+2\varepsilon_1 c - u_0\varepsilon_2 - 2\varepsilon_2 d\right]|_p = |(u_0 - 1)\varepsilon_1(a+1)$$
$$+\varepsilon_1(a+1) + 2\varepsilon_1(c-1) + 2\varepsilon_1 - (u_0 - 1)\varepsilon_2 - \varepsilon_2$$
$$-2\varepsilon_2(d-1) + 2\varepsilon_2|_p =$$
$$|\varepsilon_1(a - 1 + 2) + 2\varepsilon_1 + \varepsilon_2|_p = |4\varepsilon_1 + \varepsilon_2|_p.$$

If $|4\varepsilon_1 + \varepsilon_2|_p \le p^{-1}$ then $|c-1|_p = |d-1|_p < |a-1|_p$, so we have $|u_0 - 1|_p < |a-1|_p$. Therefore the proof is completed which means that $u_0 \in \varepsilon_p$.

Remark 4.7: 1) If $|c-1|_p$; $|b-1|_p$; $|d-1|_p < |a-1|_p$ then $|u_0 - 1|_p = |a-1|_p$.
2) If $|b-1|_p > |a-1|_p > |d-1|_p$; $|c-1|_p$ then $-u_0^2(b-1) - 2u_0(b-1)d = -(b-1)(u_0^2 + 2u_0 d)$ at that rate we obtain $|u_0 - 1|_p = |b-1|_p > |a-1|_p$ is a contradiction with $|u_0 - 1|_p < |a-1|_p$.

Now, we check the existence of $\sqrt{\Delta}$ for $\Delta = 4p^{\gamma_a}(-2\varepsilon_a + p^{\bar{\gamma}}\eta)$, whereas assumed $\min\{\gamma_a, \gamma_b, \gamma_0, \gamma_1\} = \gamma_a < \gamma_b, \gamma_0, \gamma_1$ and γ_a is even. Then we can write ε_1 and ε_2 in the canonical form of $\varepsilon_1 = a_0 + a_1 p + a_2 p^2 + ...$, where $a_0 \neq 0$, $\varepsilon_2 = b_0 + b_1 p + b_2 p^2 + ...$, where $b_0 \neq 0$ and $4\varepsilon_1 + \varepsilon_2 = 4a_0 + b_0 + p\bar{\eta}$ then $|4a_0 + b_0|_p \le p^{-1}$ or $4a_0 + b_0 \equiv 0(\mathrm{mod}\, p)$.

Examples: Now let us check the existence of $\sqrt{\Delta}$ in $\mathbb{Q}_p$ for $p = 3$; $p = 5$; $p = 7$. Let $p = 3$ and $a_0 = 1; 2$ then $u^2 \equiv 1(\mathrm{mod}3)$ so $u = 1$ is a solution. Let $p = 5$ and $a_0 = 1; 2; 3; 4$ then we obtain quadratic congruence $u^2 \equiv 4(\mathrm{mod}5)$ where $a_0 = 3$, then $u = 3$ is a solution of the quadratic congruence $u^2 \equiv 4(\mathrm{mod}5)$. Let $p = 7$ and $a_0 = 1; 2; 3; 4; 5; 6$ then $u^2 \equiv 1(\mathrm{mod}7)$, where $a_0 = 3$ then $u = 1$ is a solution and so on. Therefore, we say that for each $p \ge 3$, $\sqrt{\Delta}$ exists. This means that for Eq. (12) there are two solutions u_1, u_2 different from u_0.

Due to Proposition 4.4, we infer that there are three p-adic Gibbs measures μ_0, μ_1 and μ_2 corresponding to the fixed points u_0, u_1 and u_2, respectively. So, we have proved the existence of p-adic Gibbs measures for mixed type p-adic Ising λ-model.

5. Conclusions

In the present paper, we have defined mixed type $p-$adic Ising λ-model and then we have proved that Eq. (5) satisfies the Kolmogorov consistency condition. We have shown that this equation has 3 different fixed points under the restrictions. We have proved the existence of the p-adic quasi Gibbs measures based on the fixed points of Eq. (7) in $\mathbb{Q}_p$. In the next studies we are going to investigate the existence of the phase transition for the model (2).

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