New Applications of the $(G'/G, 1/G)$-Expansion Method

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In this paper, we study general solutions of the new fifth order nonlinear evolution and the Burgers KP equations with the aid of the two variables $(G'/G, 1/G)$-expansion method. The kink, bell-shaped solitary wave, periodic and singular periodic solutions are obtained. Finally, the numerical simulations add to these obtained solutions.

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1. Introduction

Nonlinear partial differential equations (NPDEs) have an important place in applied mathematics and physics. These equations are mathematical models of physical phenomenon that arise in engineering, chemistry, biology, mechanics, and physics. The general solutions of NPDEs give information about the character of the physical phenomena. Moreover, the solutions of NPDEs play an important role in soliton theory [1, 2]. The obtaining of general solution of NPDEs through using symbolical computer programs such as Maple, Matlab, and Mathematica has attracted the attention of scientists. Hence, many methods have been developed to investigate the character of physical models. Some of these methods can be written as: Bäcklund transformation [3], Cole–Hopf transformation [4], generalized Miura transformation [5], inverse scattering method [6], Darboux transformation [7], Painlevé analysis [8], similarity reduction method [9], sine-cosine method [10], first integral method [11], exp-function method [12] and so on. Besides these methods, there are many methods which are reached to solutions of NPDEs by using an auxiliary equation. For example, extended tanh function method [13], generalized tanh function method [14], Jacobi elliptic function method [15] contain an auxiliary Riccati equation. Furthermore, G'/G expansion method [16] is based on the second order linear equation.

The G'/G expansion method [16] was proposed by Wang et al. Guo and Zhou developed extended G'/G expansion method [17]. Later, Lü improved generalized G'/G expansion method [18]. In all of these methods there is used the second order linear equation as $G'' + \lambda G' + \mu G = 0$. The difference between these methods is solution functions which are chosen for considering problem. Recently, Li et al. [19] have presented the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method. Furthermore, there are many studies on solutions of NPLEs in literature [20–24].

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The aim of this paper is to apply the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method to find the general solutions of the new fifth order nonlinear evolution [25] and BKP equations [26].

2. An analysis of the method

Before starting to present the $\left(\frac{G'}{G}, \frac{1}{G}\right)$ expansion method, we shall give a simple description of the $\left(\frac{G'}{G}, \frac{1}{G}\right)$ expansion method [19, 27, 28]. Let us consider the second order linear ordinary differential equation (ODE):

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{1}$$

where $\phi = \frac{G'}{G}$ and $\psi = \frac{1}{G}$, then we have

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{2}$$

Remark 2.1. If $\lambda < 0$, the general solutions of Eq. (1):

$$G(\xi) = A_1 \sinh\left(\sqrt{-\lambda}\xi\right) + A_2 \cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}, \tag{3}$$

where A_1 and A_2 are arbitrary constants. Thus, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{4}$$

where $\sigma = A_1^2 - A_2^2$.

Remark 2.2. If $\lambda > 0$, the general solutions of Eq. (1):

$$G(\xi) = A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}, \tag{5}$$

and therefore,

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{6}$$

where $\sigma = A_1^2 - A_2^2$.

Remark 2.3. If $\lambda = 0$, the general solutions of Eq. (1):

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{7}$$

and therefore,

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right). \tag{8}$$

Now, let us illustrate how this method works. Therefore, let us consider that an NLPDE is given by

$$Q = (u, u_t, u_x, u_{xx}, u_{tt}, \ldots), \tag{9}$$

where $u = u(x,t)$ is an unknown function. Using the transformation $u(x,t) = u(\xi)$, $\xi = x - Vt$ then we get a nonlinear ODE for $u(\xi)$:

$$Q' = (u, u', u'' \ldots) = 0. \tag{10}$$

Assume that the solutions of Eq. (10) can be expressed by a polynomial ϕ and ψ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i \phi^i + \sum_{i=1}^{N} b_i \phi^{i-1} \psi, \tag{11}$$

where $a_i\,(i = 0, 1, \ldots, N)$ and $b_i\,(i = 1, \ldots, N)$ are constants to be determined later. N is a positive integer that can be determined by balancing the highest order derivate and with the highest nonlinear terms in Eq. (10). Substituting Eq. (11) into Eq. (10) along with Eq. (2) and Eq. (4), Eq. (10) can be converted into a polynomial in ϕ and ψ. Equating the coefficients of each power of $\phi^i\psi^j$ to zero yields a system of algebraic equation for $a_i b_i V \mu A_1 A_2$ and λ. We solve this algebraic equation with the aid of Mathematica 7.0. Thus, we obtain the general solutions in terms of the hyperbolic functions for $\lambda < 0$. We find the general solutions in terms of the trigonometric functions for $\lambda > 0$ and we have the general solutions in terms of the rational function for $\lambda = 0$.

3. Applications

Example 1. Let's consider the new fifth order nonlinear evolution equation [25] of the form

$$u_{ttt} - u_{txxxx} - 4u_{xxx}u_t - 12u_{xx}u_{xt} - 8u_x u_{xxt} = 0. \tag{12}$$

If we can use transformation $u(x,t) = u(\xi)$, $\xi = x - Vt$, Eq. (12) become

$$-V^3 u''' + V u^{(5)} + 12 V u' u''' + 12 V (u'')^2 = 0, \tag{13}$$

and we integrate twice Eq. (13), we have

$$-V^3 u' + V u''' + 6V (u')^2 = 0, \tag{14}$$

where integration constants are taken as zero. When balancing u''' with $(u')^2$, we obtain $N = 1$. Thus, we choose solution of Eq. (14) as

$$u(\xi) = a_0 + a_1 \phi(\xi) + b_1 \psi(\xi). \tag{15}$$

Case 1.1. For $\lambda < 0$, substituting Eq. (15) into Eq. (14) and by using Eq. (2) and Eq. (4) yields a set of algebraic equations for $a_0, a_1, b_1, \mu, \sigma, \lambda$ and V. These systems are

$$a_1 V^3 \lambda - 2a_1 V \lambda^2 + 6a_1^2 V \lambda^2 + \frac{1}{\mu^2 + \lambda^2\sigma} 3a_1 V \lambda^2 \mu^2 - \frac{1}{\mu^2 + \lambda^2\sigma} 6a_1^2 V \lambda^2 \mu^2 = 0,$$
$$-a_1 V^3 \mu + 5a_1 V \lambda \mu - 12a_1^2 V \lambda \mu - \frac{1}{\mu^2 + \lambda^2\sigma} 6a_1 V \lambda \mu^3 + \frac{1}{\mu^2 + \lambda^2\sigma} 12a_1^2 V \lambda \mu^3 = 0,$$
$$b_1 V^3 - 5b_1 V \lambda + 12a_1 b_1 V \lambda + \frac{1}{\mu^2 + \lambda^2\sigma} 12 b_1 V \lambda \mu^2 - \frac{1}{\mu^2 + \lambda^2\sigma} 24 a_1 b_1 V \lambda \mu^2 = 0,$$
$$a_1 V^3 - 8a_1 V \lambda + \frac{3a_1 V \lambda \mu^2}{\mu^2 + \lambda^2\sigma} - \frac{1}{\mu^2 + \lambda^2\sigma} 6a_1^2 V \lambda \mu^2 + 12a_1^2 V \lambda - \frac{1}{\mu^2 + \lambda^2\sigma} 6b_1^2 V \lambda^2 = 0,$$
$$12a_1 V \mu - 12a_1^2 V \mu + \frac{1}{\mu^2 + \lambda^2\sigma} 12 b_1^2 V \lambda \mu = 0,$$
$$-6b_1 V + 12a_1 b_1 V = 0,$$
$$-6a_1 V + 6a_1^2 V - \frac{1}{\mu^2 + \lambda^2\sigma} 6b_1^2 V \lambda = 0,$$
$$-\frac{1}{\mu^2 + \lambda^2\sigma} 6b_1 V \lambda^2 \mu + \frac{1}{\mu^2 + \lambda^2\sigma} 12a_1 b_1 V \lambda^2 \mu = 0,$$
$$-\frac{1}{\mu^2 + \lambda^2\sigma} 6b_1 V \lambda \mu + \frac{1}{\mu^2 + \lambda^2\sigma} 12a_1 b_1 V \lambda \mu = 0. \tag{16}$$

We obtain the roots of Eq. (16) with the aid of Mathematica as

$$a_1 = \frac{1}{2},\ \lambda \neq 0,\ b_1 = \pm\frac{\sqrt{-\mu^2 - \lambda^2\sigma}}{2\sqrt{\lambda}},\ V = \pm \mathrm{i}\sqrt{\lambda},$$
$$\mu^2 + \lambda^2\sigma \neq 0,\ \mathrm{i} = \sqrt{-1}. \tag{17}$$

Substituting Eq. (17) into Eq. (15), we have the following solutions of Eq. (12):

Family 1.1.1

$$u(x,t) = a_0 + \frac{1}{2} \times \left(\frac{A_1\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right) + A_2\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}} \right) \pm \frac{\sqrt{-\mu^2 - \lambda^2\sigma}}{2\sqrt{\lambda}} \times \frac{1}{\left(A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right) + \mu/\lambda\right)}, \tag{18}$$

where $\sigma = A_1^2 - A_2^2$ and $\xi = x - (i\sqrt{\lambda})t$.

Family 1.1.2

For A_1= 0, A_2> and $\mu = 0$ in Eq. (18), we obtain the solitary solution

$$u(x,t) = a_0 + \frac{1}{2A_2}\sqrt{-\lambda\sigma}\,\mathrm{sech}\left(\sqrt{-\lambda}\xi\right) + \frac{1}{2}\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right), \tag{19a}$$

$$u(x,t) = a_0 - \frac{1}{2A_2}\sqrt{-\lambda\sigma}\,\mathrm{sech}\left(\sqrt{-\lambda}\xi\right) + \frac{1}{2}\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right). \tag{19b}$$

Family 1.1.3

For A_2= 0, A_1> and $\mu = 0$ in Eq. (18), we have the solitary solution

$$u(x,t) = a_0 + \frac{1}{2}\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) + \frac{1}{2A_1}\sqrt{-\lambda\sigma}\,\mathrm{csch}\left(\sqrt{-\lambda}\xi\right), \tag{20a}$$

$$u(x,t) = a_0 + \frac{1}{2}\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) - \frac{1}{2A_1}\sqrt{-\lambda\sigma}\,\mathrm{csch}\left(\sqrt{-\lambda}\xi\right). \tag{20b}$$

Setting $\lambda = -1$, $\sigma = -1$, $A_2 = 2$, $a_0 = 1$, $A_1 = 1$, $\mu = 1$, we have solution of Eq. (18) (see Fig. 1).

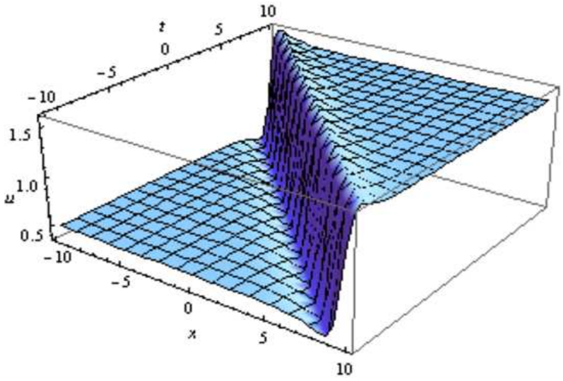

Fig. 1. The profile of solution Eq. (18).

Setting $\lambda = -1$, $\sigma = 1$, $A_2 = 2$, $a_0 = 1$, $A_1 = 1$, $\mu = 1$, we have solution of Eq. (18) (see Fig. 2).

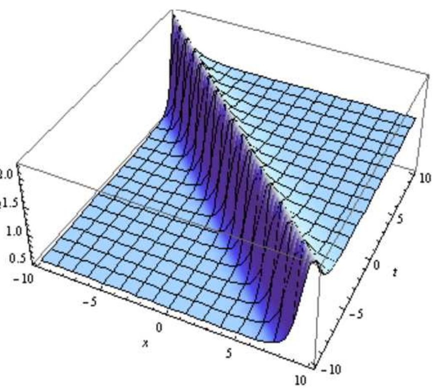

Fig. 2. The profile of solution Eq. (18).

Setting $\lambda = -1$, $\sigma = 1$, $A_2 = 1$, $a_0 = 1$, $A_1 = 0$, $\mu = 1$, we have solution of Eq. (18) (see Fig. 3).

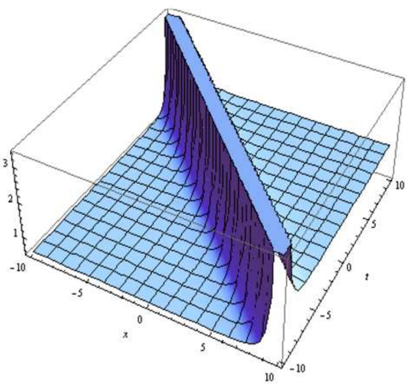

Fig. 3. The profile of solution Eq. (18).

Case 1.2. For $\lambda > 0$, substituting Eq. (15) into Eq. (14) and by using Eq. (2) and Eq. (4) yields a set of algebraic equations for a_0, a_1, b_1, μ, σ, λ and V. These systems are

$$a_1V^3\lambda - 2a_1V\lambda^2 + 6a_1^2V\lambda^2 - \frac{1}{-\mu^2+\lambda^2\sigma}3a_1V\lambda^2\mu^2 + \frac{1}{-\mu^2+\lambda^2\sigma}6a_1^2V\lambda^2\mu^2 = 0,$$

$$-a_1V^3\mu + 5a_1V\lambda\mu - 12a_1^2V\lambda\mu + \frac{1}{-\mu^2+\lambda^2\sigma}6a_1V\lambda\mu^3 - \frac{1}{-\mu^2+\lambda^2\sigma}12a_1^2V\lambda\mu^3 = 0,$$

$$\frac{1}{-\mu^2+\lambda^2\sigma}6b_1V\lambda^2\mu - \frac{1}{-\mu^2+\lambda^2\sigma}12a_1b_1V\lambda^2\mu = 0,$$

$$\frac{1}{-\mu^2+\lambda^2\sigma}6b_1V\lambda\mu - \frac{1}{-\mu^2+\lambda^2\sigma}12a_1b_1V\lambda\mu = 0,$$

$$b_1V^3 - 5b_1V\lambda + 12a_1b_1V\lambda - \frac{1}{-\mu^2+\lambda^2\sigma}12b_1V\lambda\mu^2 + \frac{1}{-\mu^2+\lambda^2\sigma}24a_1b_1V\lambda\mu^2 = 0,$$

$$a_1V^3 - 8a_1V\lambda - \frac{1}{-\mu^2+\lambda^2\sigma}3a_1V\lambda\mu^2 + \frac{1}{-\mu^2+\lambda^2\sigma}6a_1^2V\lambda\mu^2 + 12a_1^2V\lambda + \frac{1}{-\mu^2+\lambda^2\sigma}6b_1^2V\lambda^2 = 0,$$

$$12a_1V\mu - 12a_1^2V\mu - \frac{12b_1^2V\lambda\mu}{-\mu^2+\lambda^2\sigma} = 0,$$

$$-6b_1V + 12a_1b_1V = 0,$$

$$-6a_1V + 6a_1^2V + \frac{1}{-\mu^2+\lambda^2\sigma}6b_1^2V\lambda = 0. \tag{21}$$

We obtain the roots of Eq. (21) with the aid of Mathematica as

$$a_1 = \frac{1}{2},\ \lambda \neq 0,\ b_1 = \pm\frac{\sqrt{-\mu^2+\lambda^2\sigma}}{2\sqrt{\lambda}},\ V = \pm\mathrm{i}\sqrt{\lambda},\quad \mu^2+\lambda^2\sigma \neq 0,\ \mathrm{i} = \sqrt{-1}, \tag{22}$$

Substituting Eq. (22) into Eq. (15), we find the following solutions of Eq. (12):

Family 1.2.1

$$u(x,t) = a_0 + \frac{1}{2}\times\left(\frac{A_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\xi\right) - A_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}\xi\right)}{A_1\sin\left(\sqrt{\lambda}\xi\right) + A_2\cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda}}\right) \pm\frac{\sqrt{-\mu^2+\lambda^2\sigma}}{2\sqrt{\lambda}} \frac{1}{\left(A_1\sin\left(\sqrt{\lambda}\xi\right) + A_2\cos\left(\sqrt{\lambda}\xi\right) + \mu/\lambda\right)}, \tag{23}$$

where $\sigma = A_1^2 + A_2^2$ and $\xi = x - (\mathrm{i}\sqrt{\lambda})t$.

Family 1.2.2

For $A_1 = 0$, $A_2 > 0$ and $\mu = 0$ in Eq. (23), we have the solitary solution

$$u(x,t) = a_0 + \frac{1}{2A_2}\sqrt{\lambda\sigma}\sec\left(\sqrt{\lambda}\xi\right) - \frac{1}{2}\sqrt{\lambda}\tan\left(\sqrt{\lambda}\xi\right), \tag{24a}$$

$$u(x,t) = a_0 - \frac{1}{2A_2}\sqrt{\lambda\sigma}\sec\left(\sqrt{\lambda}\xi\right) - \frac{1}{2}\sqrt{\lambda}Tan\left(\sqrt{\lambda}\xi\right). \tag{24b}$$

Family 1.2.3

For $A_2 = 0$, $A_1 > 0$ and $\mu = 0$ in Eq. (23), we have the solitary solution

$$u(x,t) = a_0 + \frac{1}{2}\sqrt{\lambda}\cot\left(\sqrt{\lambda}\xi\right) + \frac{1}{2A_1}\sqrt{\lambda\sigma}\csc\left(\sqrt{\lambda}\xi\right), \tag{25a}$$

$$u(x,t) = a_0 + \frac{1}{2}\sqrt{\lambda}\cot\left(\sqrt{\lambda}\xi\right) - \frac{1}{2A_1}\sqrt{\lambda\sigma}\csc\left(\sqrt{\lambda}\xi\right). \tag{25b}$$

Setting $\lambda = 1$, $\sigma = -1$, $A_2 = 1$, $a_0 = 1$, $A_1 = 0$, $\mu = 1$, we have solution of Eq. (23) (see Figs. 4, 5).

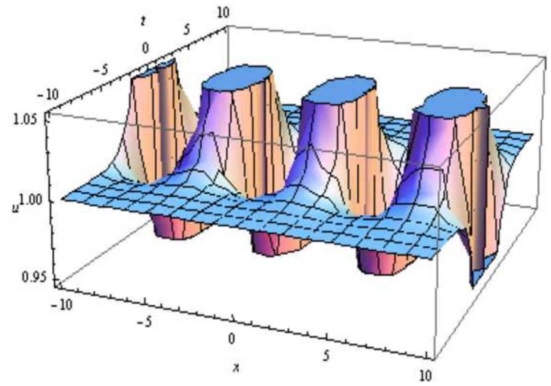

Fig. 4. The profile of solution Eq. (23), (real part).

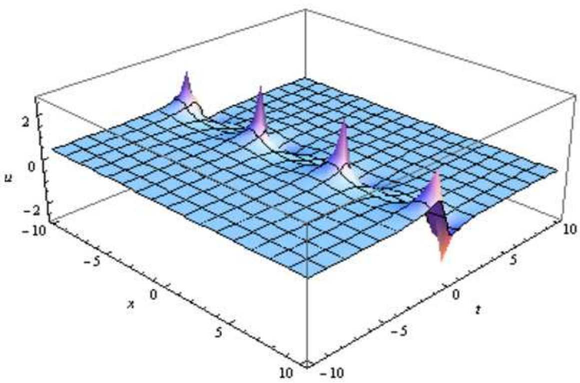

Fig. 5. The profile of solution Eq. (23), (imaginary part).

Case 1.3. For $\lambda = 0$, substituting Eq. (15) into Eq. (14) and using Eq. (2) and Eq. (4) yield a set of algebraic equations for a_0, a_1, b_1, μ, σ, λ and V. These systems are

$$-a_1V^3\mu + \frac{1}{A_1^2 - 2A_2\mu}6a_1V\mu^3 - \frac{1}{A_1^2 - 2A_2\mu}12a_1^2V\mu^3 = 0,$$

$$\frac{1}{A_1^2 - 2A_2\mu}6b_1V\mu - \frac{1}{A_1^2 - 2A_2\mu}12a_1b_1V\mu = 0,$$

$$b_1V^3 - \frac{1}{A_1^2 - 2A_2\mu}12b_1V\mu^2 + \frac{1}{A_1^2 - 2A_2\mu}24a_1b_1V\mu^2 = 0,$$

$$-\frac{1}{A_1^2 - 2A_2\mu}3a_1V\mu^2 + \frac{1}{A_1^2 - 2A_2\mu}6a_1^2V\mu^2 + a_1V^3 = 0,$$

$$12a_1V\mu - 12a_1^2V\mu - \frac{1}{A_1^2 - 2A_2\mu}12b_1^2V\mu = 0,$$

$$-6b_1V + 12a_1b_1V = 0 - 6a_1V + 6a_1^2V + \frac{1}{A_1^2 - 2A_2\mu}6b_1^2V = 0. \tag{26}$$

We have the roots of Eq. (26) with the aid of Mathematica as

$$a_1 = \frac{1}{2}, \quad b_1 = \pm\frac{1}{2}\sqrt{A_1^2 - 2A_2\mu}, \quad V = 0, A_1^2 - 2A_2\mu \neq 0. \tag{27}$$

Family 1.3.1

It is not possible to write solution of Eq. (12) because of $V = 0$.

Remark 3.1. Wazwaz [26, 29] obtained multiple kink solutions of Eq. (12) by using simplified Hirota's method and found multiple soliton solutions by using the Hereman–Nuseir method. Also, we acquire different periodic wave solutions from Wazwaz's solutions.

Remark 3.2. It is shown that the obtained solutions satisfy Eq. (12). This process is carried out using Mathematica.

Example 2. Let us consider the BKP equation [26] of the form

$$u_{xt} + u_x^2 + uu_{xx} + u_{xxx} + u_{yy} = 0, \tag{28}$$

If we can use transformation $u(x,t) = u(\xi)$, $\xi = x + y - Vt$, Eq. (28) becomes

$$-Vu'' + (u')^2 + uu'' + u''' + u'' = 0, \tag{29}$$

and if we integrate twice Eq. (29), we have

$$-Vu + \frac{1}{2}u^2 + u' + u = 0, \tag{30}$$

where integration constants are taken as zero. When balancing u' with u^2, we have $N = 1$. Thus, we choose solution of Eq. (30) as

$$u(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi). \tag{31}$$

Case 2.1. For $\lambda < 0$, substituting Eq. (31) into Eq. (30) and using Eq. (2) and Eq. (4) yield a set of algebraic equations for a_0, a_1, b_1, μ, σ, λ and V. These systems are

$$a_0 + \frac{a_0^2}{2} - a_0 V - a_1\lambda - \frac{1}{2\left(\mu^2+\lambda^2\sigma\right)} b_1^2\lambda^2 = 0,$$

$$b_1 + a_0 b_1 - b_1 V + a_1\mu + \frac{1}{\mu^2+\lambda^2\sigma} b_1^2\lambda\mu = 0,$$

$$a_1 + a_0 a_1 - a_1 V = 0,$$

$$-b_1 + a_1 b_1 = 0,$$

$$-a_1 + \frac{1}{2}a_1^2 - \frac{1}{2(\mu^2+\lambda^2\sigma)} b_1^2\lambda = 0. \tag{32}$$

We obtain the roots of Eq. (32) with the aid of Mathematica as

$$a_0 = \pm i\sqrt{\lambda},\ a_1 = 1,\ \lambda \neq 0,\ b_1 = \pm\frac{\sqrt{-\mu^2-\lambda^2\sigma}}{\sqrt{\lambda}},$$
$$V = 1 + a_0,\ \mu^2+\lambda^2\sigma \neq 0,\ i = \sqrt{-1}. \tag{33}$$

Substituting Eq. (33) into Eq. (31), we obtain the following solutions of Eq. (28):

Family 2.1.1

$$\begin{aligned} u(x,y,t) &= \pm \mathrm{i}\sqrt{\lambda} \\ &+ \left(\frac{A_1\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right) + A_2\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}}\right) \\ &\pm \frac{\sqrt{-\mu^2-\lambda^2\sigma}}{\sqrt{\lambda}} \\ &\times \frac{1}{\left(A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right) + \mu/\lambda\right)}, \end{aligned} \tag{34}$$

where $\sigma = A_1^2 - A_2^2$ and $\xi = x + y - (1 + \mathrm{i}\sqrt{\lambda})t$.

Family 2.1.2

For $A_1 = 0$, $A_2 > 0$ and $\mu = 0$ in Eq. (34), we have the solitary solutions

$$u(x,y,t) = \mathrm{i}\sqrt{\lambda} \pm \frac{1}{A_2}\sqrt{-\lambda\sigma}\,\mathrm{sech}\left(\sqrt{-\lambda}\xi\right) + \sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right), \tag{35a}$$

$$u(x,y,t) = -\mathrm{i}\sqrt{\lambda} \pm \frac{1}{A_2}\sqrt{-\lambda\sigma}\,\mathrm{sech}\left(\sqrt{-\lambda}\xi\right) + \sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right). \tag{35b}$$

Family 2.1.3

For $A_2 = 0$, $A_1 > 0$ and $\mu = 0$ in Eq. (34), we have the solitary solutions

$$u(x,y,t) = \sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) + \left(\frac{\mathrm{i}\lambda}{\sqrt{\lambda}} + \frac{1}{A_1}\sqrt{-\lambda\sigma}\right)\mathrm{csch}\left(\sqrt{-\lambda}\xi\right), \tag{36a}$$

$$u(x,y,t) = \sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) - \left(\frac{\mathrm{i}\lambda}{\sqrt{\lambda}} + \frac{1}{A_1}\sqrt{-\lambda\sigma}\right)\mathrm{csch}\left(\sqrt{-\lambda}\xi\right). \tag{36b}$$

Setting $\lambda = -1$, $\sigma = 1$, $A_2 = 2$, $a_0 = 1$, $A_1 = -2$, $\mu = 1$, $y = 0$, we have solution of Eq. (34) (see Fig. 6).

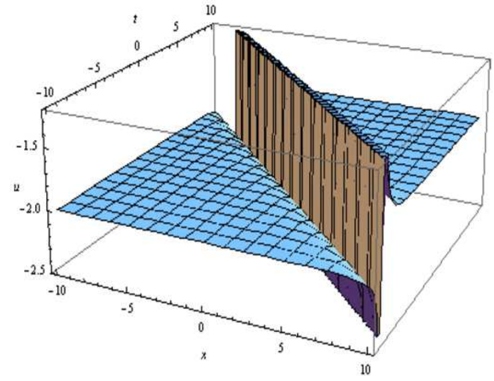

Fig. 6. The profile of solution Eq. (34).

Setting $\lambda = -1$, $\sigma = -1$, $A_2 = 1$, $a_0 = 1$, $A_1 = 0$, $\mu = -1$, $y = 0$, we have solution of Eq. (34) (see Fig. 7).

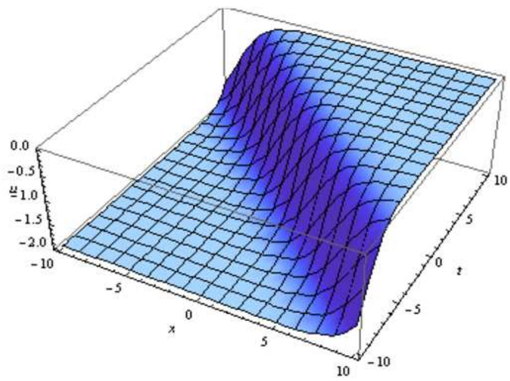

Fig. 7. The profile of solution Eq. (34).

Setting $\lambda = -1$, $\sigma = -1$, $A_2 = \mathrm{i}$, $a_0 = 1$, $A_1 = -1$, $\mu = 10$, $y = 0$, we have solution of Eq. (34) (see Figs. 8, 9).

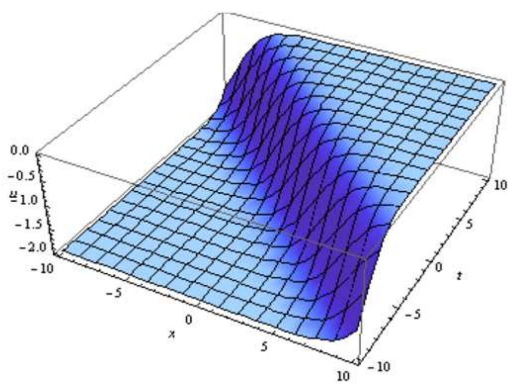

Fig. 8. The profile of solution Eq. (34), (real part).

Case 2.2. For $\lambda > 0$, substituting Eq. (31) into Eq. (30) and using Eq. (2) and Eq. (4) yields a set of

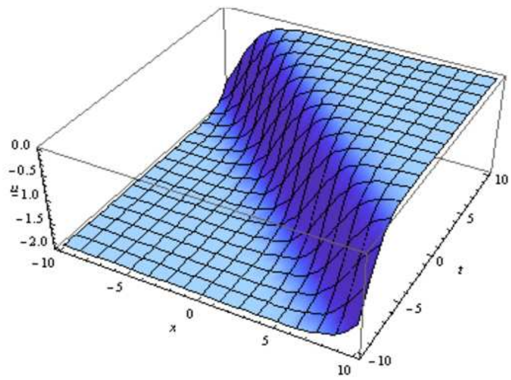

Fig. 9. The profile of solution Eq. (34), (imaginary part).

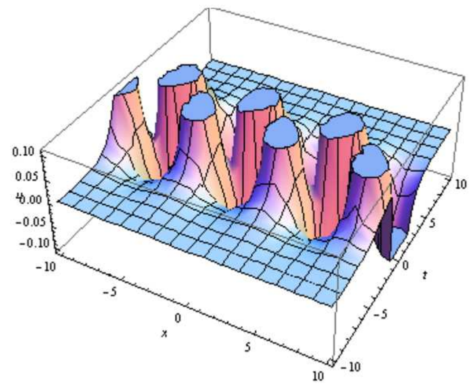

Fig. 10. The profile of solution Eq. (39), (real part).

algebraic equations for a_0, a_1, b_1, μ, σ, λ and V. These systems are

$$a_0+\frac{a_0^2}{2}-a_0V-a_1\lambda+\frac{1}{2\left(-\mu^2+\lambda^2\sigma\right)}b_1^2\lambda^2=0,$$

$$b_1+a_0b_1-b_1V+a_1\mu-\frac{1}{-\mu^2+\lambda^2\sigma}b_1^2\lambda\mu=0,$$

$$a_1+a_0a_1-a_1V=0,$$

$$-b_1+a_1b_1=0,$$

$$-a_1+\frac{1}{2}a_1^2+\frac{1}{2(-\mu^2+\lambda^2\sigma)}b_1^2\lambda=0. \quad (37)$$

We find the roots of Eq. (37) with the aid of Mathematica as

$$a_0=\pm\mathrm{i}\sqrt{\lambda},\ a_1=1,\ \lambda\neq 0,\ b_1=\pm\frac{\sqrt{-\mu^2+\lambda^2\sigma}}{\sqrt{\lambda}},$$
$$V=1+a_0,\ \mu^2+\lambda^2\sigma\neq 0,\ \mathrm{i}=\sqrt{-1}. \quad (38)$$

Substituting Eq. (38) into Eq. (31), we obtain the following solutions of Eq. (28):

Family 2.2.1

$$u(x,y,t)=\pm\mathrm{i}\sqrt{\lambda}+\left(\frac{A_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\xi\right)-A_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}\xi\right)}{A_1\sin\left(\sqrt{\lambda}\xi\right)+A_2\cos\left(\sqrt{\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)\pm\frac{\sqrt{-\mu^2+\lambda^2\sigma}}{\sqrt{\lambda}}\times\frac{1}{\left(A_1\sin\left(\sqrt{\lambda}\xi\right)+A_2\cos\left(\sqrt{\lambda}\xi\right)+\mu/\lambda\right)}, \quad (39)$$

where $\sigma=A_1^2+A_2^2$ and $\xi=x+y-(i\sqrt{\lambda})t$.

Setting $\lambda=1$, $\sigma=-1$, $A_2=0$, $a_0=1$, $A_1=\mathrm{i}$, $\mu=1$, $y=0$, we have solution of Eq. (39) (see Figs. 10, 11).

Family 2.2.2

For $A_1=0$, $A_2>0$ and $\mu=0$ in Eq. (39), we have the solitary solution

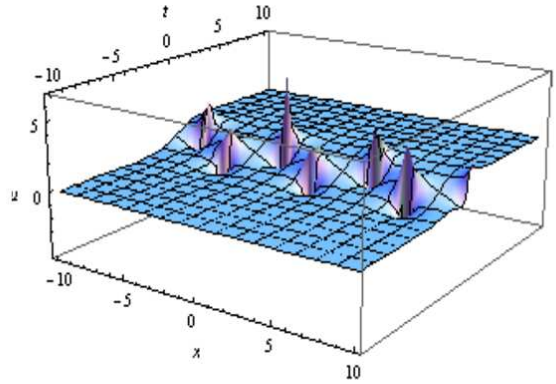

Fig. 11. The profile of solution Eq. (39), (imaginary part).

$$u(x,y,t)=\frac{1}{A_2}\sqrt{\lambda\sigma}\sec\left(\sqrt{\lambda}\xi\right)-\frac{\lambda}{\sqrt{\lambda}}\left(-\mathrm{i}+\tan\left(\sqrt{\lambda}\xi\right)\right). \quad (40)$$

Family 2.2.3

For $A_2=0$, $A_1>0$ and $\mu=0$ in Eq. (39), we have the solitary solution

$$u(x,y,t)=\frac{\lambda}{\sqrt{\lambda}}\left(\mathrm{i}+\cot\left(\sqrt{\lambda}\xi\right)\right)+\frac{1}{A_1}\sqrt{\lambda\sigma}\csc\left(\sqrt{\lambda}\xi\right). \quad (41)$$

Case 2.3. For $\lambda=0$, substituting Eq. (31) into Eq. (30) and by using Eq. (2) and Eq. (4) yields a set of algebraic equations for a_0, a_1, b_1, μ, σ, λ and V. These systems are

$$a_0+\frac{a_0^2}{2}-a_0V=0,$$

$$b_1+a_0b_1-b_1V+a_1\mu-\frac{1}{A_1^2-2A_2\mu}b_1^2\mu=0,$$

$$a_1+a_0a_1-a_1V=0,$$

$$-b_1+a_1b_1=0,$$

$$-a_1+\frac{1}{2}a_1^2+\frac{1}{2(A_1^2-2A_2\mu)}b_1^2=0. \quad (42)$$

We find the roots of above algebraic system with the aid of Mathematica as

$$a_0 = 0,\ a_1 = 1,\ b_1 = \pm\sqrt{A_1^2 - 2A_2\mu},\ V = 1,$$

$$A_1^2 - 2A_2\mu \neq 0. \quad (43)$$

Substituting Eq. (43) into Eq. (31), we obtain the following solutions of Eq. (28):

Family 2.3.1

$$u(x,y,t) = \left(\frac{\mu\xi + A_1}{\frac{\mu}{2}\xi^2 + A_1\xi + A_2}\right) + \sqrt{A_1^2 - 2A_2\mu}\left(\frac{1}{\frac{\mu}{2}\xi^2 + A_1\xi + A_2}\right), \quad (44)$$

where $\xi = x + y - t$.

Setting $A_2 = 1$, $A_1 = 1$, $\mu = -1$, $y = 0$, we have solution of Eq. (44) (see Fig. 12).

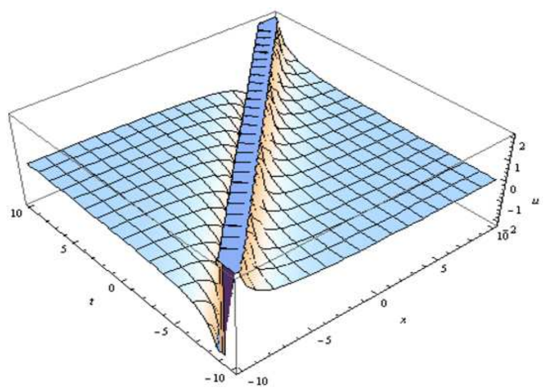

Fig. 12. The profile of rational function solution Eq. (44).

Remark 3.3. Taghizadeh et al. [30] obtained solutions of type of tan and tanh for Eq. (28) by using the first integral method. Wazwaz [31] found kink solutions and periodic solutions for Eq. (28) by using the tanh-coth method. Also, we find more different solutions and rational solution.

Remark 3.4. It is shown that the obtained solutions satisfy Eq. (28). This process is carried out using Mathematica.

4. Conclusions

In this study, we obtain some exact solutions of the new fifth order nonlinear evolution and the BKP equations by using the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method. The method used in this work can be used to search for the solutions of other NPDEs. Furthermore, this method has been successfully applied to solve some nonlinear wave equations. The effectiveness of this method is reliable and gives more solutions than other analytical methods. These equations with time-dependent coefficient and the stochastic perturbation terms will be reported in next works by using the same method. Additionally, the numerical results that are obtained in this study are in conjunction with the analytical development here (see Figs. 1–12).

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