

# Paths vs. Winding Numbers for the Two-Magnon Sector of the XXX Heisenberg Magnet

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Bethe Ansatz is the famous method of determination of eigenstates and eigenenergies for a wide range of quantum problems, e.g. for the Heisenberg XXX  $s = 1/2$  model. The Bethe equations applied to solve the problem of  $N$  nodes and  $r$  overturned spins on a magnetic chain are labeled by sets of winding numbers  $\{n_i\}$ , however the condition for admissible sets give an overcomplete number of results. On the other hand, combinatorial objects, so called “paths”, give the exact number of eigenvectors for the problem described by  $(N, r)$  values. The paper presents the method of determining the set of winding numbers from the appropriate path for the sector of  $r = 2$  spin deviations.

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## 1. Introduction

1-dimensional spin models are still the subject of interest of many researchers all over the world [1–3].

The problem of the Heisenberg XXX model can be considered with various approaches, i.e. analytical, algebraic, as well as combinatoric [4–6]. They are related with each other by appropriate bijections, which link up corresponding parameters according to a chosen method. The Robinson–Schensted–Knuth algorithm (RSK) [7, 8] joins together the set of all magnetic configurations  $f$  with standard Young tableaux (SYT), while the Kerov–Kirillov–Reshetikhin algorithm (KKR) [9, 10] shows how to relate the set of SYT with the set of rigged string configurations RC. In the present paper the focus is only on a small fragment of theory, which allows one to obtain a set of winding numbers  $\{n_i\}$  as an element of algebraic and analytic solutions [11], with combinatoric objects, called paths [12–14].

The motivation for the subject of the paper was the fact that the condition describing admissible sets of winding numbers in the Bethe equations results in overcomplete number of solutions, while such a problem does not appear in the case of combinatoric paths. The paper proposes a procedure of reading out of peaks of pyramids containing a path and collecting it in the form of a set of winding numbers.

In general, the problem is described by sets  $(\tilde{N}, \tilde{r})$ , where  $\tilde{N} = \{j = 1, 2, \dots, N\}$  is the set of nodes of the magnetic ring, and  $\tilde{r} = \{\alpha = 1, 2, \dots, r\}$  stands for the set of overturned spins on a ring. Then, the Hamiltonian of the XXX Heisenberg magnet with the nearest neighbours interaction, and spin  $s = 1/2$  on each node, can be written as [15]:

$$\hat{H} = -2 \sum_{j \in \tilde{N}} \hat{s}_j \cdot \hat{s}_{j+1} \quad (1)$$

with  $\hat{s}_j$  indicating a spin operator at the  $j$ -th position in a chain. In the general case of  $r$  spin deviations from the saturation state  $|+++ \dots +\rangle$ , eigenstates of  $H$  can be introduced as

$$|\Psi\rangle = \sum_{1 \leq j_1 < j_2 < \dots < j_r \leq N} a(j_1, j_2, \dots, j_r) |j_1 j_2 \dots j_r\rangle, \quad (2)$$

with  $\mathbf{j} = |j_1, j_2, \dots, j_r\rangle$  indicating positions of  $r$  overturned spins in the chain.  $\mathbf{j}$  can be also presented in the form of a magnetic configuration  $f = |+-+ + - \dots +\rangle$ , with “-” indicating a spin deviation at the  $j_\alpha$ -th position on the chain. A key role in (2) is being played by coefficients  $a(j_1, j_2, \dots, j_r)$ ,

$$a(j_1, j_2, \dots, j_r) = \quad (3)$$

$$\sum_{\pi \in \Sigma_r} \exp \left( i \left( \sum_{\alpha=1}^r p_{\pi(\alpha)} j_\alpha + \frac{1}{2} \sum_{\substack{1 \leq \alpha' < \alpha \leq r \\ \pi(\alpha') > \pi(\alpha)}} \phi_{\pi(\alpha')\pi(\alpha)} \right) \right),$$

known as *Bethe Ansatz* [16, 17]. Here, the sum runs over all permutations  $\pi$  of positions  $j_\alpha$  of pseudoparticles, while pseudomomenta  $p_\alpha$  and phases  $\phi_{\alpha, \alpha'}$  of interacting particles are related with each other by the Bethe equations, which in transcendental form read

$$2 \cot \frac{\phi_{\alpha, \alpha'}}{2} = \cot \frac{p_\alpha}{2} - \cot \frac{p_{\alpha'}}{2}, \quad \phi_{\alpha, \alpha'} = -\phi_{\alpha', \alpha} \quad (4)$$

and

$$N p_\alpha = 2\pi n_\alpha + \sum_{\alpha' \neq \alpha} \phi_{\alpha, \alpha'}, \quad \alpha \in \tilde{r}. \quad (5)$$

Winding numbers  $\{n_\alpha\}$ ,  $\alpha \in \tilde{r}$ , play the role of the set of quantum numbers, which enumerate the given Bethe eigenstate. From the form of the Bethe substitution (3) it turns out that for  $n_\alpha$  and  $n_\alpha + N$  one gets the same physical state. Thus, it is sufficient to limit considerations to ordered set  $\{n_\alpha\}$  satisfying the condition

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$$-N/2 < n_1 \leq n_2 \leq \dots \leq n_r \leq N/2. \quad (6)$$

Then, the total quasimomentum of the problem, in accordance with (6), becomes

$$k = \frac{N}{2\pi} \left( \sum_{\alpha} p_{\alpha} \text{mod} N \right) = \left( \sum_{\alpha} n_{\alpha} \right) \text{mod} N. \quad (7)$$

In the general case of  $r$  overturned spins on  $N$  nodes system, eigenenergies of the states (2) are of the form

$$E = -4 \sum_{\alpha=1}^r (1 - \cos p_{\alpha}). \quad (8)$$

For the problem of  $N$  nodes and  $r$  overturned spins, in the Hilbert space  $\mathcal{H}^{(r)}$ , the number of eigenstates is

$$\dim \mathcal{H}^{(r)} = \binom{N}{r}, \quad (9)$$

where the case  $(N, r)$  also includes solutions with  $r' = 0, 1, \dots, r-1$ , i.e. states with the lower weight (for  $r = r'$  one gets the highest weight states). Next section contains considerations for the highest weight states in  $r = 2$  sector.

### 2. Combinatorial objects — paths

Each Heisenberg eigenstate can be parametrized by a combinatorial object called *path*. One can consider the set of all points on the  $(j, l)$  plane given by non-negative integer coefficients, with  $j \geq l$ . Then, a path is defined recursively on grounds of  $j = 1, 2, \dots, N$  as a solid line consisting of consecutive segments [18, 19], such that a segment related to a node  $(j-1) \in \tilde{N}$  connects the point  $(j-1, l)$  with  $(j, l+1)$  (a step up), or with  $(j, l-1)$  (a step down). Thus, a path can be defined as a sequence  $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_N)$ , with

$$\mathcal{P}_j \in \{ / \text{ a step "up", } \backslash \text{ a step "down"} \}, j \in \tilde{N}, \quad (10)$$

where the first step,  $\mathcal{P}_1$ , is always "up". An exemplary path is introduced in Fig. 1.

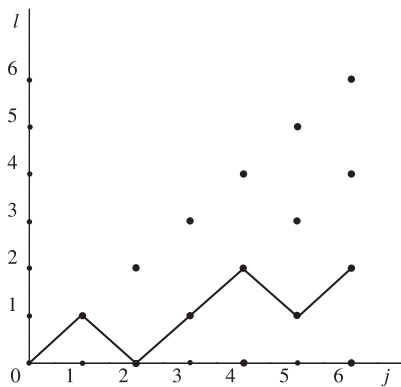


Fig. 1. An exemplary path  $\mathcal{P}$  in the plane  $(j, l)$ .

The detailed description of drawing a path is introduced in [19]. For the purpose of the present paper, there have been chosen only the most crucial conclusions from the paths theory.

Drawing a path may result in obtaining so-called *strings*. If a magnetic configuration  $f$  contains a set of  $l$  minuses followed by a set of  $l$  pluses, then such an object is called a *string* of length  $2l$ :

$$f = | \dots \underbrace{- \dots -}_l \underbrace{+ \dots +}_l \dots \rangle. \quad (11)$$

Every admissible path  $\mathcal{P} = \mathcal{P}(f) = (\mathcal{P}_1(f), \mathcal{P}_2(f), \dots, \mathcal{P}_N(f))$  is associated with an appropriate magnetic configuration  $f$  in a bijective way. The idea of drawing consecutive steps of a path in the  $(l, j)$  plane can be summarized in a brief description. Each sector  $\mathcal{P}_j(f)$  is related to “-”, or “+” in a magnetic configuration  $f$ , according to following conditions:

$$\mathcal{P}_j(f) = \begin{cases} / & \text{if } f(j) = \text{"-"}, \\ & \text{or if } f(j) = \text{"+" and it does not} \\ & \text{belong to a string,} \\ \backslash & \text{if } f(j) = \text{"+" and it belongs} \\ & \text{to a string.} \end{cases} \quad (12)$$

Any path  $\mathcal{P}(f)$  in the plane  $(j, l)$  reveals the shape of slopes of pyramids, which consist of  $l$  steps up, followed by consecutive  $l$  steps down.

Figure 2 shows an exemplary isolated  $l$ -string consisting of  $l$  Bethe pseudoparticles followed by  $l$  nodes with spin “+”.

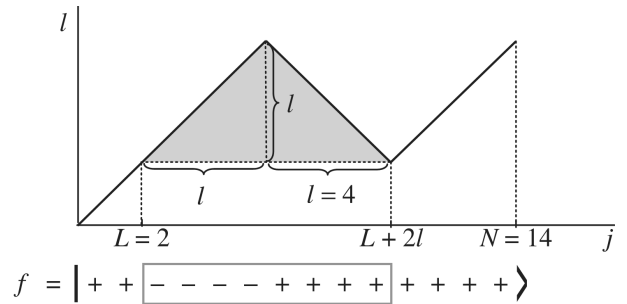


Fig. 2. An exemplary path  $\mathcal{P}(f)$  for  $N = 14$  containing 4-string ( $l = 4$ ). The height of the pyramid is also equal 4. The picture of the string on the magnetic configuration  $f$  is drawn as the contour  $\square$ .

### 3. From paths to winding numbers

Analytical solutions of the Bethe equations, as well as combinatoric approach, describe the same physical problem, so that one could expect that a set  $\{n_i\}$  of winding numbers can be somehow related with parameters describing a path. This section contains a proposition how to link up algebraic and combinatoric elements of the problem being discussed.

The procedure consists in determining an algorithm of reading out positions of peaks of a pyramid of height  $l$  in a path  $\mathcal{P}$  on the  $(j, l)$  plane, and presenting individual values  $j_{\alpha} \text{mod} N$  according to the order given by the condition (6). For the case of  $r = 2$  spin deviations the only possible values for the the height of a pyramid (or pyramids) within the frame of a given path are  $l = 1$  or  $l = 2$ . Then, the problem can be introduced as follows:

•  $l = 1$

In such a case it is necessary to read out consecutive values  $j_\alpha \bmod N$ , denoting appropriate peaks of pyramids and write it down in an increasing order. Then, neighbouring values in the set  $\{n_i\}$  differ at least by 2. An example for two  $l = 1$  pyramids is depicted in Fig. 3a;

•  $l = 2$

Here, one should read out the peak of a pyramid  $j_\alpha \bmod N$  and introduce it as a sum of the two smallest values of integers  $(n_1, n_2)$  satisfying the relation  $n_1 + n_2 = j_\alpha \bmod N$ . Thus, admissible winding numbers yield  $n_2 = n_1$ , or  $n_2 = n_1 + 1$ . Figure 3b shows an example illustrating the procedure for a  $l = 2$  pyramid.

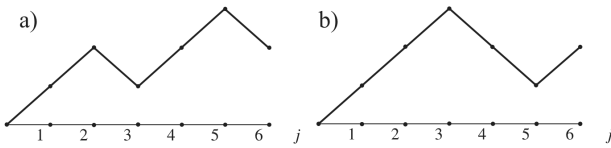


Fig. 3. Procedure of obtaining a set of winding numbers as a reading out peaks of pyramids,  $N = 6$ ,  $r = 2$ : (a) two pyramids  $(j_1, j_2) \bmod 6 = (2, 5) \bmod 6 = (2, -1) \Rightarrow (n_1, n_2) = (-1, 2)$ , (b) one pyramid  $j \bmod 6 = 3 \Rightarrow (n_1, n_2) = (1, 2)$ .

In this way one can show that a given set  $\{n_i\}$  corresponds to exactly one path  $\mathcal{P}$ . Figure 4 introduces all possible paths for the case  $(N, r) = (6, 2)$  together with appropriate magnetic configurations, as well as sets of winding numbers  $\{n_i\}$  read out from appropriate paths. The picture of each pyramid is indicated on a magnetic configuration as a contour  $\square$ .

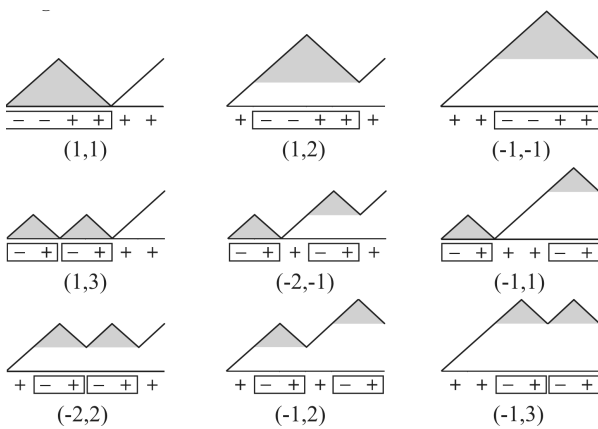


Fig. 4. Paths with pyramids and sets of winding numbers for the case  $N = 6$ ,  $r = 2$ .

4. Conclusions

The paper presents a proposition of finding a relation between quantum numbers which parametrize Bethe equations in the transcendental form with combinatoric objects, which parametrize every eigenstate of the problem being considered. The procedure of reading out peaks of pyramids introducing it in the proper order in the form of winding numbers shows that there is an exact relation “one to one”. Surely, it is possible to use different approaches of Bethe Ansatz, e.g. by using string hypothesis, however the method proposed in the paper allows to select appropriate sets of winding numbers — at least in the sector of  $r = 2$  spin deviations — thanks to combinatorial objects, like paths, and obtain parameters  $p_\alpha$  and  $\phi_{\alpha, \alpha'}$  which label Bethe eigenstates.

References

- [1] A.N. Kirillov, R. Sakamoto, *J. Phys. A Math. Theor.* **47**, 205207 (2014).
- [2] W. Hao, R.I. Nepomechie, A.J. Sommes, *Phys. Rev. E* **88**, 052113 (2013).
- [3] D. Jakubczyk, *Rep. Math. Phys.* **72**, 379 (2013).
- [4] W.J. Caspers, *Spin Systems*, World Sci., Singapore 1989.
- [5] J. Milewski, G. Banaszak, T. Lulek, M. Łabuz, R. Stagraczyński, *OSID* **19**, 1250012 (2012).
- [6] G. Banaszak, B. Lulek, T. Lulek, J. Milewski, B. Szydło, *Rep. Math. Phys.* **71**, 205 (2013).
- [7] A. Lascoux, B. Leclerc, J.-Y. Thibon, in: *Algebraic Combinatorics on Words*, Ed. M. Lothaire, Univ. Press, Cambridge 2001.
- [8] B. Lulek, T. Lulek, M. Labuz, R. Stagracyński, *Physica B* **405**, 2654 (2010).
- [9] A.N. Kirillov, A. Schilling, M. Shimozono, *Sel. Math. N.S.* **8**, 67 (2002).
- [10] P. Jakubczyk, A. Wal, D. Jakubczyk, T. Lulek, *Comp. Phys. Comm.* **183**, 1354 (2012).
- [11] W.J. Caspers, A. Wal, M. Łabuz, M. Kuźma, T. Lulek, *J. Math. Phys.* **45**, 391 (2004).
- [12] S. Dasmahapatra, O. Foda, *Int. J. Mod. Phys.* **38**, 1041 (1997).
- [13] S.O. Warnaar, *J. Stat. Phys.* **82**, 657 (1996).
- [14] T.A. Welsh, *Memoirs AMS* **175**, 827 (2005).
- [15] H. Bethe, *Z. Phys.* **71**, 205 (1931) (in German); English translation in: D.C. Mattis, *The Many-Body Problem*, World Sci., Singapore 1993, p. 689).
- [16] M. Karbach, G. Müller, *Comp. Phys.* **11**, 36 (1997).
- [17] W.J. Caspers, M. Łabuz, A. Wal, M. Kuźma, T. Lulek, *J. Phys. A Math. Gen.* **36**, 5369 (2003).
- [18] T. Lulek, B. Lulek, D. Jakubczyk, P. Jakubczyk, *Physica B Phys. Condens. Matter* **382**, 162 (2006).
- [19] T. Lulek, *Banach Center Publ.* **78**, 231 (2007).